The Intensional Many - Conservativity Revisited *

Harald Bastiaanse
Institute for Logic, Language and Computation
University of Amsterdam, Postbox 94242
1090 GE Amsterdam, The Netherlands

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Abstract. Following on Westerståhl's argument that many is not Conservative (Westerståhl, 1985), I propose an intensional account of Conservativity as well as intensional versions of EXT and Isomorphism closure. I show that an intensional reading of many can easily possess all three of these, and provide a formal statement and proof that they are indeed proper intensionalizations.
It is then discussed to what extent these intensionalized properties apply to various existing readings of many.

Keywords: Generalized Quantifiers, Many, Intensionality, Logicality

1. Introduction

In the theory of Generalized Quantifiers, much weight is given to the property of Conservativity, which for a binary quantifier $\mathbf{Q}$ can be paraphrased as

$$\mathbf{Q}AB \text{ if and only if } \mathbf{Q}A(A \text{ and } B)$$

Conservativity is often suggested as a linguistic universal (eg(Barwise and Cooper, 1981)(Keenan and Stavi, 1986)), as it seems almost trivially true for virtually every natural language determiner. For instance, all of the following seem trivial enough:

No man is perfect.	⇔	No man is a perfect man.
Seven women are running.	⇔	Seven women are women who are running.
All good philosophers are wise.	⇔	All good philosophers are good philosophers who are wise.
Many men smoke.	⇔	Many men are men who smoke.

That last one, however, is actually problematic.

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The problem

Westerståhl(Westerståhl, 1985) coined the following traditional example to demonstrate the problem. In a certain class at a certain college 10 out of the 30 students got the highest grade on a certain exam, which is unusually many. Those same 10 students are the only ones in the class who are right-handed, which is unusually few. Let A be this set of students, B_1 the set of students at the college who got the highest grade, and B_2 the set of right-handed students at the college.
Now in the traditional formalization, the interpretation of many in the context M is a function which when applied to A gives the set of sets B such that "Many A[s] [are] B." is true.
Thus, the assumptions from the example are expressed as the following:

$$B_1 \in \|\mathbf{many}\|^M(A), B_2 \notin \|\mathbf{many}\|^M(A)$$

If Conservativity were true of many, from this we could then conclude.

$$A \cap B_1 \in \|\mathbf{many}\|^M(A), A \cap B_2 \notin \|\mathbf{many}\|^M(A)$$

But of course $A \cap B_1$ and $A \cap B_2$ are in fact the same set. Hence many can not be Conservative, or at least not without using two different interpretations to arbitrarily fix the problem.

Issues

It is hard to argue with the formal part of this argument, but it does leave something to be desired. With Conservativity being untenable in this example, one of the following is now true: either

- Not many students in the class are students in the class who got the highest grade on the exam (While we at the same time accept many students in the class did get the highest grade),

or

- Many students in the class are right-handed students in the class. (While we at the same time accept not many students in the class are right-handed.)

Neither of these is a particularly attractive statement to endorse, and then there is the question of which of the two we should pick. Westerståhl offers no answer to this question, and it is hard to see how anyone could; they seem equally counterintuitive and "resolve" the inconsistency equally well.
Because of these issues, I would say that rather than a straightforward

case against Conservativity for many, what the example really provides is a paradox arising from a different problem.

What I think this problem is should not come as a surprise to anyone: it is that the theory of Generalized Quantifiers as formalized by Barwise & Cooper (Barwise and Cooper, 1981) and Westerståhl (and most others in that and other systems) is inherently extensional: while it involves possible universes and how quantifiers deal with them, it does not allow properties to be identified as more than subsets of a specific universe.[1] We can use it to talk about "right-handed students at the college, in M", but not of right-handedness as a property in its own right identified independent of any one universe. We are limited to identifying properties by their local extensions, whereas many requires an intensional approach.

This, of course, is not a particularly new thought. The fact that many is intensional has been generally agreed upon after being pointed out by Keenan and Stavi (Keenan and Stavi, 1986). What *is* interesting here is that we shall see that when it is treated in this way, Conservativity is reclaimed.

2. Intensional Conservativity

On notation

Without getting into too much formalism, let us sketch what such an intensional treatment would look like. In the traditional approaches capital letters like A and B get used to denote extensions, extensions of predicates which most of the time are not themselves used much at all. We will intensionalize this, similarly using such letters to denote properties instead, while mostly not mentioning the (complex) predicates of which these properties are the intension. Of course the related extensions are still far from irrelevant to us; for a given property X, we denote the relevant extension in the model m as $[\![X]\!]^m$.[2]

As usual, models are given by domains and the extensions of properties there. Unless otherwise noted, models are denoted in lower-case and their respective domains with the same name in upper-case.

The essential non-cosmetic change is that quantifiers are applied to properties rather than extensions. This cannot be appropriately done if

[1] Or rather, it has some of that in the background, but does not give quantifiers access to more than the local extensions.

[2] If you wish, you may think of properties as functions assigning to each model an extension in that model.

quantifiers are contextualized first, so this happens only after applying them: instead of $[\![Q]\!]^m(A)$ we have $[\![Q(A)]\!]^m$. Finally, in line with more recent treatments such as (Westerståhl, 2007), we use $Q_m AB$ as short for $B \in [\![Q(A)]\!]^m$.

2.1. Intensional Conservativity

It is now a straightforward task to rephrase the definition of Conservativity into Intensional Conservativity, which we define as follows:

$$\text{For all } m \text{ and all A, B}, \mathbf{Q}_m AB \Leftrightarrow \mathbf{Q}_m A(A \wedge B)$$

Note that since A and B are now intensional properties rather than sets, conjunction replaces intersection. As a matter of course, let us assume that $[\![A \wedge B]\!]^m = [\![A]\!]^m \cap [\![B]\!]^m$ as a rule. More importantly, note that B and $A \wedge B$ are not immediately extensionalized; if we did so, we would get a half-extensional definition that would fare no better against the original problem.

To see that Conservativity is now possible, let us take the same model m and let A, B_1 and B_2 denote the same properties as before (but now in an intensional way). Furthermore let $m \in C$, where C is a set of possible models relative to which things can be interpreted. Under these circumstances, can many be interpreted by a quantifier Q such that Q is intensionally conservative but also satisfies

$$\mathbf{Q}_m AB_1, \neg\mathbf{Q}_m AB_2 \textbf{ ?}$$

It can. From these assumptions, Intensional Conservativity lets us conclude that

$$\mathbf{Q}_m A(A \wedge B_1), \neg\mathbf{Q}_m A(A \wedge B_2)$$

Since $A \wedge B_1$ and $A \wedge B_2$ are not the same intensional properties, this does not lead to a contradiction. While this is technically enough to conclude the argument, it will carry more weight when we have an actual single interpretation **Q** that is a reasonable reading of many and satisfies these conditions.
For this we use just one further simplifying assumption, that C is finite. Given this, consider the following definition, which says roughly that many students have property X iff the relative number of students who have that property is larger than the average of that same number

taken over all possible universes:[3]

$$[\![Q(A)]\!]^m = \left\{ X \;\middle|\; \frac{|[\![X]\!]^m \cap [\![A]\!]^m|}{|[\![A]\!]^m|} > \frac{1}{|C|} \sum_{n \in C} \frac{|[\![X]\!]^n \cap [\![A]\!]^n|}{|[\![A]\!]^n|} \right\}$$

For any reasonable unbiased choice of C this definition will yield $\mathbf{Q}_m AB_1$, $\neg\mathbf{Q}_m AB_2$. It also clearly satisfies Intensional Conservativity. To see this, it is enough to note that

$$|[\![X]\!]^m \cap [\![A]\!]^m| = |([\![A]\!]^m \cap [\![X]\!]^m) \cap [\![A]\!]^m| = |[\![A \wedge X]\!]^m \cap [\![A]\!]^m|.$$

This of course is but a single possible interpretation of a single possible reading of many, but it seems likely that a variety of other options will work equally well, and we will see later that this is indeed the case. Thus, when intensionality is properly accounted for, Conservativity does not need to be given up as a universal property of natural language determiners, not even for many.

2.2. Extension and Isomorphism closure

Conservativity is not the only property taken to apply to virtually all natural language determiners. Two important others are Extension (which I will mostly refer to by the abbreviation EXT[4]) and Isomorphism closure. Let us see how well many does on intensionalized versions of those.
Before getting to this, I should state that while there may seem to be a good deal of arbitrariness involved in creating these intensionalized versions, this is not the case. All three of them are natural and true broadenings of their original counterparts, in that any extensional quantifier Q has a straightforward intensional lift Q^* such that the following is true:

- Q^* satisfies Intensional Conservativity if and only if Q is Conservative
- Q^* satisfies Intensional EXT if and only if Q satisfies EXT*, where EXT* is like EXT but applies for any M, M' such that $A, B \subseteq M$, $A, B \subseteq M'$
- Q^* satisfies Intensional Isomorphism closure if and only if Q satisfies Isomorphism closure

[3] To get around division by zero, we may harmlessly use $\frac{0}{0} = 1$. Also, it would probably be appropriate to add the condition that $[\![X \cap A]\!]^M$ is non-empty.
[4] Given how much here revolves around intensions and extensions, to do otherwise could invite confusion.

For the proof –and the definition of Q^*– see Appendix A.

We start with Extension. In the traditional definition, the point of EXT is domain restriction, making everything in $M-(A\cup B)$ irrelevant to the interpretation of $Q_M AB$. Under the circumstances one might well think that the highly context- dependent many stands a poor chance of satisfying any version of EXT. Yet it is quite possible.
We define the version of Intensional EXT we use here as follows:

$$\text{If } [\![A]\!]^m = [\![A]\!]^{m'}, [\![B]\!]^m = [\![B]\!]^{m'}, \text{ then } \mathbf{Q}_m AB \Leftrightarrow \mathbf{Q}_{m'} AB$$

This essentially says that universes where A and B are interpreted the same way will always agree on whether $\mathbf{Q}AB$.[5]
This sounds like a tall order, but it is satisfied by the interpretation from our earlier example. To see this, it suffices to note that

$$\frac{|[\![B]\!]^m \cap [\![A]\!]^m|}{|[\![A]\!]^m|} = \frac{|[\![B]\!]^{m'} \cap [\![A]\!]^{m'}|}{|[\![A]\!]^{m'}|}.$$

There are some important caveats to this result. First of all, Intensional EXT does *not* mean the quantifier only "has access to" the interpretations in the local universe. It also has access to the intensions, which in this case it used to create the appropriate comparison standard. This is not undesirable, and is in fact part of the point of using an intensionalized definition, but it should be kept in mind all the same. What it does mean is that insofar as the quantifier has access to more than the local interpretations of A and B, it only has such access in a universe-independent way.
Second, Intensional EXT does not necessarily prevent all context-dependence. Some interpretations may call for using a different set C of possible universes for different kinds of comparison, largely bypassing the requirement above.
Third, even this intensional version might not be possible or desirable for every reading we want to model. Those who compare things against alternatives (eg (Cohen, 2001)(Tanaka, 2003)) risk running foul of it. More on this in Section 3.

Next, we consider Intensional Isomorphism closure. Recall the extensional version, which says that where f is a bijection from M to

[5] This can be seen as a more general version of the traditional EXT, where one universe needs to be an extension of the other. Of course this difference is often unimportant as you can consider the universe $A \cup B$ itself.

M', we have $\mathbf{Q}_m AB \Leftrightarrow \mathbf{Q}_{m'} f[A]f[B]$, for a binary Q.This gets slightly problematic in an intensional system where Q depends on comparison to a standard derived from the intensions of A and B; this is not so much because such a standard might change under a bijection, but because we get extra baggage in the form of A and B's own interpretations in m', which gets in the way. To get around this, we replace those interpretations with the f-images, resulting in the following *lift* f^* of f:

$$[\![f^*(X)]\!]^n = \begin{cases} f[[\![X]\!]^n] & \text{if } n = m' \\ [\![X]\!]^n & \text{otherwise} \end{cases}$$

Our definition of Intensional Isomorphism closure now becomes

For any bijection $f : M \to M', \mathbf{Q}_m f^*(A)f^*(B) \Leftrightarrow \mathbf{Q}_{m'} f^*(A)f^*(B)$.

Like the other two properties, Intensional Isomorphism closure is satisfied by our earlier reading of many. Instead of proving this directly, we will in the next section prove it for many other possible readings at the same time.

2.3. General form

The intensionalized properties described above obviously apply to far more than the simple example reading of many. Generally speaking, it is enough to have a comparison of the (actual) cardinalities of A and B with a world-independent standard determined by their intensions. The general form we define here will give a good impression of how far we can go, and will be useful for proofs later.

Definition-Proposition: Similar to regular Logicality in (Westerståhl, 2007), let a binary quantifier Q be *Intensionally Logical* if and only if it satisfies Intensional Conservativity, Intensional EXT and Intensional Isomorphism Closure. Then for a binary quantifier Q to be Intensionally Logical, it is a sufficient condition that it be of the form

$$\mathbf{Q}_m AB \Leftrightarrow R(f(|[\![A]\!]^m|, |[\![A \wedge B]\!]^m|, C), g(A, A \wedge B, C)),$$

where R is a relationship on the real numbers and where g does not depend on the model m, but both f and g may depend in a general sense on the set of all possible models C. Note that f here depends only on the *cardinality* of $[\![A]\!]^m$ and $[\![A \wedge B]\!]^m$, not on those sets themselves. To see that this guarantees Intensional Isomorphism closure, use the definition above and let $h : M \to M'$ be a bijection. Then it is the case that

$$|[\![h^*(A)]\!]^{m'}| = |h[[\![A]\!]^m]| = |[\![A]\!]^m| = |[\![h^*(A)]\!]^m|,$$

and similarly for $A \wedge B$. Thus, f is invariant under replacing m by m'. This is true for g by definition, and so we are done.
The proof that it guarantees Intensional EXT is even simpler, as the condition from EXT guarantees directly that the input for f is the same in both universes. And of course Intensional Conservativity is a trivial result of only using B under conjunction with A.

3. Other readings of Many

As already known, readings of many without an intensional component can only be Conservative at the price of being almost arbitrarily context dependent. To reduce this arbitrariness, a number of readings with such a component have been proposed, some of which we will look at presently. In fact, these treatments are often mostly extensional except for the intensional component, so in a number of places we will have to imagine $[\![A]\!]^m$ where they say A, an so on. In order to preserve readability I do not generally make these substitutions in my paraphrasings.

Fernando & Kamp

Fernando and Kamp's account(Fernando and Kamp, 1996) states that "...the arguments of many ...cannot be interpreted simply by their extensions" and uses a probability-based method for the intensional component. In their simpler reading, the quantifier is given by

$$\text{Many}x(A,B) \text{ iff } \bigvee_{n\geq 1}((\exists_{\geq n}x)(A\wedge B)\wedge n\text{-is-many}_x(A,B))$$

where n-is-many$_x(A,B)$ is given as

$$p(\{w : |A\wedge B|_{x,w} < n\}) > c,$$

for a world-independent probability function p and constant c. What this means is essentially that n is many if the probability of being in a world where there are less than n things with property $A \wedge B$ is high.
This reading can be intensionalized a bit and expressed in our general form by rephrasing it as

$$|[\![A\wedge B]\!]_m|_x \geq \min_n n\text{-is-many}_x(A, A\wedge B)$$

and thus has the interesting properties. Unfortunately it is also symmetrical. Thus we are more interested in the more advanced reading,

where n-is-many$_x(A, B)$ is conditionalized against the probability of A being true of as many things as it is in the actual world, as follows:

$$p(\{w : |A \wedge B|_{x,w} < n\} | \{w : |A|_{x,w} = |A|_x\}) > c$$

Because of this actual world-dependent component, this reading cannot be rephrased as gainfully as above. But it is still expressible as a function of A, $A \wedge B$, their local interpretations and the independent c, and thus the Intensional Conservativity, EXT and Isomorphism closure must hold.

COHEN

In the Relative Proportional reading introduced by Cohen(Cohen, 2001) we take many(X, Y) to be true iff the following holds:

$$\frac{|X \cap Y|}{|X \cap \bigcup A|} > \frac{|\bigcup A \cap Y|}{|\bigcup A|}$$

Here A is a set of pairs of alternatives for X and Y, given by

$$A = \{X' \wedge Y' | X' \in \mathrm{ALT}(X), Y' \in \mathrm{ALT}(Y)\}$$

where ALT(X) gives a set of propositions considered to be alternatives to X. It would be only natural to allow access to A as part of knowing the intensions of X and Y, so it makes sense to check for Intensional Conservativity.[6] So does the reading have this property?
Let us assume that the extension of $X \wedge Y$ is the same as $X \cap Y$. To avoid confusion, let A remain as before and let A' be the version of A obtained when X is replaced by $X \wedge Y$. This raises the question what kind of alternatives are in ALT$(X \wedge Y)$. I would say the obvious choice is to let ALT$(X \wedge Y) = A$. Hence we get

$$A' = \{X' \wedge Z' | X' \in \mathrm{ALT}(Y), Z' \in A\}.$$

But because of the nature of Z', it always either implies or contradicts X'. Therefore what we in fact end up with is $A' = A$. We now obtain

$$\begin{aligned} \mathrm{many}(X, X \wedge Y) &\Leftrightarrow \frac{|X \cap X \cap Y|}{|X \cap \bigcup A'|} > \frac{|\bigcup A' \cap X \cap Y|}{|\bigcup A'|} \\ &\Leftrightarrow \frac{|X \cap Y|}{|X \cap \bigcup A'|} > \frac{|X \cap Y|}{|\bigcup A'|} \\ &\Leftrightarrow |X \cap \bigcup A'| < |\bigcup A'| \end{aligned}$$

[6] On the other hand, Cohen might disagree here, and for the same reason he already calls it non-conservative in his abstract.

The latter is a tautology, so Intensional Conservativity does not hold. With the reading depending so much on the extensions of alternatives, we shouldn't expect Intensional EXT to hold either, and it doesn't. Let M be such that $\|\text{many}(X,Y)\|^M$ is true. Let M' be like M, except that every alternative to X or Y (except X and Y themselves) has empty extension. Then relative to M' many(X,Y) reduces to

$$\frac{|X\cap Y|}{|X\cap Y|} > \frac{|X\cap Y|}{|X\cap Y|},$$

a contradiction.[7]

Tanaka

Similar to Cohen, Tanaka's account is based on sets of alternatives, based on taxonomic knowledge.(Tanaka, 2003) It distinguishes between taking alternatives to the subject or the predicate, and between comparing alternatives of the same level (the Sister-alt reading) or a higher level (the Mother-alt reading). This leads to four possible readings[8] of "Many A's are B", which can be paraphrased as follows:

	$S-ALT$	$M-ALT$
Subject	$\frac{\lvert A\cap B\rvert}{\lvert B\rvert} > \frac{\lvert \text{sisters}(A)\cap B\rvert}{\lvert B\rvert}$	$\frac{\lvert A\cap B\rvert}{\lvert B\rvert} > \frac{\lvert A\rvert}{\lvert \text{mother}(B)\rvert}$
Predicate	$\frac{\lvert A\cap B\rvert}{\lvert A\rvert} > \frac{\lvert A\cap \text{sisters}(B)\rvert}{\lvert A\rvert}$	$\frac{\lvert A\cap B\rvert}{\lvert A\rvert} > \frac{\lvert B\rvert}{\lvert \text{mother}(A)\rvert}$

None of these turn out to be (Intensionally) Conservative. It's easy enough to see that both M-ALT readings turn into a tautology if B is replaced by $A\cap B$. For the S-ALT readings, keep in mind that sisters are disjoint, so that

$$\text{sisters}(A)\cap A = \emptyset = A\cap \text{sisters}(A\wedge B).$$

Therefore replacing B with $A\cap B$ will turn the right-hand side of both S-ALT readings to zero, making them tautologies as well.

To make matters particularly odd, consider that Tanaka makes it a point to propose a revised notion of Conservativity, wherein focal mapping determines which element is conservative. This could mean that for some or all of the readings above, he would have us replace not B but A by $A\cap B$ to test for Conservativity. But the fact of the matter is that this changes nothing. Replacing A by $A\cap B$ above turns all four

[7] This proof might fail if the intensions imply certain logical relationships that make constructing M' impossible. Still, it suffices to show that this reading cannot as a rule satisfy Intensional EXT.

[8] In addition to two absolute readings we're not interested in here.

readings into tautologies in essentially the same ways. As it stands I fail to see how his readings could satisfy the notion he introduces.
As for Intensional EXT, it fails for much the same reason it fails for Cohen's reading. The proof for this is left as an exercise for the reader.

Lappin

Lappin provides the only thoroughly intensional treatment I am aware of(Lappin, 2000), and it holds up well. Lappin broadly defines many as follows:

$$\|B\|^{sa} \in \|\text{many}\|(\|A\|^{sa}) \text{ iff } S \neq \emptyset, \text{ and for every } sn \in S,$$
$$|\|A\|^{sa} \cap \|B\|^{sa}| \geq |\|A\|^{sn} \cap \|B\|^{sn}|$$

Here sa is the actual situation, and S a set of normative situations used for comparison. This account looks deceptively good and simple, but is held back by a highly underdefined S, which could depend on anything. This means that we can only say much of interest about the properties of many by looking at the effect of restrictions on S.
To make things easier for us, the above can be rephrased as

$$|[\![A]\!]^{sa} \cap [\![B]\!]^{sa}| \geq \max_{sn \in S} |[\![A]\!]^{sn} \cap [\![B]\!]^{sn}|$$

(Except for the extreme case where $S = \emptyset$, but we can take $\max_{\emptyset} = \infty$ or even undefined there.) This matches our normal form, provided the right-hand side doesn't require too much. Specifically, we get the restriction that S must be a function of A and $A \wedge B$ and must not depend on sa.

Solt

Like Lappin, Solt(Solt, 2009) provides an account where much is put into a somewhat underspecified parameter: all the possible readings and context-dependencies are interpreted as affecting the neutral range N_S of cardinalities considered neither many nor few. When reduced and rewritten enough to fit our normal-form, the truth condition for "Many A are B" becomes

$$|[\![A \wedge B]\!]^{m}| \geq \sup N_S$$

Thus, similar to what we see with Lappin, it is enough to require that N_S does not depend on the actual world and is a function of A and $A \wedge B$ (and possibly some world-independent things).

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Appendix

A. Reductions

For a standard generalized quantifier $\mathbf{Q}$, define its intensional lift Q^* as follows:

$$[\![Q^*(X)]\!]^m = \{Y | \mathbf{Q}_m [\![X]\!]^m [\![Y]\!]^m\}$$

Also, for any set X in domain M define the lift $l(X)$ as follows:

$$[\![l(X)]\!]^n = \begin{cases} X & \text{if } X \subseteq N \\ \emptyset & \text{otherwise} \end{cases}$$

Theorem

- Q^* satisfies Intensional Conservativity if and only if Q is Conservative
- Q^* satisfies Intensional EXT if and only if Q satisfies EXT*, where EXT* is like EXT but applies for any M, M' such that $A, B \subseteq M$, $A, B \subseteq M'$
- Q^* satisfies Intensional Isomorphism closure if and only if Q satisfies Isomorphism closure

Conservativity is the easiest. If Intensional Conservativity is assumed, we find

$$\begin{aligned}\mathbf{Q}_m AB &\Leftrightarrow Q^*_m l(A)l(B)\\ &\Leftrightarrow Q^*_m l(A)(l(A)\cap l(B))\\ &\Leftrightarrow Q^*_m l(A)l(A\cap B)\\ &\Leftrightarrow \mathbf{Q}_m A(A\cap B)\end{aligned}$$

If regular Conservativity is assumed (with A, B now intensional), we find

$$\begin{aligned}Q^* AB &\Leftrightarrow \mathbf{Q}_m [\![A]\!]^m [\![B]\!]^m\\ &\Leftrightarrow \mathbf{Q}_m [\![A]\!]^m([\![A]\!]^m \cap [\![B]\!]^m)\\ &\Leftrightarrow \mathbf{Q}_m [\![A]\!]^m([\![A\wedge B]\!]^m)\\ &\Leftrightarrow Q^* A(A\wedge B)\end{aligned}$$

For EXT*, let $A, B \subseteq M, M'$. Then it is clearly the case that $[\![l(A)]\!]^m = [\![l(A)]\!]^{m'}, [\![l(B)]\!]^m = [\![l(B)]\!]^{m'}$. Thus,

$$\begin{aligned}\mathbf{Q}_m AB &\Leftrightarrow Q^*_m l(A)l(B)\\ &\Leftrightarrow Q^*_{m'} l(A)l(B)\\ &\Leftrightarrow \mathbf{Q}_{m'} AB\end{aligned}$$

and letting A, B be intensional for the other direction

$$\begin{aligned}Q^*_m AB &\Leftrightarrow \mathbf{Q}_m [\![A]\!]^m [\![B]\!]^m\\ &\Leftrightarrow \mathbf{Q}_{m'} [\![A]\!]^{m'} [\![B]\!]^{m'}\\ &\Leftrightarrow Q^*_{m'} AB\end{aligned}$$

For Isomorphism closure, let f be a bijection from M to M'. First we assume that Q satisfies Isomorphism closure, yielding the following:

$$\begin{aligned}Q^*_m f^*(A)f^*(B) &\Leftrightarrow \mathbf{Q}_m [\![f^*(A)]\!]^m [\![f^*(B)]\!]^m\\ &\Leftrightarrow \mathbf{Q}_m [\![A]\!]^m [\![B]\!]^m\\ &\Leftrightarrow \mathbf{Q}_{m'} f[[\![A]\!]^m] f[[\![B]\!]^m]\\ &\Leftrightarrow \mathbf{Q}_{m'} [\![f^*(A)]\!]^{m'} [\![f^*(B)]\!]^{m'}\\ &\Leftrightarrow Q^*_{m'} f^*(A)f^*(B)\end{aligned}$$

(The step from the first to the second line is invalid if $m = m'$, but then Intensional Isomorphism closure is trivial, plus there then is a trivial step from the first to the fourth line instead.) For the other direction,

assume Q^* satisfies Intensional Isomorphism closure to obtain

$$\begin{aligned}
\mathbf{Q}_m AB &\Leftrightarrow \mathbf{Q}_m [\![f^*(l(A))]\!]^m [\![f^*(l(B))]\!]^m \\
&\Leftrightarrow Q^*_m f^*(l(A)) f^*(l(B)) \\
&\Leftrightarrow Q^*_{m'} f^*(l(A)) f^*(l(B)) \\
&\Leftrightarrow \mathbf{Q}_{m'} [\![f^*(l(A))]\!]^{m'} [\![f^*(l(B))]\!]^{m'} \\
&\Leftrightarrow \mathbf{Q}_{m'} f[[\![l(B)]\!]^m] f[[\![(l(A))]\!]^m] \\
&\Leftrightarrow \mathbf{Q}_{m'} f[A] f[B]
\end{aligned}$$

Extensional Intensional Quantifiers

It is a matter of some interest to see under which conditions a given intensional quantifier can be interpreted as a lift of an extensional one. As one might expect, the answer is that this is so iff the truth value in a given model depends only on that model and the local extensions there, modulo some considerations about definedness. The following two lemmas demonstrate this. **Lemma:** If an intensional quantifier $\mathbf{Q}$ is such that $\mathbf{Q}_m XY$ is a function of $[\![X]\!]^m, [\![Y]\!]^m$ and m, and such that $\mathbf{Q}_m XY$ is well-defined for any X, Y that are the lifts of sets, then there is an extensional quantifier $\mathbf{Q}^2$ such that $\mathbf{Q}_m AB \Leftrightarrow (\mathbf{Q}^2)^*_m AB$ whenever the left side is well-defined.
For the proof, define

$$\mathbf{Q}^2_m(X) := \{Y | \mathbf{Q}^1_m l(X) l(Y)\}.$$

This gives

$$\begin{aligned}
(\mathbf{Q}^2)^*_m AB &\Leftrightarrow \mathbf{Q}^2_m [\![A]\!]^m [\![B]\!]^m \\
&\Leftrightarrow \mathbf{Q}_m l([\![A]\!]^m) l([\![B]\!]^m) \\
&\Leftrightarrow \mathbf{Q}_m AB
\end{aligned}$$

The other direction follows straightforwardly from the definition.
Lemma: For any lift Q^* of a standard generalized quantifier $\mathbf{Q}$, $Q^*_m XY$ is a function of $[\![X]\!]^m, [\![Y]\!]^m$ and m, and is well-defined for any X, Y that are the lifts of sets.