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In Defense of Trusts: R&D Cooperation in Global Perspective*

Jeroen Hinloopen† Grega Smrkolj‡ Florian Wagener†

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Abstract

We examine the trade-off between the benefits of allowing firms to cooperate in R&D and the corresponding increased potential for product market collusion. For that we utilize a dynamic model of R&D whereby we consider all possible initial marginal cost levels (technologies), including those that exceed the choke price. This global analysis yields four possibilities: initial marginal costs are above the choke price and the technology is, or is not, developed further, and initial marginal costs are below the choke price and the technology is, or is not, (eventually) taken off the market. We show that an extension of the cooperative agreement towards collusion in the product market is not necessarily welfare reducing: if firms collude, they (i) develop further a wider range of initial technologies, (ii) invest more in R&D such that process innovations are pursued more quickly, and (iii) abandon the technology for a smaller set of initial marginal costs. We also discuss the implications of our analysis for antitrust policy.

Keywords: Antitrust policy, Bifurcations, Collusion, R&D cooperatives, Spillovers
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1 Introduction

A prominent reason for allowing firms to set up R&D cooperatives is that these "organizations, jointly controlled by at least two participating entities, whose primary

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purpose is to engage in cooperative R&D" (Caloghirou *et al.*, 2003) internalize technological spillovers - the free flow of knowledge from the knowledge creator to its competitors.[1] Sustaining R&D cooperatives is thus perceived to diminish the failure of the market for R&D.[2] However, as Scherer (1980) observes: "the most egregious price fixing schemes in American history were brought about by R&D cooperatives", an observation that constitutes the prime counterargument for a permissive antitrust treatment of R&D markets (Pfeffer and Nowak (1976), Grossman and Shapiro (1986), Brodley, 1990).[3] In this paper we qualify the commonly held view that these price fixing schemes necessarily reduce consumer surplus and total surplus.

The channels through which cooperation in R&D facilitates product market collusion have been examined in a number of theoretical studies (Martin (1995), Greenlee and Cassiman (1999), Cabral (2000), Lambertini *et al.* (2002), Miyagiwa, 2009). As Fisher (1990, p. 194) puts it: "...[firms] cooperating in R&D will tend to talk about other forms of cooperation. Furthermore, in learning how other firms react and adjust in living with each other, each cooperating firm will get better at coordination. Hence, competition in the product market is likely to be harmed." But Geroski (1992) argues that it is the feedback from product markets that directs research towards profitable tracks and that, therefore, for an innovation to be commercially successful there must be strong ties between marketing and development of new products. And Jacquemin (1988) puts forward that R&D cooperatives are fragile and unstable. He reasons that when there is no cooperation in the product market, there exists a continuous fear that one partner in the R&D cooperative may be strengthened in such a way that it will become too strong a competitor in the product market. Preventing firms from collaborating in the product market may therefore destabilize R&D cooperatives, or prevent their formation in the first place.

Our focus is on private incentives to develop cost saving technologies. We find that product market collusion fosters investment in R&D because more of the ensuing economic rents can be appropriated by the investing firms. As a result, if firms collude in the product market, more initial technologies will be brought to full maturation. And this is unambiguously welfare enhancing.

[1] Bloom *et al.* (2007) estimate that a 10% increase in a competitor's R&D is associated with up to a 2.4% increase in a firm's own market value. Internalizing technological spillovers is one of the prime reasons for firms to join an R&D cooperative (Hernan *et al.*, 2003; see also Roeller *et al.*, 2007).

[2] See Martin (1997) for an overview of the policy treatment of R&D cooperatives in the E.U., the U.S., and Japan.

[3] Goeree and Helland (2008) find that in the U.S. the probability that firms join an R&D cooperative has gone down due to a revision of antitrust leniency policy in 1993. This revision is perceived as making collusion less attractive. Goeree and Helland (2008) conclude that "Our results are consistent with RJVs [research joint ventures] serving, at least in part, a collusive function." Related evidence is reported by Duso *et al.* (2010). They find that the combined market share declines if partners in an RJV compete on the same product market ("horizontal RJVs"), while it increases if members of the RJV are not direct rivals ("vertical RJVs"). The laboratory experiments of Suetens (2008) show that members of an RJV are more likely to collude on price.

The notion that product market collusion fosters R&D investments is pioneered by d'Aspremont and Jacquemin (1988). Their static model of R&D predicts that firms are willing to invest more in R&D if the intensity of product market competition is diminished.[4] Their model also predicts that this type of collusion always reduces consumer surplus and total surplus, in spite of the increased R&D efforts, a result that has been replicated in many follow-up studies (see De Bondt (1997) for an overview). But a static view of the world necessarily ignores an important aspect of R&D: time. It takes time for an initial idea to be developed towards a marketable product; continuous process innovations only gradually reduce production costs (Utterback, 1994). Perhaps more important is that static models necessarily assume marginal costs to be below the choke price (the lowest price for which there is no demand).[5] This assumptions excludes all situations where the initial costs of a new technology (the cost, say, to produce a prototype) exceed the highest willingness to pay in the market, a situation that is bound to hold in the early stages of development. Therefore, to examine if product market collusion necessarily reduces welfare we develop a dynamic model of R&D whereby no initial level of marginal cost is assumed away.[6]

For our analysis we develop a dynamic R&D model that is rooted in Hinloopen *et al*. (2013). We expand their analysis in three significant directions. First of all, we consider a duopoly rather than a monopoly. Second, we examine two different scenarios: one in which firms cooperate in R&D and compete in the product market, and one in which firms cooperate both in R&D and in setting price. Third, rather than relying on simulations we prove a set of propositions that characterize the dynamics throughout the entire parameter space.

For our global analysis we use bifurcation theory.[7] This yields a bifurcation diagram that indicates for every possible parameter combination the qualitative features of any market equilibrium as well as of the transient dynamics towards it. As in Hinloopen *et al*. (2013), we do not limit ourselves to an analysis of equilibrium

[4]This touches upon the debate of *Schumpeter Mark I* ("...new combinations are, as a rule, embodied, as it were, in new firms which generally do not arise out of the old ones but start producing beside them;...in general it is not the owner of stage-coaches who builds railways"; Schumpeter, 1934, p. 66) versus *Schumpeter Mark II* ("As soon as we go into the details and inquire into the individual items in which progress was most conspicuous, the trail leads not to the doors of those firms that work under conditions of comparatively free competition but precisely to the doors of the large concerns...and a shocking suspicion dawns upon us that big business may have had more to do with creating that standard of living than with keeping it down"; Schumpeter, 1934, p. 82).

[5]This also holds for (stationary) repeated game models of R&D (Cabral (2000), Lambertini *et al*., 2002).

[6]The related literature consists of Petit and Tolwinski (1999), Cellini and Lambertini (2009), Lambertini and Mantovani (2009), and Kovac *et al*. (2010), which all assume initial levels of marginal costs to be below the choke price As will become clear below, this restriction excludes a crucial part of the parameter space.

[7]Solution structures may change qualitatively due to variations in parameter values (indifference points may appear, steady states may lose their stability, and so on). These qualitative changes due to smooth variations in parameters are called *bifurcations*. For an introduction, see Grass *et al*. (2008), or Kiseleva and Wagener (2010, 2011).

paths but we consider all trajectories that are candidates for an optimal solution. This enables us to determine *critical parameter values* - points in parameter space at which the optimal investment function changes qualitatively. In particular, we determine the value of marginal costs for which R&D investments are terminated, and for which they are not initiated at all. We show that these critical cost levels are affected by firm conduct. As a result, extending the R&D cooperative agreement to product market collusion can lead to qualitatively different long-run solutions, in spite of starting from an identical initial technology.

In general, our framework yields four possible outcomes (cf. Hinloopen *et al.* (2013)). First of all, a 'promising technology' arrives, whereby the initial technology is developed through ensuing R&D investments. This can occur for initial cost levels both below and above the choke price. In the latter case production starts only after some time, because early R&D efforts have to bring down marginal cost below the choke price. Second, a 'strained market' arises: initial marginal cost is below the choke price and firms invest in R&D but only to leave the market after some time.[8] In case of an 'uncertain future', the third situation, it is not immediately clear whether the long-run steady state will be reached, or that it is optimal to gradually leave the market. Only time will tell. Fourth, an 'obsolete technology' can emerge: whatever the initial marginal cost, the technology is either not developed, or developed only to be taken off the market. The long-run steady state will not be reached in either case.

According to our analysis, if firms collude in the product market: (i) the range of initial marginal cost that leads to the creation of a new market is larger, (ii) the speed with which new technologies enter the product market increases, and (iii) the set of initial marginal cost that induces firms to abandon the technology in time is smaller. We also show that there are parameter configurations that lead to a long-run steady state in both scenarios whereby the collusive scenario yields higher total surplus.

Our results suggest that for the implementation of antitrust policies, it is important to understand the wider effect of these policies. A ban on collusion not only affects current markets, but also markets that have not yet materialized. Preventing firms from colluding in the product market reduces the number of potential R&D trajectories that successfully lead to the development of new markets. In itself this constitutes a welfare loss. However, because not developing further an initial technology does not surface as a direct surplus loss, this welfare loss remains hidden. Also, prohibiting firms to collude reduces the speed with which new technologies enter the product market. As a result, marginal cost are unnecessarily high, which creates a social waste. Moreover, collusion yields more R&D investments. In so far higher R&D investments as such are desirable, the case for prohibiting collusion

[8]This situation resembles the 'sailing ship effect' of Cooper and Schendel (1976) (see also Howells, 2002), whereby the arrival of a new, possibly superior technology spurs the development of the old technology. In our case, there is no rival technology that induces continued investment in a technology that is bound to leave the market. Rather, it is the technology itself (characterized by the size of the initial marginal cost) that makes it optimal for firms to gradually take it off the market in due time.

in the product market is further weakened. On the other hand, colluding firms tend to hold on longer to technologies that are destined to leave the market. This is not desirable from a social welfare point of view in so far that it prevents the development of new, superior technologies.

2 The model

Time t is continuous: $t \in [0, \infty)$. There are two *a priori* fully symmetric firms which both produce a homogenous good at constant marginal costs $c(t)$. At every instant, market demand is

$$p(t) = A - Q(t), \tag{1}$$

where $Q(t) = q_1(t) + q_2(t)$, with $q_i(t)$ the quantity produced by firm i at time t, and where $p(t)$ and A are respectively the market price at time t and the choke price.

Each firm i can reduce its marginal cost $c_i(t)$ by investing in R&D. In particular, firm i exerts R&D effort $k_i(t)$ such that its marginal cost evolves as

$$\frac{dc_i}{dt}(t) \equiv \dot{c}_i(t) = c_i(t)\left(-k_i(t) - \beta k_j(t) + \delta\right), \tag{2}$$

where $k_j(t)$ is the R&D effort exerted by its rival and where $\beta \in [0, 1]$ measures the degree of spillover. Note that efficiency of production is assumed to decrease at a constant rate, as captured by $\delta > 0$. This depreciation is due to the (exogenous) aging of technology and organizational forgetting (Besanko *et al.* (2010), Lambertini and Mantovani, 2009). As Benkard (2004) observes: "...an aircraft producer's stock of production experience is constantly being eroded by turnover, lay offs and simple losses of proficiency at seldom repeated tasks. When producers cut back output, this erosion can even outpace learning, causing the stock of experience to decrease"(Benkard, 2004, p. 590). In our model, it is R&D investments that yields know-how gains (not production), but the logic of the argument is the same. Complementary inputs that are typically purchased also constitute a fraction of production cost. Incorporating these inputs becomes ever more costly due to their inherent evolution over time, especially for firms that are relatively sluggish in R&D as R&D efforts also determine any firm's 'absorptive capacity' (Cohen and Levinthal, 1989).[9]

Both firms are endowed with an identical initial technology $c_i(0) = c_j(0) = c_0$, which in this article is assumed to be drawn by Nature. Per unit of time, the costs of R&D efforts are

$$\Gamma_i(k_i) = bk_i^2, \tag{3}$$

[9]A non-positive depreciation rate yields trivial equilibria. Every initial technology will be developed in case δ is negative, as there is an exogenous reduction in marginal cost over time. For $\delta = 0$ consider δ to be marginally positive. In that case, the value of initial marginal cost that would make it optimal not to invest in R&D is far above the choke price because only an infinitesimally small investment in R&D is then needed to reduce marginal cost over time.

where $b > 0$ is inversely related to the cost-efficiency of the R&D process. The R&D process is thus assumed to exhibit decreasing returns to scale (Schwartzman, 1976; see also the discussion in Hinloopen *et al.*, 2013). Both firms discount the future with the same constant rate $\rho > 0$. Either firm's instantaneous profit therefore equals

$$\pi_i(q_i, Q, k_i, c_i) = (A - Q - c_i)q_i - bk_i^2, \tag{4}$$

with total discounted profit

$$\Pi_i(q_i, Q, k_i, c_i) = \int_0^\infty \pi_i(q_i, Q, k_i, c_i)e^{-\rho t}dt. \tag{5}$$

The model has five parameters: A, β, b, δ, and ρ. To simplify the analysis, we rescale the model such that it has only three parameters (the proof of Lemma 1 is analogous to the proof of Lemma 1 in Hinloopen *et al.*, 2013).

Lemma 1. *By choosing the units of t, q_i, q_j, c_i, c_j, k_i, and k_j appropriately, we can assume $A = 1$, $b = 1$, and $\delta = 1$. This yields the following rescaled version of the model:*

$$\tilde{\pi}_i(\tilde{q}_i, Q, \tilde{k}_i, \tilde{c}_i) = (1 - \tilde{Q} - \tilde{c}_i)\tilde{q}_i - \tilde{k}_i^2, \tag{6}$$

$$\tilde{\Pi}_i(\tilde{q}_i, \tilde{Q}, \tilde{k}_i, \tilde{c}_i) = \int_0^\infty \tilde{\pi}_i(\tilde{q}_i, \tilde{Q}, \tilde{k}_i, \tilde{c}_i)e^{-\tilde{\rho}\tilde{t}}d\tilde{t} \tag{7}$$

$$\dot{\tilde{c}}_i = \tilde{c}_i\left(1 - \left(\tilde{k}_i + \beta\tilde{k}_j\right)\phi\right), \quad \tilde{c}_i(0) = \tilde{c}_0, \quad \tilde{c}_i \in [0,\infty)\,\forall\,\tilde{t} \in [0,\infty) \tag{8}$$

$$\tilde{q}_i \geq 0, \quad \tilde{k}_i \geq 0 \tag{9}$$

$$\tilde{\rho} > 0, \quad \phi > 0 \tag{10}$$

with conversion rules: $q_i = A\tilde{q}_i$, $q_j = A\tilde{q}_j$, $k_i = \frac{A}{\sqrt{b}}\tilde{k}_i$, $k_j = \frac{A}{\sqrt{b}}\tilde{k}_j$, $c_i = A\tilde{c}_i$, $c_j = A\tilde{c}_j$, $\pi_i = A^2\tilde{\pi}_i$, $\pi_j = A^2\tilde{\pi}_j$, $\phi = \frac{A}{\delta\sqrt{b}}$, $t = \frac{\tilde{t}}{\delta}$, $\tilde{\rho} = \frac{\rho}{\delta}$.

This rescaling introduces a new parameter: ϕ. It is one-to-one related to the profit potential of a technology. Higher potential revenues come with a higher A, and each unit of R&D effort costs more if b increases, while it reduces marginal cost by less the higher is δ. In sum, a lower (higher) ϕ corresponds to a lower (higher) profit potential. For notational convenience we henceforth omit tildes.

3 Competition and Collusion

This section derives the necessary conditions for optimal production and investment schedules in case firms cooperate in R&D but compete in the product market (a scenario labelled 'competition'), and in case firms cooperate in R&D and collude in the product market (a scenario labelled 'collusion').

3.1 Competition

Both firms operate their own R&D laboratory and production facility, and while they select their output levels non-cooperatively, they adopt a strictly cooperative behavior in determining their R&D efforts so as to maximize joint profits. These assumptions amount to imposing *a priori* the symmetry condition $k_i(t) = k_j(t) = k(t)$.[10] As $c_i(0) = c_j(0) = c_0$, this implies that $c_i(t) = c_j(t) = c(t)$. Equation (8) thus reads as

$$\dot{c} = c(1 - (1+\beta)\phi k). \tag{11}$$

It may seem reasonable to assume that when firms cooperate in R&D, they also fully share information, that is, to assume the level of spillover to be at its maximum ($\beta = 1$; see Kamien *et al.*, 1992). For the sake of generality, we do not *a priori* fix the value of β at its maximal value. There are also intuitive arguments for not doing so as there might still be some *ex post* duplication and/or substitutability in R&D outputs if firms operate separate laboratories (see the discussion in Hinloopen, 2003).

The instantaneous profit of firm i is

$$\pi_i(q_i, Q, k, c) = (1 - Q - c)q_i - k^2, \tag{12}$$

with $Q = q_1 + q_2$, yielding its total discounted profit over time

$$\Pi_i(q_i, Q, k, c) = \int_0^\infty \pi_i(q_i, Q, k, c)e^{-\rho t}dt. \tag{13}$$

As firms jointly decide on their R&D efforts, the only independent decisions are those of production. However, as quantity variables do not appear in the equation for the state variable (11), production feedback strategies of a dynamic game are simply static Cournot-Nash strategies of each corresponding instantaneous game.

Maximizing π_i over $q_i \geq 0$ gives us standard Cournot best-response functions for the product market

$$q_i(q_j) = \begin{cases} \frac{1}{2}(1 - c - q_j) & \text{if} \quad q_j < 1 - c, \\ 0 & \text{if} \quad q_j \geq 1 - c. \end{cases} \tag{14}$$

Note that the constraint $q_i \geq 0$ is binding when $q_j \geq 1 - c$. Solving for Cournot-Nash production levels, we obtain

$$q^N = \begin{cases} \frac{1}{3}(1 - c) & \text{if} \quad c < 1, \\ 0 & \text{if} \quad c \geq 1. \end{cases} \tag{15}$$

Consequently, the instantaneous profit of each firm is

$$\pi(c, k) = \begin{cases} \frac{1}{9}(1 - c)^2 - k^2 & \text{if} \quad c < 1, \\ -k^2 & \text{if} \quad c \geq 1. \end{cases} \tag{16}$$

[10]Throughout the paper we consider symmetric equilibria only. See Salant and Shaffer (1998) for a specific example of a static model of R&D in which it is optimal for firms in an R&D cooperative to make unequal investments.

The dynamic optimization problem of the R&D cooperative boils down to finding an R&D effort schedule k^* for either firm that maximizes the total discounted joint profit, taking into account the state equation (11), the initial condition $c(0) = c_0$, and the control constraint $k(t) \geq 0$ which must hold at all times. Note that according to (11), if $c_0 > 0$, then $c(t) > 0$ for all t. The state space of this problem is the interval $[0, \infty)$ of marginal cost levels.

To solve this problem, we introduce the current-value Pontryagin function (also called the un-maximized Hamilton or pre-Hamilton function)[11]

$$P(c, k, \lambda) = \begin{cases} \frac{1}{9}(1-c)^2 - k^2 + \lambda c(1 - (1+\beta)\phi k) & \text{if} \quad c < 1, \\ \quad\quad\quad\quad\;\; - k^2 + \lambda c(1 - (1+\beta)\phi k) & \text{if} \quad c \geq 1, \end{cases} \tag{17}$$

where λ is the current-value co-state variable of a firm in the R&D cooperative. The co-state (or shadow value) measures the marginal worth of the increment in the state c for each firm at time t when moving along the optimal path. We expect $\lambda(t) \leq 0$ along optimal trajectories because marginal cost is a "bad".

We use Pontryagin's maximum principle to obtain the solution to our optimization problem. Maximizing over the control $k \geq 0$ yields

$$k = \max\left\{0, -\frac{1}{2}\lambda c(1+\beta)\phi\right\}. \tag{18}$$

The maximum principle states further that the optimizing trajectory necessarily corresponds to the trajectory of the state-costate system

$$\dot{c} = \frac{\partial P}{\partial \lambda}, \quad \dot{\lambda} = \rho\lambda - \frac{\partial P}{\partial c}, \tag{19}$$

where k is replaced by its maximizing value. For $\lambda \leq 0$, relation (18) gives a one-to-one correspondence between the co-state λ and the control k. We use this relation to transform the state-costate system into a state-control system which an optimizing trajectory has to satisfy necessarily as well. This system consists of two regimes (following the two part composition of the Pontryagin function). The first one corresponds to $c < 1$ and positive production ($q = (1-c)/3$). The second one corresponds to $c \geq 1$ and zero production.[12] The state-control system with positive production consists of the following two differential equations:[13]

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[11] We omit a factor of 2 for joint profits to obtain the solution expressed in per-firm values. Due to symmetry, maximizing the per-firm total profit corresponds to maximizing joint total profit.

[12] Recall from Lemma 1 that $A = 1$ in the rescaled model. In the non-rescaled model, the analogous conditions for positive and zero production are $c(t) < A$ and $c(t) \geq A$, respectively.

[13] Our closed-loop solution differs from that of Cellini and Lambertini (2009), who consider the case when marginal cost is always lower than the choke price. This is so because their proof that the open-loop and closed-loop solutions coincide is flawed by the fact that in their derivation of the closed-loop solution, players' output choices are not properly treated as functions of the state variable. Cellini and Lambertini (2009) implicitly assume that if marginal cost within the R&D cooperative changes, rivals' quantity does not change, which is in violation of the feedback principle underlying the closed-loop solution. It is also counterintuitive as firms in the R&D cooperative jointly decide on their R&D efforts taking into account that marginal cost in any period affects the ensuing Nash-equilibrium profits.

$$\begin{cases} \dot{c} = c\,(1-(1+\beta)\phi k)\,, \\ \dot{k} = \rho k - \frac{(1+\beta)\phi}{9} c(1-c). \end{cases} \tag{20}$$

The state-control system with zero production is given by

$$\begin{cases} \dot{c} = c\,(1-(1+\beta)\phi k)\,, \\ \dot{k} = \rho k. \end{cases} \tag{21}$$

3.2 Collusion

If firms collude, they determine jointly their R&D efforts and their output levels. This amounts to imposing *a priori* the symmetry conditions $k_i(t) = k_j(t) = k(t)$ and $q_i(t) = q_j(t) = q(t)$. Equation (8) reads again as Equation (11). The profit of each firm at every instant is

$$\pi(q,k,c) = (1-2q-c)q - k^2, \tag{22}$$

yielding its total discounted profit over time

$$\Pi(q,k,c) = \int_0^\infty \pi(q,k,c)e^{-\rho t}dt. \tag{23}$$

The optimal control problem of the two colluding firms is to find controls q^* and k^* that maximize the profit functional Π subject to the state equation (11), the initial condition $c(0) = c_0$, and two control constraints that must hold at all times: $q \geq 0$ and $k \geq 0$.[14] Notice again that according to (11), if $c_0 > 0$, then $c(t) > 0$ for all t.

The current-value Pontryagin function in case of collusion reads as:

$$P(c,q,k,\lambda) = (1-2q-c)\,q - k^2 + \lambda c\,(1-(1+\beta)\phi k)\,, \tag{24}$$

where λ is the current-value co-state variable. It now measures the marginal worth at time t of an increment in the state c for a colluding firm when moving along the optimal path.

The necessary conditions for the solution to the dynamic optimization problem consist again of a state-control system which has two regimes. As in the competitive case, the first regime corresponds to $c < 1$ and positive production ($q = (1-c)/4$), while the second regime corresponds to $c \geq 1$ and zero production.

The state-control system in the region with positive production reads as

$$\begin{cases} \dot{c} = c\,(1-(1+\beta)\phi k)\,, \\ \dot{k} = \rho k - \frac{(1+\beta)\phi}{8} c(1-c), \end{cases} \tag{25}$$

whereas the state-control system with zero production is

$$\begin{cases} \dot{c} = c\,(1-(1+\beta)\phi k)\,, \\ \dot{k} = \rho k. \end{cases} \tag{26}$$

[14] Again, due to symmetry, maximizing per-firm total profit corresponds to maximizing joint total profit.

4 Analysis

Consider the system

$$\begin{cases} \dot{c} = c\,(1 - (1+\beta)\phi k), \\ \dot{k} = \rho k - \alpha\phi(1+\beta)c(1-c)\chi(c), \end{cases} \tag{27}$$

where $\chi(c) = 1$ if $0 < c < 1$ and $\chi(c) = 0$ if $c \geq 1$ (or $c \leq 0$). Systems (20) – (21) and (25) – (26) are instances of the system (27), with $\alpha = 1/9$ for the competitive scenario and $\alpha = 1/8$ for the collusive scenario.[15]

The first result gives the properties of the steady states of the state-control system (see Appendix A.1 for the proof).

Proposition 1. *Let*

$$D = \frac{1}{4} - \frac{\rho}{\alpha(1+\beta)^2\phi^2}.$$

Depending on the value of D, there are three different situations.

1. *If $D > 0$, the state-control system with positive production* (25) *has three steady states:*

   i. $(c^e, k^e) = (0, 0)$ *is an unstable node,*

   ii. $(c^e, k^e) = \left(\frac{1}{2} + \sqrt{D}, \frac{1}{(1+\beta)\phi}\right)$ *is either an unstable node or an unstable focus, and*

   iii. $(c^e, k^e) = \left(\frac{1}{2} - \sqrt{D}, \frac{1}{(1+\beta)\phi}\right)$ *is a saddle-point steady state.*

2. *At $D = 0$, there are two steady states:*

   i. $(c^e, k^e) = (0, 0)$*, which is an unstable node, and*

   ii. $(c^e, k^e) = \left(\frac{1}{2}, \frac{1}{(1+\beta)\phi}\right)$*, which is a semi-stable steady state.*

3. *If $D < 0$, the origin $(c^e, k^e) = (0, 0)$ is the unique steady state of the state-control system with positive production, which is unstable.*

The system consequently exhibits a saddle-node bifurcation at $D = 0$.

The stable manifold of the saddle-point steady state is one of the candidates for an optimal solution. As however neither the Mangasarian nor the Arrow concavity conditions are satisfied, the stable manifold is not necessarily optimal. Note that Proposition 1 already implies that there should be other candidates for optimality as there is a parameter region for which there is no saddle point, and hence no stable manifold to it. The following result clarifies the matter (Appendix A.2 contains the proof).

[15] The monopoly system in Hinloopen *et al.* (2013) is also a special case of system (27), with $\alpha = 1/4$.

Proposition 2. *The set of candidates for an optimal solution consists of the stable paths W^s_- of the saddle-point steady state and the trajectory E through the point $(c, k) = (1, 0)$.*

The thick black lines W^s_- and E in Figure 1 indicate these candidates. In this

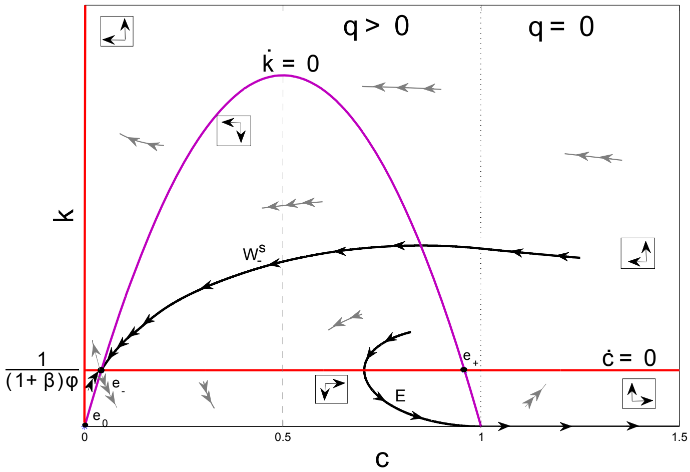


Figure 1: *Candidate maximizing trajectories W^s_- and E in the state-control space.*

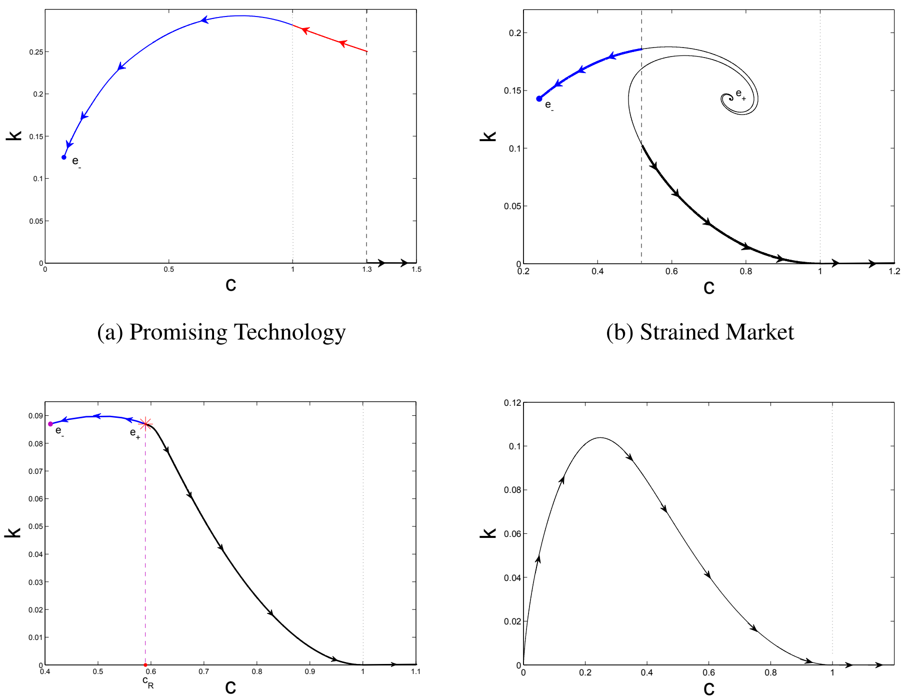
figure, the dotted vertical line $c = 1$ separates the region with zero production from the region of positive production. We label the trajectory E the "exit trajectory", as following this trajectory implies that firms eventually leave the region with positive production.

Proposition 2 only reduces the set of trajectories by applying necessary conditions for optimality, but there is no guarantee that an optimal solution exists. The next proposition summarizes when an optimal solution exists (the proof is in Appendix A.3).

Proposition 3. *For all admissible values of the parameters, the following is true. At all initial points, the optimal control problem has at least one solution, which is among the candidates specified in Proposition 2. Moreover, there is at most one initial state $\hat{c}$ such that there are two optimizing trajectories starting at $\hat{c}$.*

To assess the dependence of the solution structure on the model parameters, we carry out a bifurcation analysis. This consists of identifying those parameter values for which the qualitative structure of the optimal dynamics changes. These 'bifurcating' values bound open parameter regions such that the optimal dynamics are qualitatively identical for all parameter values in a region (see Wagener, 2003; Kiseleva & Wagener, 2010, 2011). Put differently, for all points in a region, a sufficiently small change in parameter values will not lead to a qualitative change of the optimal dynamics; regions characterize *stable* types of dynamics.

Hinloopen *et al.* (2013) identify four distinct stable types for their monopoly system. Those types carry over to system (27). Figure 2 illustrates the four types; Figure 3 shows the corresponding bifurcation diagram if firms compete in the product market.



(a) Promising Technology

(b) Strained Market

(c) Uncertain future

(d) Obsolete technology

Figure 2: The four stable types of dynamics of system (27).

The first type is a “Promising Technology”, where there is an *indifference threshold*[16] in the region of no production. In an optimal control problem, an indifference threshold is a point in state space where the decision maker is indifferent between two optimal trajectories that have distinct long-term limit behavior. In case of a Promising Technology, there is a point $\hat{c} > 1$, such that for $0 < c_0 \leq \hat{c}$, it is optimal to start developing the initial technology, ending up in the saddle-point steady state in the region of positive production. In case $1 < c_0 \leq \hat{c}$, initially firms invest only in R&D; production begins whenever $c(t) < 1$. If $c_0 \geq \hat{c}$, it is optimal not to initiate R&D efforts as in this case potential future profits do not suffice to compensate for losses that would be incurred in the initial periods during which firms would invest in R&D but would not produce yet. Note that for $c_0 = \hat{c}$, there are two entirely different R&D investment policies, which are, nevertheless, both optimal.

[16] Also known as Skiba, Dechert-Nishimura-Skiba or DNSS point; see Grass *et al.* (2008).

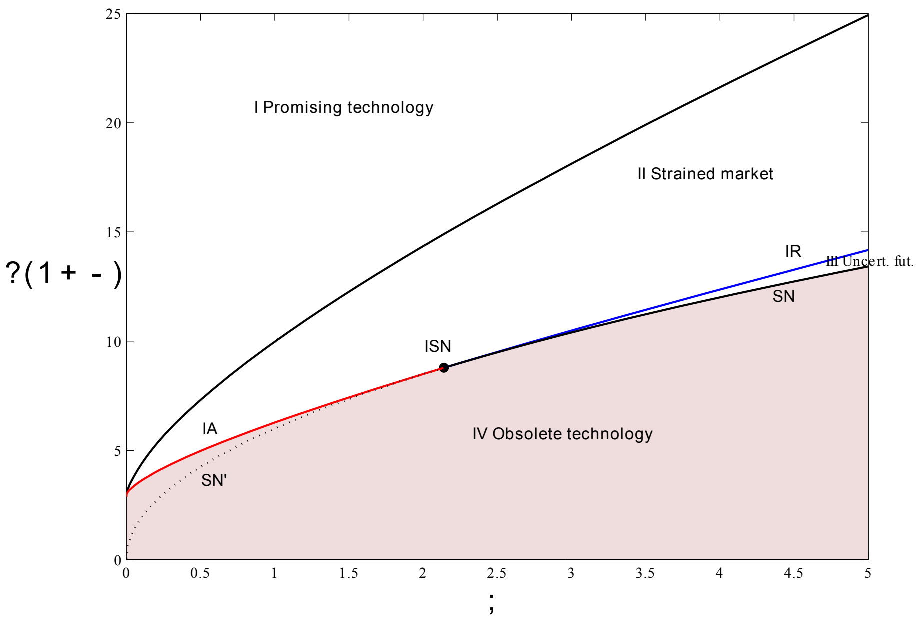


Figure 3: *Bifurcation diagram (competitive scenario).*

The second type corresponds to a "Strained Market", where there is an indifference threshold in the region of positive production: $0 < \hat{c} < 1$. In this case, if $0 < c_0 < \hat{c}$, that is, if the initial marginal cost level is sufficiently small, the firm develops the technology and ends up in the saddle-point steady state as before. If however $\hat{c} < c_0 < 1$, that is, if the initial marginal cost level is sufficiently small to make production immediately profitable, but above the critical level $\hat{c}$, the firm does invest in R&D, but only to follow the exit-trajectory. In this situation the R&D investments serve to slow down the technological decay.

In a small part of the parameter space the third type arises: an "Uncertain Future". Initial states (that either optimally converge to the steady state with positive production, or to the exit from the market) are now divided by a repelling steady state (rather then an indifference point). If the system starts exactly at the repelling point, it stays there indefinitely; when it starts close to it, it stays there for a long period of time, after which it converges to one of the two attractors (i.e., to the steady state with positive production or to the exit from the market).

The fourth type typifies the dynamics of an "Obsolete Technology". Whatever the initial state, the firms let the technology decay and (eventually) leave the market. In the region of positive production, the decay is slowed down by R&D investments.

In the bifurcation diagram, the uppermost curve represents parameter values for which the indifference point is exactly at $c = 1$. At the saddle-node curve (SN), an optimal repeller and an optimal attractor collide and disappear. The curve SN' corresponds to saddle-node bifurcations in the state-control system that

do not correspond to optimal dynamics. At the indifference-attractor bifurcations (IA), an indifference point collides with an optimal attractor and both disappear.[17] Finally, at an indifference-repeller bifurcation (IR), an indifference point turns into an optimal repeller. The central indifference-saddle-node (ISN) bifurcation point at $(\rho, \phi(1+\beta)) \approx (2.14, 8.78)$ organizes the bifurcation diagram. The curve representing indifference points at $c = 1$ obtains a value of $\phi(1+\beta) \approx 2.998$ for $\rho = 1 \times 10^{-5}$.

5 Collusion and the incentives to innovate

Having characterized the global optimum of both the competitive and the collusive scenario, we can compare their respective bifurcation diagrams. These are superimposed in Figure 4. Qualitatively, there is no difference between the diagrams. There are, however, important quantitative differences. To state them quantitatively, introduce the notation $I_1, II_1, \ldots$ for the regions $I, II, \ldots$ of the competitive scenario, and $I_2, II_2, \ldots$ for the corresponding regions in the collusive scenario.

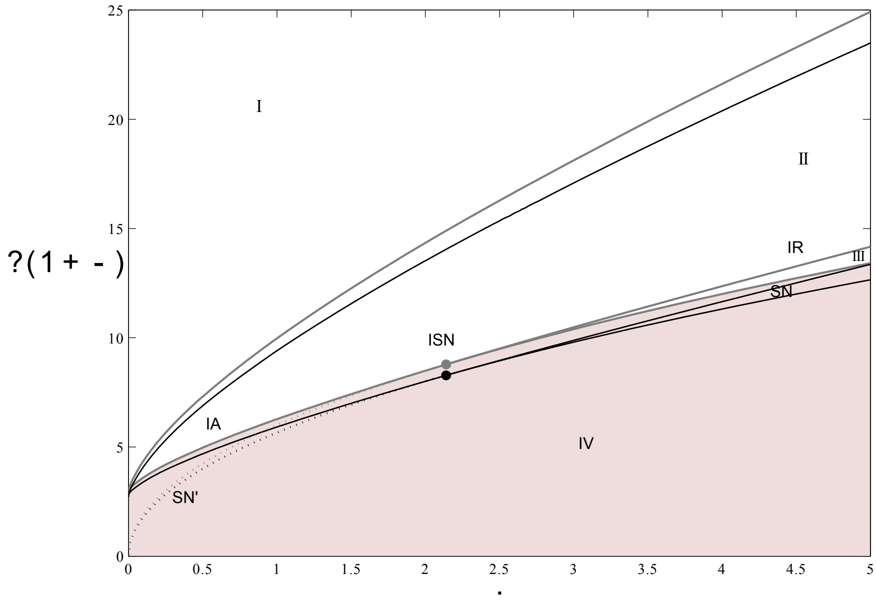


Figure 4: *Bifurcation diagram.* Curves of the competitive (collusive) scenario are grey (black).

[17]For a detailed exposition of the terminology, see Kiseleva & Wagener (2010, 2011).

Proposition 4. *The following inclusions hold:*

$$I_1 \subset I_2,$$
$$I_1 \cup II_1 \subset I_2 \cup II_2,$$
$$I_1 \cup II_1 \cup III_1 \subset I_2 \cup II_2 \cup III_2.$$

See Appendix A.4 for the proof of this proposition. The first inclusion states that the "Promising Technology" region is larger if firms collude. Put differently, if firms collude, the situation where firms first invest in R&D, and only after some initial development period start producing, is more likely to occur. Moreover if firms collude, the "Obsolete Technology" region IV_2 is smaller. That is, due to collusion, it is less likely that firms either do not develop an initial technology, or that they invest in R&D only to abandon the technology in time.

To state the next proposition, it is convenient to have the threshold level of initial marginal cost $\hat{c}$ between 'eventual exit' and 'eventual positive production' defined for all situations; to do this, set formally $\hat{c} = 0$ in the "Obsolete Technology" region. Introduce $\hat{c}_1$ and $\hat{c}_2$ as the threshold levels for the competitive and the collusive scenarios respectively.

Proposition 5. *For all parameter values, either*

$$\hat{c}_2 > \hat{c}_1$$

or $\hat{c}_1 = \hat{c}_2 = 0$.

The implications of Proposition 5 (proved in Appendix A.5.2) are twofold. First, if firms collude, the set of initial technologies that are developed and that lead to the saddle-point steady state is larger. Figure 5 illustrates this implication. If the initial technology c_0 falls in the non-empty interval $(\hat{c}_1, \hat{c}_2)$ both firms will develop the technology and this will eventually give rise to a new market, but only if firms collude. If they compete, neither firm will develop the technology.

Note that a higher value of initial marginal cost implies larger early-stage losses because there is no profitable production yet. Obviously, these losses are more quickly off-set by future profits if firms collude, due to higher mark-ups. Therefore, when colluding, firms can afford to invest more in R&D prior to production, and thereby to bring down over time a higher initial level of marginal cost.

For a welfare comparison, we introduce total discounted values of profits (Π), consumer surplus (CS), and total surplus (TS)

$$\Pi = \int_0^\infty \pi(t) e^{-\rho t} dt, \tag{28}$$

$$CS = \int_0^\infty \frac{1}{2}(1 - p(t))Q(t)e^{-\rho t} dt = \int_0^\infty 2q(t)^2 e^{-\rho t} dt, \tag{29}$$

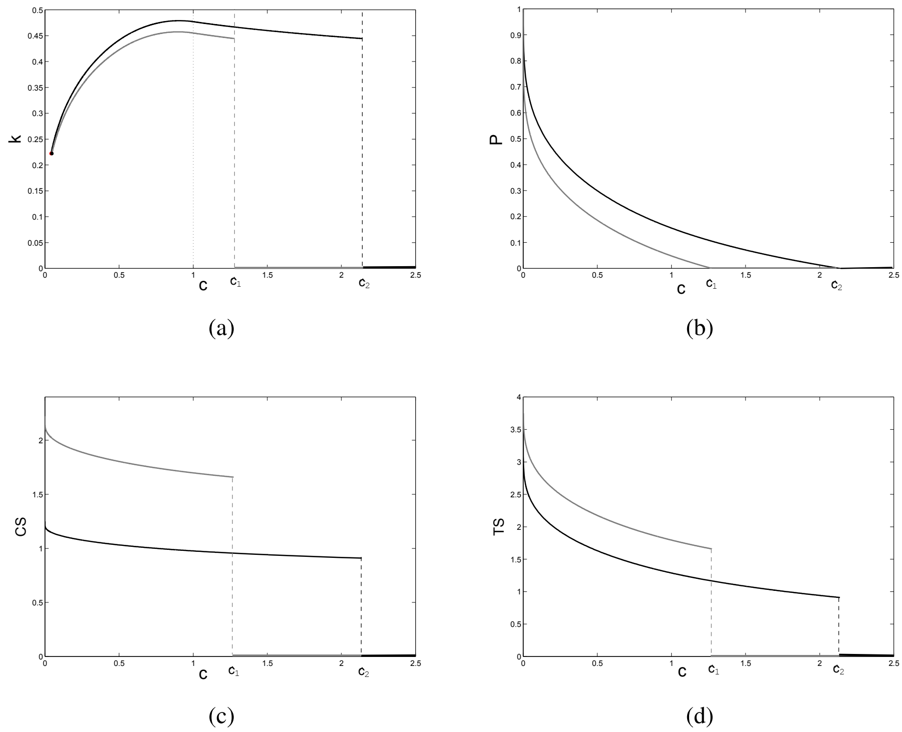


Figure 5: *State-control space (a), total discounted profit (b), consumer surplus (c), and total surplus (d), when the indifference point is in the region with zero production.* Parameters: $(\beta, \rho, \phi) = (1, 0.1, 2.25)$. Curves of the competitive (collusive) scenario are grey (black).

$$TS = 2\Pi + CS, \tag{30}$$

where at time $t = 0$ firms start with c_0 and then invest along the optimal trajectory $\gamma(t) = (c(t), k(t))$ as $t \to \infty$. Plots (b)–(d) in Figure 5 show how these discounted values vary with different initial values of c_0.

Figure 6 illustrates some comparative statics of the indifference points for a Promising Technology. Obviously, these points are positively related to market size and R&D efficiency. Note, however, that also $\Delta\hat{c}$ (the difference between $\hat{c}_1$ and $\hat{c}_2$) increases if the R&D process becomes more efficient and/or if the market size becomes larger, the more so the lower the discount rate is. In Figure 6, this corresponds to a larger slope of the convex curves. Because future mark-ups are positively related to both market size and R&D efficiency, an increase in either one has a larger (positive) effect on future profits if firms collude. And these future benefits feature more prominently in total discounted profits if the discount rate is lower. Put differently, indifference points occur at smaller values if the discount rate goes up, all else equal (cf. the relative location of C_1 and C_2 in Figure 6).

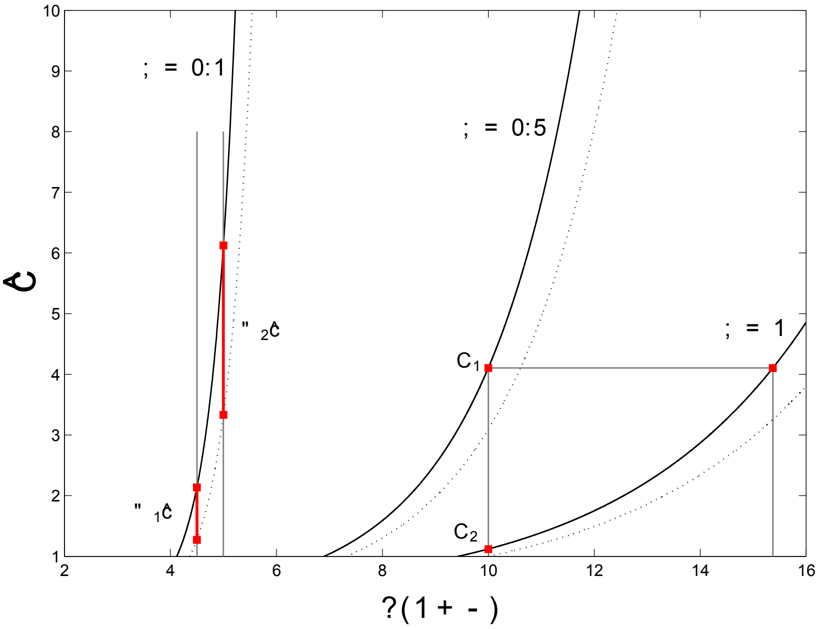
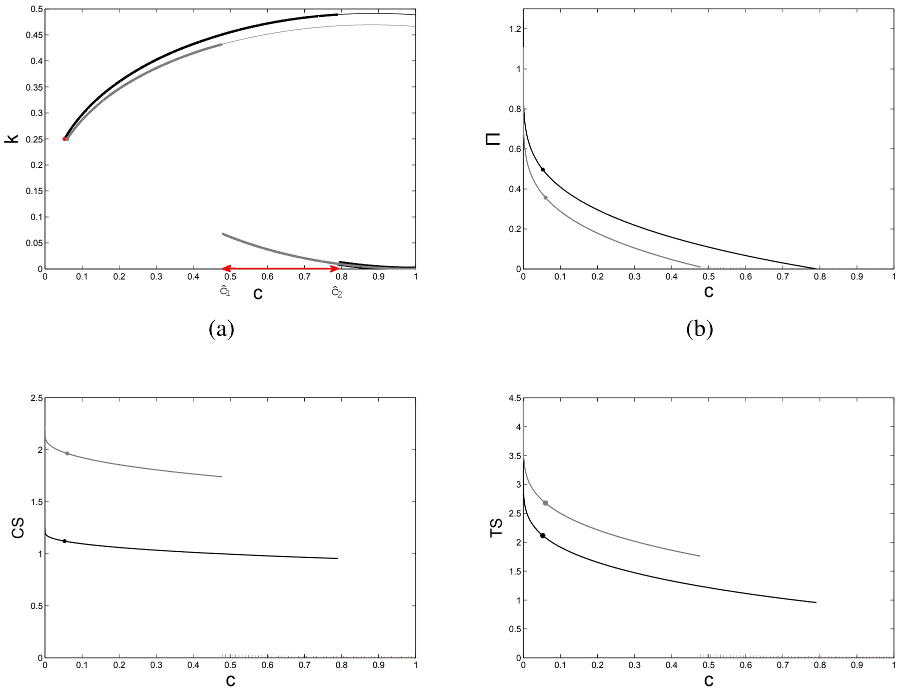


Figure 6: *Dependence of the indifference point $\hat{c}$ on model parameters.* Curves of the competitive (collusive) scenario are dotted (solid).

A particular situation arises when the indifference point with collusion is above the choke price, while it is below the choke price if firms compete. This is the case for all points in Figure 4 in between the two bifurcation curves that separate a Promising Technology from a Strained Market. In any such a situation, only colluding firms may develop a technology which requires investments in advance of production; competing firms would find it optimal to select the exit trajectory. Obviously, the latter scenario yields a lower total surplus.

Second, if firms collude, the set of initial technologies that triggers no investment in R&D at all or that induces firms to select the exit trajectory is smaller. Figure 7 illustrates this for a Strained Market. The strained investment circumstances induce competing firms to exit the market in due time for all $c_0 > \hat{c}_1$. In contrast, colluding firms exit the market only for $c_0 > \hat{c}_2$, which is again due to larger mark-ups in the product market. Initial technologies c_0 in the interval $(\hat{c}_1, \hat{c}_2)$ are therefore only brought to full maturation by colluding firms, which yields a direct welfare gain of collusion.

So far we can conclude that due to collusion (i) it is more likely that we have a Promising Technology, and if so, that it is more likely to be developed further, (ii) it is less likely that we have an Obsolete Technology, and if so, it is more likely that firms invest in R&D, albeit temporarily, and (iii) if the technology causes a Strained Market or if it induces an Uncertain Future, it is less likely that it will be taken off the market in due time. In sum, due to collusion it is more likely that firms invest



(a) (b)

(c) (d)

Figure 7: *State-control space (a), total discounted profit (b), consumer surplus (c), and total surplus (d), when the indifference point is within the region with positive production.* Parameters: $(\beta, \rho, \phi) = (1, 0.1, 2)$. Curves of the competitive (collusive) scenario are grey (black); curves of the stable path (exit trajectory) are solid (dotted). Dots indicate the saddle-point steady state.

in R&D, and that these investments eventually lead to a steady state with positive production.

For a more complete comparison between the competitive and collusive scenario, we also look at the intensity of the R&D process as such.

Proposition 6. *Investment in R&D in the collusive scenario is always at least as high as in the corresponding competitive scenario.*

Proposition 6 (the proof of which is in Appendix A.5.1) implies the following. First, whenever both scenarios lead to the saddle-point steady state, marginal costs in the collusive scenario are lower than in case of competition, because colluding firms have invested more in cost-reducing R&D to arrive at the long-run equilibrium. Put differently, collusion yields a higher production efficiency. Second, if the initial technology leads to production after some initial development period only, colluding firms will enter this production phase more quickly. That is, at every instant of the pre-production phase, colluding firms invest more in R&D in order to bring some

initial level of marginal costs below the choke price. As a result, less favorable initial technologies will be brought to the market if firms collude. Third, colluding firms abandon obsolete technologies at a lower pace. This implication, that a monopolist holds on longer to a technology that is bound to leave the market, has a similar vein as the argument of Arrow (1962), that a monopolist has less incentive to invest in R&D than an otherwise identical but perfectly competitive market, because by doing so the monopolist replaces current monopoly profits by future (higher) monopoly profits. Here, of course, the alternative for the colluding firms is to exit the market more quickly (rather than staying in the market as a monopolist, as in Arrow, 1962), an alternative that for them is not optimal (see Figure 8).

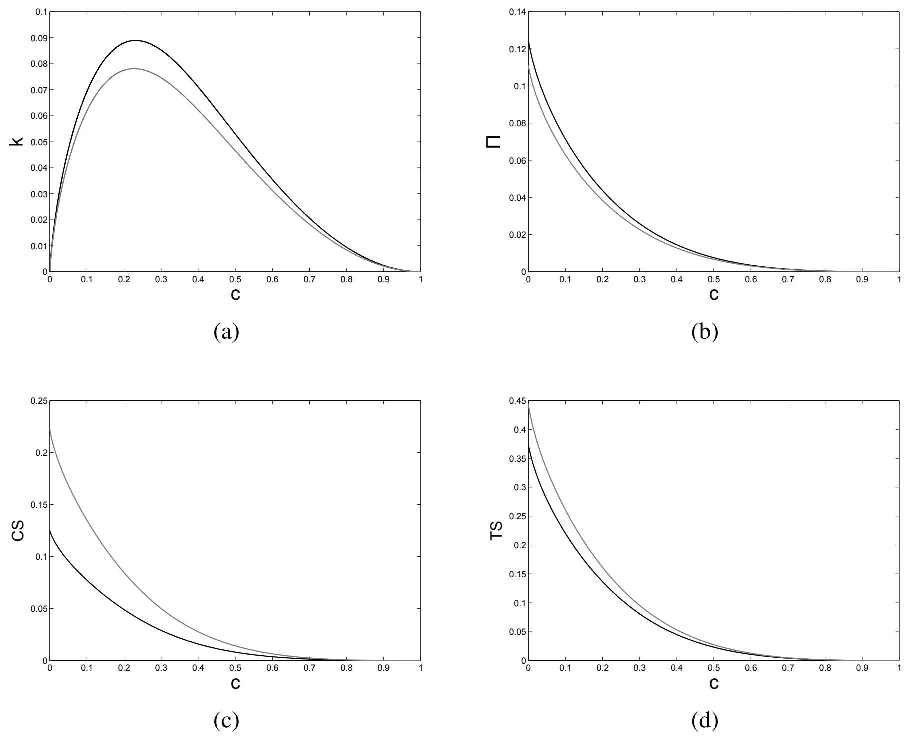


Figure 8: *State-control space (a), total discounted profit (b), consumer surplus (c), and total surplus (d), when the exit trajectory is an optimal solution.* Parameters: $(\beta, \rho, \phi) = (1, 1, 2)$. Curves of the competitive (collusive) scenario are grey (black).

6 Antitrust policies

Summarizing the results of the previous section, we have found that the collusive scenario is more R&D intensive: R&D investment levels are higher and the set of initial technologies that is developed is larger. The price to be paid for this increased

innovation intensity is the higher mark-up in the product market. Indeed, the welfare comparison of the two scenarios yields a mixed picture.

First, as alluded to in the previous section, if firms develop an initial technology that leads to a positive production steady state, a higher total surplus is obtained over the alternative of no R&D investment at all. Indeed, in Figure 5, for all $c_0 \in (\hat{c}_1, \hat{c}_2)$, the collusive scenario is the better alternative. That is[18]

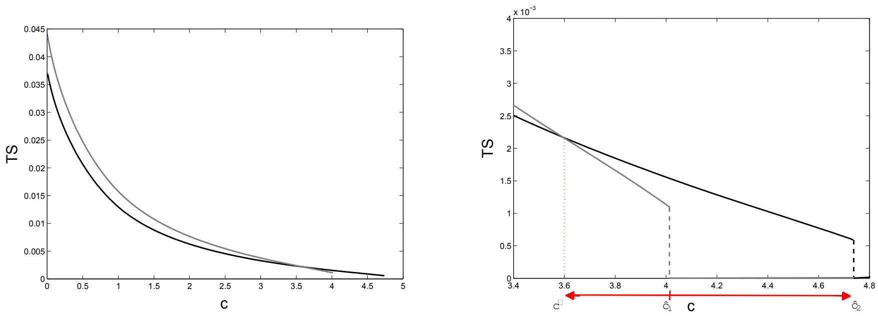
Proposition 7. *Whenever both scenarios have an indifference point above the choke price, the collusive scenario yields higher consumer surplus and total surplus than the competitive scenario for all initial technologies in between the two indifference points.*

This proposition qualifies the argument that R&D cooperatives make it easier for firms to collude in the concomitant product market and that this is necessarily welfare reducing. Obviously, this fails to be the case for all c_0 in the interval $(\hat{c}_1, \hat{c}_2)$. It is also not necessarily valid in situations where collusion induces firms to select the stable path while competition induces them to exit the market (recall Figure 7).

For competition authorities, a particularly difficult situation arises when the initial draw c_0 out of $(\hat{c}_1, \hat{c}_2)$ is above the choke price $(c_0 > 1)$. The welfare costs of prohibiting firms to collude in the product market do then not surface because no production is affected by this prohibition. There is no production yet, and because collusion is prohibited, there will be no production in the future. Yet, in this case, prohibiting firms of an R&D cooperative to collude in the product market is welfare reducing. To the extent that antitrust policies are designed to enhance total surplus, a general prohibition of product market collusion is not first-best *per se*. At the same time, and more in line with traditional views, Figures 5 and 7 suggest that if both scenarios induce firms to select the stable path towards the saddle-point steady state, the competitive scenario yields a higher total surplus (Figure 8 contains a similar suggestion in case both scenarios induce firms to select the exit trajectory).[19] However, this is not necessarily the case, as Figure 9 illustrates. Although both scenarios would induce firms to select the trajectory towards the saddle-point steady state, for all $c_0 \in (c^\star, \hat{c}_2)$, total surplus is higher if firms collude in the product market. In this example, the discount rate is high: $\rho = 10$, which corresponds, for instance, to $\delta = 0.01$ and $\rho = 0.1$ (non-rescaled variables). Also, the initial marginal costs have to be 'high' for the collusive scenario to outperform the competitive scenario in terms of consumer surplus and total surplus. In such an environment, the higher R&D investments and the reduced importance that is attached to future surplus are favorable for the collusive scenario: if firms collude, they reach the production stage more quickly, a benefit that more than off-sets

[18]The proof of Proposition 7 follows trivially from the fact that $i)$ for all values of c above the indifference point in the region where $c \geq 1$, both $q = 0$ and $k = 0$ for all $t \in [0, \infty)$, and $ii)$ for all values of c below the indifference point, $\Pi > 0$ and, sooner or later, also $q > 0$ as $t \to \infty$.

[19]As noted above, over the entire trajectory, collusion yields more R&D investments. Insofar higher investment levels as such are desirable, the case for prohibiting collusion in the product market is weakened.



(a) (b)

Figure 9: *Total surplus when the indifference point is in the region with zero production.* Parameters: $(\beta, \rho, \phi) = (1, 10, 50)$. Grey curves correspond to competition, whereas the black ones correspond to collusion. $c^\star \approx 3.6$, $\hat{c}_1 \approx 4.01$, $\hat{c}_2 \approx 4.74$. For all $c_0 \in (c^\star, \hat{c}_2)$, total surplus is higher if firms collude in the product market.

the welfare loss of increased mark-ups in the future.[20] To illustrate further what difficulties competition authorities face, consider Figure 10. Among others, it shows the development of the Lerner index over time towards its long-run level of 0.92 for the parameter configuration of Figure 5, where the initial draw $c_0 = 2$ is from the interval $(\hat{c}_1, \hat{c}_2)$. This case illustrates what has been alluded to by Lindenberg and Ross (1981, p. 28): "[The Lerner index] does not recognize that some deviation of P from MC comes from ... the need to cover fixed costs and does not contribute to market value in excess of replacement cost."[21] In our model, this deviation arises in part from the need to cover expenses incurred in advance of production. The high value of the Lerner index is due to collusion, which, in this case, is welfare enhancing. Indeed, this example suggests that the court was right in its ruling of US vs. Eastman Kodak (1995) when it concluded that "Kodak's film business is subject to enormous expenses that are not reflected in its short-run marginal costs."[22] More generally, it illustrates the difficulty in designing optimal antitrust policies for high-tech industries. This is illustrated further if one considers instantaneous profits and total discounted profits, as in Panel (b) of Figure 10. Clearly, after a while, the former are much larger than the latter. But the high instantaneous mark-ups should not be considered as a signal of potential welfare losses, because if it had not been

[20]More precisely, a higher rescaled discount rate $\tilde{\rho} = \rho/\delta$, referred to above, implies either a higher discount rate ρ or a lower δ. With a lower δ, cost reductions take longer, such that the time difference in reaching the production stage between the scenarios becomes more pronounced.

[21]See Elzinga and Mills (2011) for a critical assessment of the use of the Lerner index; see also Armentano (1999).

[22]In 1995, Kodak accounted for 75 percent of amateur color negative film sales in the US; it was followed by Fuji, Konica, Agfa, and 3M. Claiming that the sales price of Kodak film was twice the short-run marginal cost, the Department of Justice accused Kodak of having market power in the US market for photographic film. The court found the evidence of a large Lerner Index insufficient to infer guilt, referring to fixed costs and expenses not reflected in short-run marginal costs.

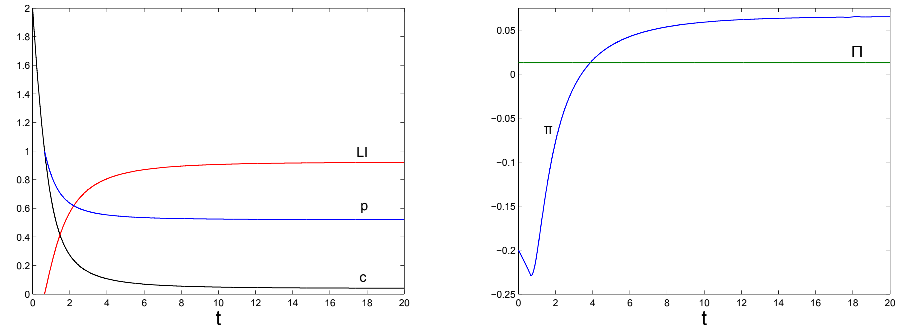


Figure 10: *Marginal cost, price, and Lerner Index (a); total discounted profit and instantaneous profit (b), for collusion.* Parameters: $(\beta, \rho, \phi) = (1, 0.1, 2.25)$ and starting point $c_0 = 2$.

for these mark-ups, in the long run there would have been no market at all.

7 Conclusion

We present an analysis of R&D cooperatives whereby the phase prior to production is taken into account, because it is well known that collusion triggers the incentives to invest in R&D. Our global analysis reveals that if firms collude in the product market, the set of initial technologies that is developed further increases, and that, in particular, more initial technologies are brought to full maturation. This is a direct welfare gain of product market collusion. Also, the probability that an initial technology induces firms to leave the market altogether is reduced, which again is welfare enhancing.

Our analysis suggests a potential problem for antitrust policy as it shows that prohibiting collusion in the product market *per se* is not univocally welfare enhancing. It also shows that the associated welfare costs might not surface because a prohibition of product market collusion affects R&D investments. Any decision not to develop some initial technology does not materialize as a welfare cost because no production is affected (yet).

A Appendix

A.1 Proof of Proposition 1

Second rescaling of the problem. Recall the dynamic optimization problem: to maximize

$$\Pi = \int_0^\infty \left(\alpha(1-c)^2\chi(c) - k^2\right) \mathrm{e}^{-\rho t}\mathrm{d}t,$$

subject to the dynamic restriction

$$\dot{c} = (1 - \phi(1+\beta)k)c.$$

This problem is rewritten by introducing constants

$$K = \frac{1}{\phi(1+\beta)} \quad \text{and} \quad \mu = \frac{\alpha\phi^2(1+\beta)^2}{4\rho}, \tag{31}$$

and the variable u through

$$k = Ku.$$

It is then seen to be equivalent to the problem to maximize

$$V = \frac{\Pi}{K^2} = \int_0^\infty \left(4\rho\mu(1-c)^2\chi(c) - u^2\right) \mathrm{e}^{-\rho t}\mathrm{d}t, \tag{32}$$

subject to the dynamic restriction

$$\dot{c} = (1-u)\,c$$

and the control restriction

$$u \geq 0.$$

The Pontryagin function of this problem is

$$P = 4\rho\mu(1-c)^2\chi(c) - u^2 + \lambda c(1-u),$$

which is maximized at

$$u = \max\left\{0, -\frac{c}{2}\lambda\right\}. \tag{33}$$

This yields the Hamilton function

$$H = 4\rho\mu(1-c)^2\chi(c) + \lambda c + \begin{cases} \frac{(\lambda c)^2}{4} & \text{if } \lambda \leq 0; \\ 0 & \text{if } \lambda > 0. \end{cases}$$

If $\lambda \leq 0$, the associated state-costate equations read as

$$\dot{c} = H_\lambda = \frac{\lambda c^2}{2} + c, \tag{34}$$

$$\dot{\lambda} = \rho\lambda - H_c = \rho\lambda + 8\rho\mu(1-c)\chi(c) - \frac{\lambda^2}{2}c - \lambda, \tag{35}$$

whereas if $\lambda > 0$, they simplify to

$$\dot{c} = c, \quad \dot{\lambda} = (\rho - 1)\lambda + 8\rho\mu(1-c)\chi(c). \tag{36}$$

Using the relation (33) as a variable transformation whenever $\lambda \leq 0$, we can put the system into state-control form

$$\dot{c} = F_1(c,u) = c\,(1-u)\,, \tag{37}$$

$$\dot{u} = F_2(c,u) = \rho\,(u - 4\mu c(1-c)\chi(c))\,. \tag{38}$$

For later use, we note that in (c,u) variables, the Hamilton function takes the form

$$H_{\text{control}}(c,u) = 4\rho\mu(1-c)^2\chi(c) + u^2 - 2u. \tag{39}$$

A.1.1 Steady states

To determine the steady states of the state-control system (37)–(38), we solve the equations $\dot{c} = 0$, $\dot{u} = 0$. It is immediate that this system has no solutions in $c > 1$.

If $0 \leq c \leq 1$, the equation $\dot{c} = 0$ is satisfied if $c = 0$ or $u = 1$. Substitution into $\dot{u} = 0$ of the former yields the steady state $(c, u) = (0, 0)$. Substitution of the latter leads to the quadratic equation

$$c^2 - c + \frac{1}{4\mu} = 0,$$

which can be written as

$$\left(c - \frac{1}{2}\right)^2 - D = 0,$$

with

$$D = \frac{1}{4}\left(1 - \frac{1}{\mu}\right). \tag{40}$$

Note that $D < \frac{1}{4}$, as all parameters are assumed to have positive values. For $D > 0$, the quadratic equation has two real solutions

$$c_{\pm} = \frac{1}{2} \pm \sqrt{D} = \frac{1 \pm \sqrt{1 - 1/\mu}}{2},$$

both satisfying $0 < c_{\pm} < 1$; for $D = 0$, there is a single real solution $c = 1/2$, while for $D < 0$, there is no real solution.

Summarizing, if $0 \leq c \leq 1$ we have the steady states

$$(c, u) = e_0 = (0, 0)$$

and, for $D \geq 0$,

$$(c, u) = e_{\pm} = (c_{\pm}, u_{\pm}) = \left(\frac{1}{2} \pm \sqrt{D}, 1\right). \tag{41}$$

A.1.2 Stability

To analyze stability, we have to determine the eigenvalues of

$$DF = \begin{pmatrix} 1 - u & -c \\ 4\rho\mu(2c - 1) & \rho \end{pmatrix}$$

at the steady states e_0, e_+ and e_-. As

$$DF(e_0) = \begin{pmatrix} 1 & 0 \\ -4\rho\mu & \rho \end{pmatrix},$$

which has eigenvalues ρ and 1, the point e_0 is always an unstable node.

Denote the eigenvalues of the matrix

$$DF(e_\pm) = \begin{pmatrix} 0 & -c_\pm \\ \pm 8\rho\mu\sqrt{D} & \rho \end{pmatrix} \tag{42}$$

by $\lambda^i_\pm$, $i = 1, 2$. They satisfy

$$\lambda^1_\pm + \lambda^2_\pm = \text{trace}\, DF(e_\pm) = \rho$$

and

$$\lambda^1_\pm \lambda^2_\pm = \det DF(e_\pm) = \pm 8\rho\mu c_\pm \sqrt{D}.$$

We have seen before that $c_\pm > 0$ whenever it is real. If $D > 0$, it follows that the eigenvalues λ^1_-, λ^2_- have opposite sign, and e_- is a saddle, whereas λ^1_+ and λ^2_+ have the same sign and positive sum, implying that e_+ is a unstable node.

Expressing these results in the original variables, we obtain the results announced in the proposition.

A.1.3 Bifurcation analysis

It remains to prove the occurrence of a saddle-node bifurcation. If $\mu = \mu_b = 1$, then $D = 0$ and the point $e_b = (c_b, u_b) = (1/2, 1)$ is a steady state with eigenvalues 0 and ρ respectively.

We use a result from Sotomayor (1973) (quoted as Theorem 3.4.1 in Guckenheimer and Holmes, 1986), which for planar dynamical systems states that if the family

$$\dot{x} = F(x; \mu)$$

parametrised by μ satisfies the following three conditions

1. $D_x F(x_0; \mu_0)$ has a simple eigenvalue 0 with right eigenvector v and left eigenvector w;
2. $w D_\mu F(x_0; \mu_0) \neq 0$;
3. $w \left[D^2_x F(x_0; \mu_0)(v, v) \right] \neq 0$;

then it features a non-degenerate saddle-node bifurcation at $x = x_0$ for $\mu = \mu_0$.

As $DF(e_b; \mu_b) = \begin{pmatrix} 0 & -1/2 \\ 0 & \rho \end{pmatrix}$, it follows that $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $w = \begin{pmatrix} 2\rho & 1 \end{pmatrix}$ are respectively left and right eigenvectors associated to the eigenvalue 0. Furthermore

$$w D_\mu F(e_b; \mu_b) = w \begin{pmatrix} 0 \\ -\rho \end{pmatrix} = -\rho \neq 0$$

and, as $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$w \left[D^2_x F(e_b; \mu_b)(v, v) \right] = w \frac{\partial^2}{\partial c^2} F = w \begin{pmatrix} 0 \\ 8\rho \end{pmatrix} = 8\rho \neq 0.$$

We conclude that a nondegenerate saddle-node bifurcation occurs in the system at $\mu = 1$. This completes the proof of Proposition 1.

A.2 Proof of Proposition 2

As in the proof of Proposition 1, introduce the constants

$$K = \frac{1}{\phi(1+\beta)} \quad \text{and} \quad \mu = \frac{\alpha\phi(1+\beta)}{4\rho K} = \frac{\alpha\phi^2(1+\beta)^2}{4\rho},$$

as well as the rescaled control variable $u = k/K$. The state-control system then takes the form

$$\dot{c} = c\,(1-u)\,, \quad \dot{u} = \rho\Big(u - 4\mu c(1-c)\chi(c)\Big). \tag{43}$$

Recall also the notations

$$e_0 = (0,0), \quad e_- = \left(\frac{1-\sqrt{1-1/\mu}}{2}, 1\right), \quad e_+ = \left(\frac{1+\sqrt{1-1/\mu}}{2}, 1\right)$$

for the three steady states of the system, and introduce

$$e_1 = (1,0).$$

To prove the proposition, the state-control space is partitioned into four subsets, R_1, R_2, R_3 and E. Of these, the sets R_3 and E are independent of the values of the system parameters. They are given as

$$R_3 = \{(c,u)\,:\,0 < c < 1,\, u = 0\}, \quad E = \{(c,u)\,:\,c \geq 1,\, u = 0\}.$$

Let $U = \{(c,u)\,:\,u > 0\}$ be the upper half plane. Given the set R_1, the set R_2 is equal to

$$R_2 = U \backslash R_1.$$

It remains to specify R_1, which is the first step in the proof. Then it is shown that no trajectory in either R_2 or R_3 can be optimal. The next step is to demonstrate that of the trajectories in R_1, only those can be optimal which converge either to a steady state in R_1, necessarily a saddle, or which end up in the "exit trajectory" E. Then it has to be shown that the trajectories that are not excluded up to this point, the candidate trajectories, "cover" the state space; that is, for every initial state c_0, there is at least one candidate trajectory passing through the line $c = c_0$. Using parts of the remaining candidate trajectories, we construct a viscosity solution of the Hamilton-Jacobi equation, which is then necessarily the value function. This shows the optimality of the remaining trajectories.

A.2.1 Definition of R_1

Set

$$u_0 = \max\{1, \mu\},$$

and consider the trajectory $\gamma(t) = (c(t), u(t))$ of the system (43) that passes at $t = 0$ through the point $(1, u_0)$.

If $\mu \leq 1$, then $u_0 = 1$ and R_1 is specified as

$$R_1 = \{(c, u) \,:\, 0 \leq c \leq 1,\, 0 < u \leq 1\}\,.$$

If the other possibility $\mu > 1$ obtains, then $u_0 = \mu > 1$ and $\dot{c}(0) < 0$. In this situation, let τ be the least upper bound of those negative values of t that satisfy $c(t) \leq 1$; that is, let

$$\tau = \sup\{t < 0 \,:\, c(t) \leq 1\}.$$

We claim that τ is finite. Arguing by contradiction, assume that $\tau = -\infty$. Then for all $t < 0$ we have $c(t) > 1$, and equation (43) implies that for all $t < 0$

$$u(t) = u_0 \mathrm{e}^{\rho t}.$$

In particular, there is a $t_1 < 0$ such that

$$u(t) < u_0 \mathrm{e}^{\rho t_1} =: K_1 < 1$$

for all $t < t_1$. But for those values of t, it follows that

$$\dot{c} = (1 - u)\, c > (1 - K_1)\, c =: K_2 c,$$

where $K_2 > 0$. Gronwall's lemma implies then that

$$c(t) < \mathrm{e}^{K_2(t - t_1)} c(t_1)$$

if $t < t_1$. But for t sufficiently small, this is smaller than 1, contradicting the hypothesis that $\tau = -\infty$. Hence τ is finite.

Introduce u_τ by the equation $\gamma(\tau) = (1, u_\tau)$. The set R_1 is defined as follows: it is the open region bounded by the concatenation of the curve γ taken between $t = 0$ and $t = \tau$, connecting $(1, u_0)$ and $(1, u_\tau)$, the vertical line segment connecting $(1, u_\tau)$ to e_1, the horizontal segment connecting e_1 to e_0, the vertical segment connecting e_0 to $(0, u_0)$, and the horizontal segment connecting $(0, u_0)$ to $(1, u_0)$. See Figure 11 for the possible shapes of R_1.

A.2.2 Trajectories in R_2 cannot be optimal

In the second step of the proof, the transversality condition is used to show that any trajectory that passes through points in R_2 cannot be optimal.

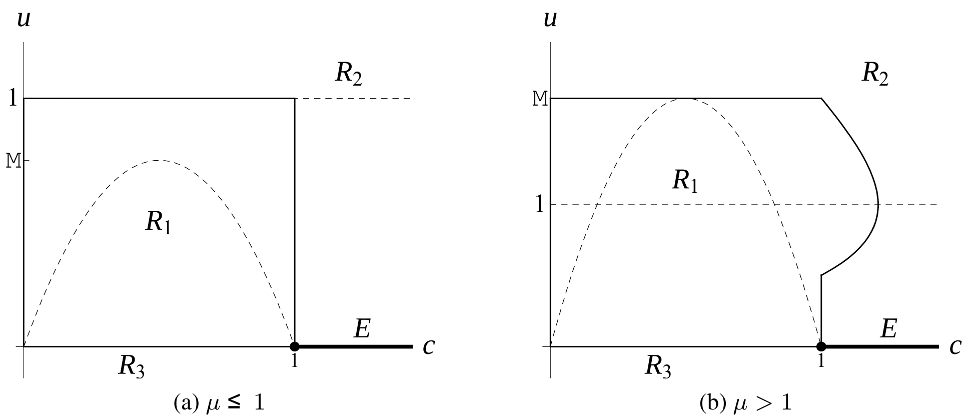


Figure 11: Definition of the set R_1. Solid curves denote the boundary of the set, dashed curves the isoclines of the system (37)–(38).

Beginning with R_2, we note that the subset

$$R_2^{(1)} = \{(c, u) \,:\, 0 \leq c \leq 1\} \cap R_2$$

of R_2 is a *forward trapping region*: once a trajectory of (43) is inside $R_2^{(1)}$, it remains inside for all subsequent times. This fact is established by demonstrating that the vector field defined by (43) is inward pointing on the boundary of $R_2^{(1)}$. For, if $u = u_0 = \max\{1, \mu\}$ and $0 \leq c \leq 1$, then

$$\dot{u} \geq \rho(\mu - 4\mu c(1 - c)) = 0.$$

If $c = 0$, then $\dot{c} = 0$, and if finally $c = 1$ and $u \geq u_0 \geq 1$, then

$$\dot{c} \leq c\,(1 - 1) = 0.$$

Actually, we can make the sharper statement that if $u > u_0$, then

$$\dot{u} > 0. \tag{44}$$

To show that no trajectory that enters $R_2^{(1)}$ can be maximizing, pick an arbitrary trajectory γ such that $\gamma(t_0) \in R_2^{(1)}$ at a given time t_0. By the Poincaré-Bendixon theorem, $\gamma(t)$ is either unbounded, or its ω-limit set is a steady state, or a limit cycle. The latter possibility is excluded, as the state-costate system, which is in one-to-one relation with the state-control system, has constant positive divergence everywhere (see [51]). There are no steady states in $R_2^{(1)}$. Hence there is a sequence $t_0, t_1, \ldots$ such that $\|\gamma(t_i)\| \to \infty$. In particular, there is $\bar{t} > t_0$ such that $u(\bar{t}) > 2u_0$. But then $u(t)$ is monotonely increasing towards infinity as $t > \bar{t}$, as a consequence of (44).

Consequently, if $t \geq \bar{t}$, then

$$\dot{c} \leq (1 - 2u_0)\, c \leq -c.$$

By Gronwall's lemma it follows that

$$c(t) \leq c(\bar{t})\mathrm{e}^{-(t-\bar{t})}. \tag{45}$$

Likewise, if $t \geq \bar{t}$, then $u(t) > 2u_0$ and

$$\dot{u} \geq \rho(u - \mu).$$

Gronwall's lemma implies then that

$$u(t) \geq \mu + (2u_0 - \mu)\mathrm{e}^{\rho(t-\bar{t})}. \tag{46}$$

If the trajectory $\gamma(t) = (c(t), u(t))$ is optimal, then by the Hamilton-Jacobi equation (see e.g. [51]), the total profit Π takes the value

$$\Pi(c(0)) = \frac{1}{\rho} H(c(0), \lambda(0)) = \frac{1}{\rho} H_{\text{control}}(c(0), u(0)). \tag{47}$$

Michel's transversality condition [39] states that along a maximizing trajectory the relation

$$\lim_{t\to\infty} \Pi(c(t))\mathrm{e}^{-\rho t} = 0$$

holds. Combining (47) and (39) yields

$$\Pi(c(t))\mathrm{e}^{-\rho t} \geq \left(4\rho\mu(1 - c(t))^2\chi(c(t)) + u(t)(u(t) - 2)\right)\mathrm{e}^{-\rho t}$$

Using that the first term between brackets is always nonnegative, and taking into account (46) yields that

$$\Pi\mathrm{e}^{-\rho t} \geq (2u_0 - \mu)\mathrm{e}^{\rho(t-\bar{t})}\left(\mu - 2 + (2u_0 - \mu)\mathrm{e}^{\rho(t-\bar{t})}\right)\mathrm{e}^{-\rho t}.$$

As $2u_0 - \mu \geq \mu > 0$, it follows that the right hand side of this inequality tends to infinity as $t \to \infty$. But then

$$\lim_{t\to\infty} \Pi(c(t))\mathrm{e}^{-\rho t} = \infty,$$

and γ cannot be a maximizing trajectory.

It remains to show that no trajectory passing through

$$R_2^{(2)} = R_2 \backslash R_2^{(1)},$$

the complement of $R_2^{(1)}$ in R_2, can be optimal. Consider therefore a trajectory γ such that $\gamma(t_0) \in R_2^{(2)}$ for some t_0. As in the definition of the region R_1, using Gronwall's lemma it can be shown that there is some $t_1 > t_0$ such that $u(t_1) > 1$, and some $t_2 > t_1$ such that $u(t_2) > 1$ and $c(t_2) = 1$. But then γ enters the trapping region $R_2^{(1)}$, and we have already seen that such trajectories cannot be optimal.

A.2.3 Trajectories intersecting R_3 cannot be optimal

If a trajectory intersects R_3, the state-control representation breaks down, and we have to switch to the state-costate representation.

Pick an arbitrary state-costate trajectory $\gamma(t) = (c(t), \lambda(t))$, with associated control $u(t) = \max\{0, -\frac{1}{2}c(t)\lambda(t)\}$ such that $\big(c(t_0), u(t_0)\big) \in R_3$ for some $t_0 \geq 0$ and $\big(c(t), u(t)\big) \in R_1$ for all $t < t_0$ that are sufficiency close to t_0. The costate λ then satisfies $\lambda(t_0) = 0$ and $\dot{\lambda}(t_0) > 0$. Note that the region

$$\tilde{R}_3 = \{(c, \lambda) \,:\, \lambda > 0\}$$

is a trapping region for the state-costate flow, as $\dot{\lambda} \geq 0$ whenever $\lambda = 0$.

Using Gronwall's lemma, we show first that

$$c(t) \geq c(t_0)\mathrm{e}^{(t-t_0)},$$

for $t > t_0$, since $\dot{c} = c \geq c$ in $\tilde{R}_3$ (equation (36)). It follows that there is $t_1 > t_0$ such that $c(t) > 1$ for all $t > t_1$.

Let $h(t) = H(c(t), \lambda(t))$. Note that for all $t > t_1$ we have $c(t) > 1$ and $\lambda(t) > 0$, and consequently $h(t) = \lambda(t)c(t) > 0$. The state-costate equations reduce to

$$\dot{c} = c, \quad \dot{\lambda} = (\rho - 1)\lambda. \tag{48}$$

Compute:

$$\dot{h} = \dot{\lambda}c + \lambda\dot{c} = \rho\lambda c = \rho h.$$

Hence

$$h(t) = h(t_1)\mathrm{e}^{\rho(t-t_1)}$$

for all $t > t_1$. But then

$$\lim_{t\to\infty} h(t)\mathrm{e}^{-\rho t} = h(t_1)\mathrm{e}^{-\rho t_1} > 0.$$

If γ is optimal, Michel's transversality condition implies that

$$\lim_{t\to\infty} \Pi(c(t))\mathrm{e}^{-\rho t} = \lim_{t\to\infty} \frac{1}{\rho} H(c(t), \lambda(t))\mathrm{e}^{-\rho t} = \lim_{t\to\infty} \frac{h(t)}{\rho}\mathrm{e}^{-\rho t} = 0.$$

As this is a contradiction, the trajectory γ cannot be optimal.

A.2.4 Trajectories in R_1 with wrong limit behavior cannot be optimal

As the set R_1 is bounded, by the Poincaré-Bendixon theorem trajectories in R_1 can either converge to a steady state, or leave R_1 (cf. the argument in Section A.2.2). Those entering either R_2 or R_3 have already been shown to be suboptimal. The remaining possibility is to leave R_1 through the point e_1 and enter the line segment E; these trajectories remain candidates for optimality.

Trajectories remaining in R_1 have to converge to a steady state. From proposition 1 we learn that e_0 and e_+ are unstable nodes, to which no trajectory can converge as $t \to \infty$. The only remaining candidate is then the saddle e_-, if $\mu < 1$, or the bifurcating point e_b if $\mu = 1$.

This completes the proof of Proposition 2.

A.3 Proof of proposition 3

A.3.1 Construction of policy functions

The first step in the proof is to construct those (parts of) trajectories of the system (43) that will turn out to optimize the profit functional. In particular, we shall construct a, possibly multivalued, *policy function* u_f such that the following holds. If (c_0, u_0) is such that $u_0 = u_f(c_0)$, then the trajectory $(c(t), u(t))$ of (43) starting at this point satisfies, for all $t \geq 0$, that $\dot{c}(t) \neq 0$ and $u(t) = u_f(c(t))$.

Again we have to distinguish between the situations $\mu < 1$ and $\mu \geq 1$.

First situation: $\mu < 1$. Here the only steady state of (43) is the origin e_0, which is an unstable node. Therefore, the only candidate optimizer is the trajectory passing through the point e_1. Note that a corollary of the analysis performed above is that the set R_1 is a *backward trapping region*: if a trajectory is in R_1 for some time, it is in R_1 for all previous times. Necessarily it converges to the origin as $t \to -\infty$.

Let $\gamma(t) = (c_\gamma(t), u_\gamma(t))$ be the trajectory such that $\gamma(0) = e_1$. As $\gamma(t) \in R_1$ for all $t < 0$, it follows that $\dot{c}_\gamma > 0$ for all $t < 0$ (recall that R_1 is open). As $u(t) = 0$ for all $t \geq 0$, it follows that $\dot{c}_\gamma > 0$ for all t, and that the map $c_\gamma : \mathtt{R} \to (0, \infty)$ is invertible, with inverse $t = t_\gamma(c)$. Define $u_f : (0, \infty) \to \mathtt{R}$ by

$$u_f(c) = u_\gamma\left(t_\gamma(c)\right)$$

Then the image of the curve $\gamma : \mathtt{R} \to \mathtt{R}^2$ is equal to the graph of the function $u_f : (0, \infty) \to \mathtt{R}$, as

$$u_\gamma(t) = u_f(c_\gamma(t))$$

for all t.

Second situation: $\mu \geq 1$. In this case, though R_1 is still a backward trapping region, there are at least two steady states in R_1: apart from the origin e_0, we also have e_- and e_+ As seen before, if $D > 0$ the first is a saddle and the second a repeller; if $D = 0$, these two points coincide in e_b.

Denote by δ_1 the part of the parabola $u = 4\mu c(1 - c)$ connecting e_0 to e_-, by δ_2 the segment of the line $u = 1$ connecting e_- to e_+, by δ_3 that part of the same parabola which connects e_+ to e_1, and by δ_4 the segment of the line $u = 0$ connecting e_1 to e_0. All curves δ_i are taken without their endpoint. Let finally $S_1 \subset R_1$ be the open subregion of R_1 that is bounded by the curves δ_i, $i = 1, \ldots, 4$. See Figure 12.

Let $\gamma(t) = (c(t), u(t))$ be the trajectory of (43) satisfying $\gamma(0) = e_1$. As the open set S_1 is bounded, the trajectory γ either converges to a steady state on the boundary of S_1, or it enters S_1 for the last time by crossing one of the curves δ_i. We analyze the possibilities one by one.

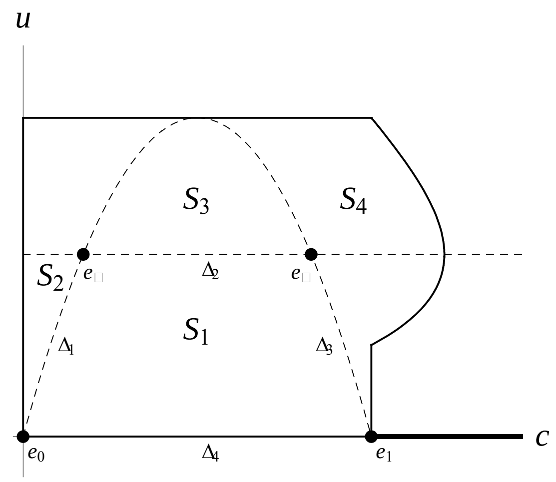


Figure 12: Subdivision of region R_1. The vertices e_0, e_1, e_- and e_+, the edges δ_i, $i = 1, \ldots, 4$, and the faces S_i, $i = 1, \ldots, 4$ are defined in the text.

The trajectory remains in S_1 for all $t < 0$ and tends to e_0. If $\gamma(t) \in S_1$ for all $t < 0$ and $\gamma(t) \to e_0$ as $t \to -\infty$, then the results of the situation $D < 0$ carry over unmodified, and we obtain a policy function $u_f : (0, \infty) \to \mathtt{R}$.

The trajectory remains in S_1 for all $t < 0$ and tends to e_-. If $\gamma(t) \in S_1$ for all $t < 0$ and $\gamma(t) \to e_-$ as $t \to -\infty$, then γ is part of the unstable manifold of e_-. Reasoning as in the situation $D < 0$, we obtain a policy function

$$u_f^{(1)} : (c_-, \infty) \to \mathtt{R}$$

with

$$\lim_{c \downarrow c_-} u_f^{(1)}(c) = u_- = 1.$$

However, this function is not defined for all $c > 0$. To construct a policy function for $0 < c < c_-$, we take a trajectory γ^s on the left half of the stable manifold of the saddle e_-.

We claim that this part of the stable manifold is contained in its entirety in the region S_2 that is bounded by δ_1, the segment of $u = 1$ connecting e_- to $(0, 1)$, and the segment of the line $c = 0$ connecting $(0, 1)$ to e_0. It is straightforward to show that S_2 is a backward trapping region; consequently, every trajectory in S_2 converges to the unstable node e_0 as $t \to -\infty$.

The stable manifold of e_- is tangent to the stable eigenspace of

$$DF(e_-) = \begin{pmatrix} 0 & -c_- \\ 8\rho\mu\sqrt{D} & \rho \end{pmatrix},$$

cf. equation (42), at e_-. Note that the vector $v = (0, 1)$ cannot be an eigenvector of this matrix, as $c_- \neq 0$. Therefore any eigenvector $v = (v_1, v_2)$ satisfies $v_1 \neq 0$; it may therefore be assumed that $v_1 = 1$.

Let $v^s = (1, v_2^s)$ be the stable eigenvector, with eigenvalue $\lambda^s < 0$. The eigenvalue equation

$$DF(e_-)v^s = \lambda^s v^s$$

then yields

$$v_2^s = -\frac{\lambda^s}{c_-} > 0.$$

Locally around the saddle, the stable manifold coincides with the graph of a function w^s, defined on a neighborhood of c_-, which is of the form

$$w^s(c) = c_- + v_2^s(c - c_-) + O((c - c_-)^2).$$

In particular, if $c_0 < c_-$ is sufficiently close to c_-, then

$$\frac{\mathrm{d}w^s}{\mathrm{d}c}(c) > 0$$

for all $c \in [c_0, c_-]$. The trajectory $\gamma(t)$ of (43) such that $\gamma(0) = (c_0, w^s(c_0))$ consequently satisfies $c_0 \le c(t) < c_-$, as well as $\dot{c}(t) > 0$ and $\dot{u}(t) > 0$ for all $t \ge 0$. We infer that necessarily

$$4\mu c(t)(1 - c(t)) < u(t) < 1$$

for all $t \ge 0$, and hence $(c(t), u(t)) \in S_2$ for all $t \ge 0$. But as S_2 is a backward trapping region, the trajectory γ is contained in S_2 for all t, hence satisfying

$$\gamma(t) \to e_0 \quad \text{as} \quad t \to -\infty, \qquad \text{and} \qquad \gamma(t) \to e_- \quad \text{as} \quad t \to \infty.$$

As in S_2, we have $\dot{c} > 0$ everywhere, and we construct as above a policy function

$$u_f^{(2)} : (0, c_-) \to \mathtt{R}, \quad \text{with} \quad \lim_{c \uparrow c_-} u_f^{(2)}(c) = u_- = 1.$$

It follows that the function

$$u_f(c) = \begin{cases} u_f^{(1)}(c) & \text{if } c > c_-, \\ u_- & \text{if } c = c_-, \\ u_f^{(2)}(c) & \text{if } 0 < c < c_-, \end{cases}$$

is a continuous policy function that is defined for all $c > 0$.

The trajectory remains in S_1 for all $t < 0$ and tends to e_+. As before, we can construct a policy function

$$u_f^{(1)} : (c_+, \infty) \to \mathtt{R}, \quad \text{with} \quad \lim_{c \downarrow c_+} u_f^{(1)}(c) = u_+ = 1.$$

The remaining part of the policy function has to be furnished by the stable manifold of e_-. As above, the left half of this stable manifold furnishes a policy function

$$u_f^{(2)} : (0, c_-) \to \mathtt{R}, \quad \text{with} \quad \lim_{c \uparrow c_-} u_f^{(2)}(c) = u_- = 1.$$

We turn to the right half of the stable manifold. For values of c_0 larger than but close to c_-, the point $(c_0, u_0) = (c_0, w^s(c_0))$ on the stable manifold is contained in the bounded open region S_3 that is bounded by the line $u = 1$ and the parabola $u = 4\mu c(1-c)$. In this region $\dot{c} < 0$ and $\dot{k} < 0$. Fix (c_0, u_0) and consider the trajectory γ of (43) such that $\gamma(0) = (c_0, u_0)$. This trajectory enters S_3 through the part of the parabola connecting its vertex $(1/2, \mu)$ with the point e_+. It enters from the region S_4 that is bounded by that same part of the parabola, the line $u = u_+$ and the boundary of R_1. In that region, $\dot{c} < 0$, but $\dot{k} > 0$. It follows that the trajectory has to enter S_4 through the line segment of $c = c_+$ connecting e_+ and (c_+, μ), or through one of the endpoints.

If $\gamma(t) \to e_+$ as $t \to -\infty$, then its graph defines a policy function

$$u_f^{(3)} : (c_-, c_+) \to \mathtt{R} \quad \text{with} \quad \lim_{c \downarrow c_-} u_f^{(3)}(c) = u_- = 1, \quad \lim_{c \uparrow c_+} u_f^{(3)}(c) = u_+ = 1.$$

A continuous policy function is then given by

$$u_f(c) = \begin{cases} u_f^{(1)}(c) & \text{if } c > c_+, \\ u_+ & \text{if } c = c_+, \\ u_f^{(2)}(c) & \text{if } 0 < c < c_-, \\ u_- & \text{if } c = c_-, \\ u_f^{(3)}(c) & \text{if } c_- < c < c_+. \end{cases}$$

Otherwise, there is a time $t_1 < 0$, such that $c(t_1) = c_+$ and $u(t_1) > u_+$. As in this case $\gamma(t)$ does not tend to a steady state in the boundary of S_4, it has to enter S_4 for some $t_2 < t_1$; the only possibility for this is through the line $u = 1$. We therefore have

$$\gamma(t_2) = (c_M, 1).$$

In this situation, the graph $\gamma([t_2, \infty))$ defines a policy function

$$u_f^{(3)} : (c_-, c_M) \to \mathtt{R} \quad \text{with} \quad \lim_{c \downarrow c_-} u_f^{(3)}(c) = u_- = 1, \quad \lim_{c \uparrow c_M} u_f^{(3)}(c) = 1.$$

On the interval (c_+, c_M), there are now two policy functions defined. Recall that the total profit at an initial state c of an R&D policy for which $u = u_f(c)$ is given by

$$\Pi(c) = \frac{1}{\rho} H_{\text{control}}(c, u) = \frac{1}{\rho} \left(4\rho\mu(1-c)^2 \chi(c) + u^2 - 2u \right).$$

For fixed values of c, the function $H_{\text{control}}(c, u)$ is minimal at $u = 1$. Hence the policy $u_f^{(3)}$ is superior to $u_f^{(1)}$ at $c = c_+$, but it is inferior to it at $c = c_M$. As both

functions are continuous, there is a value $c = \hat{c}$ such that both policies generate the same total profit. This is an indifference point, as the manager is indifferent between two policies at this state. A policy function, which is at one point two-valued, is then given by

$$u_f(c) = \begin{cases} u_f^{(1)}(c) & \text{if } c > \hat{c}, \\ u_f^{(1)}(\hat{c}) \text{ or } u_f^{(3)}(\hat{c}) & \text{if } c = \hat{c}, \\ u_f^{(2)}(c) & \text{if } 0 < c < c_-, \\ u_- & \text{if } c = c_-, \\ u_f^{(3)}(c) & \text{if } c_- < c < \hat{c}. \end{cases}$$

Note that the induced total profit $\Pi(c) = H_{\text{control}}(c, u_f(c))/\rho$ is Lipschitz continuous.

The trajectory enters S_1 for the last time through δ_1. The next situation to be investigated is that the trajectory γ satisfying $\gamma(0) = e_1$ enters S_1 through δ_1 at some time $t_1 < 0$, and remains in S_1 for all $t_1 < t < 0$. But then it has to be in the backward trapping region S_2 for all $t < t_1$, and it converges to e_0 as $t \to -\infty$. As $\dot{c} > 0$ in both S_1 and S_2, we can construct a differentiable policy function exactly as in the situation that the trajectory remains in S_1 for $t < 0$ and converges to e_0.

The trajectory enters S_1 for the last time through δ_2. Finally consider the situation that the trajectory γ that passes through e_1 at $t = 0$ enters S_1 through δ_2 for some $t_1 < 0$, and remains in S_1 for all $t_1 < t < 0$. Introduce c_m by setting $\gamma(t_1) = (c_m, 1)$. As $\dot{c}(t) > 0$ for $t_1 < t < 0$ as well as for $t \geq 0$, we can construct a continuous policy function

$$u_f^{(1)} : [c_m, \infty) \to \mathbb{R}, \quad u_f^{(1)}(c_m) = 1.$$

in the usual manner. The left branch of the stable manifold of the saddle e_- furnishes a continuous policy function

$$u_f^{(2)} : (0, c_-) \to \mathbb{R}, \quad \text{with} \quad \lim_{c \uparrow c_-} u_f^{(2)}(c) = u_- = 1,$$

and the right branch of that manifold furnishes a continuous policy function

$$u_f^{(3)} : (c_-, c_M) \to \mathbb{R}, \quad \text{with} \quad \lim_{c \downarrow c_-} u_f^{(3)}(c) = u_- = 1, \quad u_f^{(3)}(c_M) = 1,$$

where $c_+ \leq c_M$. Invoking the same arguments as above, we show that $u_f^{(3)}$ is superior to $u_f^{(1)}$ at $c = c_m$ and inferior to it at $c = c_M$. By the intermediate value theorem, there is an indifference point $\hat{c}$ such that $c_m < \hat{c} < c_M$, and such that the

manager is indifferent between the two policies at $c = \hat{c}$. A policy function defined on all points of state space is then

$$u_f(c) = \begin{cases} u_f^{(1)}(c) & \text{if } c > \hat{c}, \\ u_f^{(1)}(\hat{c}) \text{ or } u_f^{(3)}(\hat{c}) & \text{if } c = \hat{c}, \\ u_f^{(2)}(c) & \text{if } 0 < c < c_-, \\ u_- & \text{if } c = c_-, \\ u_f^{(3)}(c) & \text{if } c_- < c < \hat{c}. \end{cases}$$

Summary. For all parameters, we have constructed a policy function

$$u_f : (0, \infty) \to \mathtt{R},$$

which is single-valued, except at most at one point $\hat{c}$, the indifference point. Moreover, the values of the two trajectories originating at an indifference point are the same.

A.3.2 Policy functions generate viscosity solutions of the Hamilton-Jacobi equation

Using relation (47), we have that

$$V(c) = \frac{1}{\rho} H_{\text{control}}(c, u_f(c))$$

is well-defined at $c = \hat{c}$, continuous and continuously differentiable at all points $c > 0$ except $\hat{c}$. Moreover, the value of the total profit (32) along a trajectory γ of the state-control system (37) such that $\gamma(0) = (c, u_f(c))$ is equal to $V(c)$.

We now go back to the state-costate representation (34)–(35), and introduce the feedback costate function

$$\lambda_f(c) = -\frac{2}{c} u_f(c).$$

Note that then

$$V(c) = \frac{1}{\rho} H(c, \lambda_f(c)). \tag{49}$$

By construction, if $\gamma(t) = (c(t), \lambda(t))$ is a solution of the state-costate system such that $\lambda(0) = \lambda_f(c(0))$, then

$$\lambda(t) = \lambda_f(c(t)) \quad \text{for all} \quad t.$$

If $t > 0$, then $c(t) \neq \hat{c}$ and λ_f is differentiable at $c(t)$; by the chain rule

$$\dot{\lambda} = \lambda_f'(c)\dot{c}. \tag{50}$$

We claim that $\lambda_f(c) = V'(c)$ for all $c \neq \hat{c}$. For, differentiating (49) with respect to c yields

$$V'(c) = \frac{1}{\rho}\left(H_c + H_\lambda \lambda_f'(c)\right).$$

Evaluating this equation at $c = c(t)$, using first (50) and then (34) and (35) gives

$$\begin{aligned} V'\big(c(t)\big) &= \frac{1}{\rho}\left(H_c + H_\lambda \frac{\dot{\lambda}}{\dot{c}}\right) \\ &= \frac{1}{\rho}\left(H_c + H_\lambda \frac{\rho\lambda - H_c}{H_\lambda}\right) \\ &= \lambda(t) = \lambda_f\big(c(t)\big); \end{aligned}$$

this proves the claim.

It follows that the function V defined by (49) is a regular solution of the Hamilton-Jacobi equation

$$\rho V(c) = H(c, V'(c)) \tag{51}$$

for all $c \neq \hat{c}$.

Viscosity solutions. We quote the definition of viscosity sub- and supersolutions from [17] (Section II.11, p. 106).

Definition

1° *A function W is a* viscosity subsolution *of* (51) *at $\bar{c}$, if for every continuously differentiable function w such that the difference $W - w$ takes a local maximum at $\bar{c}$, we have*

$$\rho V(\bar{c}) - H(\bar{c}, w'(\bar{c})) \leq 0. \tag{52}$$

2° *A function W is a* viscosity supersolution *of* (51) *at $\bar{c}$, if for every continuously differentiable function w such that the difference $W - w$ takes a local minimum at $\bar{c}$, we have*

$$\rho V(\bar{c}) - H(\bar{c}, w'(\bar{c})) \geq 0. \tag{53}$$

3° *A function W is a* viscosity solution *of* (51)*, if it is both a viscosity subsolution and a viscosity supersolution.*

As V is continuously differentiable in the neighborhood of every point $\bar{c} \neq \hat{c}$, taking $w = V$ in these definitions shows that V is a viscosity solution of the Hamilton-Jacobi equation (51) at $\bar{c}$ if and only if it is a regular solution at $\bar{c}$.

It remains to show that V is a viscosity solution at an indifference point $\hat{c}$. Note that the left and right limits of $V'(c)$ exist at $\hat{c}$; we write

$$\hat{\lambda}^- = \lim_{c \uparrow \hat{c}} V'(c), \quad \hat{\lambda}^+ = \lim_{c \downarrow \hat{c}} V'(c).$$

From the analysis done above, we infer that

$$\hat{\lambda}^- < \hat{\lambda}^+$$

Let v be a continuously differentiable function such that $V - v$ takes a local minimum at $c = \hat{c}$. Then necessarily

$$\lim_{c \uparrow \hat{c}} V'(c) - v'(c) \leq 0, \quad \lim_{c \downarrow \hat{c}} V'(c) - v'(c) \geq 0,$$

implying that

$$\hat{\lambda}^- \leq v'(c) \leq \hat{\lambda}^+. \tag{54}$$

As $\hat{c}$ is an indifference point, we have that

$$H(\hat{c}, \hat{\lambda}^-) = H(\hat{c}, \hat{\lambda}^+) = \rho V(\hat{c}).$$

Moreover, the Hamilton function $H(c, \lambda)$ is convex in λ. Together with (54) this implies that

$$\rho V(\hat{c}) - H(\hat{c}, v'(\hat{c})) \geq 0.$$

Hence V is a viscosity supersolution.

Consider now the situation that v is a continuously differentiable function such that $V - v$ takes a local maximum at $\hat{c}$. Then

$$\lim_{c \uparrow \hat{c}} V'(c) - v'(c) \geq 0, \quad \lim_{c \downarrow \hat{c}} V'(c) - v'(c) \leq 0,$$

which implies that

$$v'(\hat{c}) \leq \hat{\lambda}_- < \hat{\lambda}_+ \leq v'(\hat{c}),$$

which is a contradiction. There is no differentiable function such that $V - v$ takes a local minimum; but then for all such functions, the inequality (52) holds at $\hat{c}$, and V is a viscosity subsolution.

As we know (cf. [17]) that the unique viscosity solution of the Hamilton-Jacobi equation is the value function of the problem, it follows that the trajectories defined by the policy function are optimal. This concludes the proof.

A.4 Proof of Proposition 4

This is an immediate consequence of the scaling (31). For assume that there is a bifurcation at $(\mu, \rho) = (\mu_*, \rho_*)$. Then for $\rho = \rho_*$, the value $K^{-1} = \phi(1 + \beta)$ is bifurcating if

$$K_*^{-1} = \frac{2\sqrt{\rho_* \mu_*}}{\sqrt{\alpha}}.$$

As $\alpha = 1/9$ under competition and $\alpha = 1/8$ under collusion, this implies

$$K_{*\text{comp}}^{-1} = 6\sqrt{\rho_* \mu_*} > 4\sqrt{2}\sqrt{\rho_* \mu_*} = K_{*\text{coll}}^{-1}.$$

This proves the proposition.

A.5 Proof of Propositions 5 and 6

We want to compare, for a given parameter combination, the collusive situation $\alpha = \frac{1}{8}$, and the competitive situation $\alpha = \frac{1}{9}$. Performing the scaling to (c, u) variables and (μ, ρ) parameters, this reduces to comparing a competitive situation (μ_1, ρ) with the collusive situation (μ_2, ρ), where the μ_i are related as

$$\mu_2 = \frac{9}{8}\mu_1.$$

Denote by u_f^i, $i = 1, 2$ the corresponding policy functions, and recall that their graphs are locally equal to a portion of a trajectory of (37)–(38), with u replaced by u_1 or u_2, depending on whether $\mu = \mu_1$ or $\mu = \mu_2$. Invoking the chain rule as in (50), we can derive a differential equation for $u_i = u_f^i$ as follows:

$$\frac{\mathrm{d}u_i}{\mathrm{d}c} = \frac{\dot{u}_i}{\dot{c}} = \frac{\rho\,(u_i - 4\mu c(1-c)\chi)}{c(1-u_i)};$$

here, we have written $\chi = \chi(c)$ for brevity. This is a first order non-autonomous differential equation, with singularities at $c = 0$ and $u_i = 1$.

Writing $\Delta\mu = \mu_2 - \mu_1$ and $\Delta u = u_2 - u_1$, the difference Δu satisfies the following differential relation:

$$\begin{aligned}
\frac{\mathrm{d}\Delta u}{\mathrm{d}c} &= \frac{\rho\,(u_2 - 4\mu_2 c(1-c)\chi)}{c(1-u_2)} - \frac{\rho\,(u_1 - 4\mu_1 c(1-c)\chi)}{c(1-u_1)} \\
&= \frac{\rho(1-u_1)\,(u_2 - 4\mu_2 c(1-c)\chi)}{c(1-u_1)(1-u_2)} - \frac{\rho(1-u_2)\,(u_1 - 4\mu_1 c(1-c)\chi)}{c(1-u_1)(1-u_2)} \\
&= \frac{\rho\,(u_2 - u_1u_2 - 4c(1-c)\chi(\mu_2 - u_1\mu_2))}{c(1-u_1)(1-u_2)} \\
&\qquad - \frac{\rho\,(u_1 - u_1u_2 - 4c(1-c)\chi(\mu_1 - u_2\mu_1))}{c(1-u_1)(1-u_2)} \\
&= \frac{\rho\,(\Delta u - 4c(1-c)\chi(\Delta\mu + u_2\mu_2 - u_1\mu_2 - u_2\mu_2 + u_2\mu_1))}{c(1-u_1)(1-u_2)} \\
&= \frac{\rho\,(\Delta u - 4c(1-c)\chi(\Delta\mu + \mu_2\Delta u - u_2\Delta\mu))}{c(1-u_1)(1-u_2)} \\
&= \frac{\rho\,(1 - 4\mu_2 c(1-c)\chi)}{c(1-u_1)(1-u_2)}\Delta u - \frac{4\rho(1-c)\chi}{1-u_1}\Delta\mu
\end{aligned}$$

As u_1 and u_2 are known, this relation is of the form

$$\frac{\mathrm{d}\Delta u}{\mathrm{d}c} = a(c)\Delta u + b(c),$$

where a and b are known functions. For

$$\Delta u(c_0) = \Delta_0$$

the variations of constants formula for the solution reads as

$$\Delta u(c) = \Delta_0 \mathrm{e}^{\int_{c_0}^{c} a(x)\mathrm{d}x} + \int_{c_0}^{c} b(x)\mathrm{e}^{\int_{x}^{c} a(y)\mathrm{d}y}\mathrm{d}x.$$

A.5.1 Proof of Proposition 6

Consider first the situation that there is a value $0 \leq \bar{c} \leq 1$ such that for all $c \in (\bar{c}, 1]$ the optimal trajectories for both the collusive and the competitive case leave the production region through e_1. As we know that trajectories through e_1 can be optimal only if they have not crossed the line $u = 1$ yet, the term b of the variations of constants formula satisfies

$$b(c) = -\frac{4\rho(1-c)\chi}{1-u_1}\Delta\mu \leq 0$$

for $\bar{c} < c \leq 1$. Taking $c_0 = 1$ gives $\Delta_0 = 0$, which implies that

$$\Delta(c) > 0$$

for all $\bar{c} < c \leq 1$. Hence collusive R&D effort is always larger than competitive R&D effort if both lead to eventually leaving the market.

Next, we consider the situation that there is some $\bar{c} > 0$, such that for all $c \in (0, \bar{c})$, the optimal trajectories for both the competitive and the collusive case converge to their respective steady states $e_-^1 = (c_-^1, 1)$ and $e_-^2 = (c_-^2, 1)$. As $\mu_2 < \mu_1$, it follows that $0 < c_-^2 < c_-^1 \leq 1/2$. The stable manifold tending to e_-^2 can only leave the region bounded by the parabola $u = \mu_2 c(1-c)$ and the lines $u = 1$ and $c = 1/2$ through the line segment connecting the points $(1/2, 1)$ with $(1/2, \mu_2)$. It follows that necessarily

$$u_2(c_-^1) > u_1(c_-^1), \quad \text{or equivalently,} \quad \Delta(c_-^1) > 0.$$

We have already established that trajectories tending to either e_-^1 or e_-^2 can only be optimal if they do not cross the line $u = 1$. Therefore

$$b(c) = \frac{4\rho(1-c)\chi}{u_1 - 1}\Delta\mu > 0,$$

if $0 < c < \bar{c}$, and the variations of constants formula implies

$$\Delta(c) > 0 \quad \text{for all} \quad c_-^1 \leq c < \bar{c}.$$

Moreover $u_1(c) < 1$ if $0 < c < c_-^1$, implying that $b(c) < 0$ there. Again using the variations of constants formula, we obtain

$$\Delta(c) > 0 \quad \text{for all} \quad 0 < c \leq c_-^1$$

as well.

Finally, if the optimal trajectory of the collusive case converges to e_-^2, whereas the optimal trajectory of the competitive case exit the production region through e_1, we have that the former satisfies $u \geq 1$ and the latter $u \leq 1$.

This proves Proposition 6.

A.5.2 Proof of Proposition 5

To prove Proposition 5, we again use the fact that the value of the integral Π over a trajectory starting at a point (c, u) equals

$$\begin{aligned}\Pi(c,u) &= \frac{1}{\rho}H_{\text{control}}(c,u) = \frac{1}{\rho}\left(4\rho\mu(1-c)^2\chi - 1 + (u-1)^2\right) \\ &= h(c) + C(u-1)^2. \end{aligned} \tag{55}$$

If $c = \hat{c}$ is an indifference point, there are values $\hat{u}^{(1)} > \hat{u}^{(2)}$ such that the trajectories starting at $(\hat{c}, \hat{u}^{(i)})$, for $i = 1, 2$, are both optimal and have both the same value. Note that the trajectory through $(\hat{c}, \hat{u}^{(1)})$ goes to the left, and that through $(\hat{c}, \hat{u}^{(2)})$ goes to the right. As

$$\Pi\left(\hat{c}, \hat{u}^{(1)}\right) = \Pi\left(\hat{c}, \hat{u}^{(2)}\right),$$

it follows that

$$|\hat{u}^{(1)} - 1| = |\hat{u}^{(2)} - 1|.$$

Consider a fixed value of ρ and two values μ_1, μ_2 of μ such that $\mu_2 = (9/8)\mu_1$; that is, (μ_1, ρ) describes a competitive situation, and (μ_2, ρ) is the corresponding collusive situation.

Assume first that there is an indifference point in the competitive problem; denote these points as $\hat{c}_1$, and the corresponding values of u as

$$\hat{u}_1^{(1)} < \hat{u}_1^{(2)}.$$

We have seen in the proof of Proposition 6 that necessarily the collusive trajectory going towards e_-^2 is above the competitive trajectory going towards e_-^1. Denote its intersection with the line $c = \hat{c}_1$ by $(\hat{c}_1, \hat{u}_2^{(1)})$. We have that

$$|\hat{u}_2^{(1)} - 1| > |\hat{u}_1^{(1)} - 1|.$$

We argue by contradiction. Assume that the threshold $\hat{c}_2$ in the collusive case exists and is below the threshold in the competitive case, then the collusive trajectory going right, that is, to e_1, has to intersect the line $c = \hat{c}_1$ in a point $(\hat{c}_1, \hat{u}_2^{(2)})$. Moreover, this trajectory has to be optimal at $\hat{c}_1$. Using (55), this implies that

$$|\hat{u}_2^{(1)} - 1| < |\hat{u}_2^{(2)} - 1|.$$

Finally, the collusive trajectory has to be above the competitive trajectory going to e_1, implying

$$|\hat{u}_2^{(2)} - 1| < |\hat{u}_1^{(2)} - 1|.$$

Combining these inequalities with the fact that $\hat{c}_1$ is an indifference point in the competitive situation, we arrive at

$$|\hat{u}_2^{(1)} - 1| > |\hat{u}_1^{(1)} - 1| = |\hat{u}_1^{(2)} - 1| > |\hat{u}_2^{(2)} - 1| > |\hat{u}_2^{(1)} - 1|.$$

But this is a contradiction. The proof in situation that the threshold is a repeller is similar and will be omitted.

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