REGRESSION-BASED ALGORITHMS FOR LIFE INSURANCE CONTRACTS WITH SURRENDER GUARANTEES

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Abstract. We present a general framework for pricing life insurance contracts embedding a surrender option. The model allows for several sources of risk, such as uncertainty in mortality, interest rates and other financial factors. We describe and compare two numerical schemes based on the Least Squares Monte Carlo method, emphasizing underlying modeling assumptions and computational issues.

Keywords: insurance contracts, surrender option, stochastic mortality, American contingent claims, Least Squares Monte Carlo method.

1. Introduction

Life insurance contracts usually offer policyholders a variety of options and can therefore be regarded as *options packages*, as suggested by Smith (1982) and Walden (1985). A first distinction can be made between *American* and *European* options. In the first case policyholders have the right to alter the contract before its natural termination, from which the expression 'early exercise'. Since in practice exercise can occur at regular time intervals only, these options are essentially of *Bermudan* type. In the second case exercise is admitted only at contract expiration, i.e. at the minimum between a fixed maturity and the insured's death time. Since the expiration date is in this case random, these options are called *Titanic* by Milevsky and Posner (2001). For large enough portfolios of insureds with independent and identically distributed lifetimes pooling arguments can be applied to reduce the pricing of a Titanic option to the situation of a portfolio of European options with different maturities. This is not the case when early resolution of the contract is allowed.

The most common American option that has attracted the interest of researchers in recent years is undoubtedly the *surrender option*. It gives the policyholder the right to terminate the contract before death or maturity and receive a cash amount called *surrender value*. It is therefore a *knock-out* American put option written on the residual contract, with exercise price given by the surrender value. The knock-out event is represented by the insured's death, as the option can be exercised only

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upon survival. As opposed to the Titanic option case, the analysis of surrender options cannot be reduced to a portfolio of American options with different maturities, even when pooling arguments can be used to neutralize non-systematic mortality risk. Indeed, the surrender decision involves a comparison, at any date of possible exercise and only if the insured is still alive, between the surrender value and the value of the residual contract, which simultaneously depends on financial and demographic factors. As a result, the option cannot be properly priced unless both demographic and financial risk factors are analyzed in an integrated fashion.

The valuation of surrender options is of interest to insurers because early withdrawals reduce assets under management and may generate imbalances in the mortality risk profile of remaining insureds. Any withdrawal risk is clearly increased by the presence of minimum guarantees on surrender values, while the provision of guarantees on survival and death benefits can make the contract more or less valuable to policyholders at any given time, depending on market conditions. The long term nature of insurance policies, as well as the range of financial exposures that modern insurance products entail, make the valuation of surrender options quite challenging. The literature has usually focused on purely financial contracts and on simplifying assumptions on the dynamics and the number of risk factors. Early examples are represented by the seminal papers Albizzati and Geman (1994) and Grosen and Jørgensen (1997, 2000), which paved the way for a number of following studies. Due to the high dimensionality of the problem (multiple exercise dates, several risk factors), the analysis of surrender options is usually carried out for stylized situations. When moving to more realistic models, contributions become fewer. For example, the introduction of mortality is present in a limited number of papers, which we group according to the pricing methodology employed: binomial or multinomial trees; partial differential equations and free boundary problems; Least Squares Monte Carlo (LSMC) simulation. In the first group of papers, for example, Bacinello (2003a,b) considers participating policies, while Vannucci (2003) and Bacinello (2005) consider equity-linked contracts embedding a surrender option (see Section 3.1 for a detailed description of these contracts). In both cases mortality as well as interest rates are deterministic, the single premium is computed by backward induction, and the annual premium is implicitly defined by a recursive procedure. The papers by Moore and Young (2005) and Shen and Xu (2005) are representative of the second approach, where the surrender option problem is cast in terms of a free boundary problem requiring the numerical solution of a partial differential equation. While the approach is very helpful to understand the mechanics of rational exercise in stylized situations, it becomes intractable for more realistic situations. As the number of risk factors increases, the numerical burden becomes unsurmountable and a number of simplifying assumptions are required. The third approach, which is at the heart of the present paper, includes the works by Andreatta and Corradin (2003), Baione, De Angelis and Fortunati (2006) and Bacinello, Biffis and Millossovich (2007). The first two contributions seem to combine the LSMC approach (proposed by Carrière, 1996; Longstaff and Schwartz, 2001; Tsitsiklis and Van Roy, 2001, for the valuation of purely financial American claims) with the approach proposed by Bacinello (2003a,b) to introduce mortality risk in the valuation of surrender options for participating contracts. Since it is not completely clear how mortality plays its final role in the valuation procedure, Bacinello,

Biffis and Millossovich (2007) introduced an alternative algorithm to employ the LSMC approach in the context of mortality uncertainty.

The aim of the present paper is threefold. First, we apply the LSMC approach to a general pricing framework, showing how to integrate the analysis of rational exercise and death in the early resolution of the contract. There are clearly alternative simulation methods for pricing American options (see Glasserman, 2004, and references therein), but they are not very effective in the presence of multiple state variables and several exercise dates. Since our objective is to cope with a range of features of real-world markets, such as stochastic volatility, jumps in asset prices or randomness in the force of mortality, we focus on the powerful LSMC approach.

Second, we refine and extend the procedure of Bacinello, Biffis and Millossovich (2007) by describing two algorithms applicable to setups of different degrees of generality. The first one essentially relies on the requirement that the random time of death cannot be foretold given knowledge of asset prices and demographic risk factors. The second one imposes additional structure and requires the insured's time of death to coincide with the first jump of a conditionally Poisson process. We show that the methods of Andreatta and Corradin (2003) and others must rely on the (conditionally) Poisson assumption, while the one introduced in Bacinello, Biffis and Millossovich (2007) applies more generally. More interestingly, we show that even when the (conditionally) Poisson assumption is desirable, application of the first algorithm is more efficient.

Finally, we encompass in a common framework the case of irrational withdrawal decisions and differences in policyholders' and insurers' risk preferences. Empirical evidence (e.g., FSA, 2007) shows that surrenders can be affected by factors such as distribution channels, bad publicity, etc., thus requiring some modifications in our basic framework. In addition, while insurance companies operate at portfolio level and can exploit diversification effects, policyholders are faced with their own time of death only, when making rational surrender decisions. The relevance of this angle is somewhat limited when adopting a prudent perspective (for pricing and reserving purposes), as discussed in Bacinello (2005) and in Section 4.5 of this paper, but can be important for realistic valuations.

The paper is structured as follows. In Section 2 we describe a general model for different life insurance contracts. We begin by introducing an arbitrage-free financial market where frictionless trading occurs continuously over time. We then introduce mortality uncertainty and extend the market to include life insurance contracts. In Section 3, we introduce early exercise features and describe the valuation of insurance securities embedding surrender options, providing in turn some examples of typical guarantees and options available on the market. In Section 4 we briefly describe the LSMC approach, emphasizing the key approximations involved and reviewing some convergence results. We then provide two LSMC algorithms that exploit different features of the pricing framework. We discuss computational implications and show how the first algorithm not only applies more generally, but also outperforms the second in terms of computational speed and approximation errors. Furthermore, we adapt both algorithms to the case of asymmetric insurer's and policyholder's risk preferences. In Section 5 we offer numerical examples for unit-linked or participating endowments with different types of minimum guarantees attached to survival, death and surrender benefits. Section 6 provides some concluding remarks.

2. Valuation Framework

2.1. **Financial Market.** We take as given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{P}$ is the real-world or physical probability measure and $\mathbb{F} \doteq (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions of right continuity and $\mathbb{P}$-completeness and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We will add more structure to $\mathbb{F}$ when considering more specific examples. Available for trade are $d+1$ securities with semimartingale price processes $S^0, S^1, \ldots, S^d$. Trading takes place continuously over time and without incurring transaction costs. Security S^0 represents the balance of a money market account formalizing the investment of cash at a continuously compounded locally risk-free rate r. We set $S^0_\cdot \doteq \exp(\int_0^\cdot r_s \mathrm{d}s)$ and assume that r is predictable and such that $E(\int_0^t |r_u| \mathrm{d}u) < \infty$ for all $t \geq 0$. The remaining d securities represent risky assets with cumulated dividends processes $D^1, \ldots, D^d$ of bounded variation, adapted, and null at time 0. For $i = 1, \ldots, d$, we let S^i_t denote the time-t ex-dividend price of security i, meaning that the security pays the lump sum dividend $\Delta D^i_t = D^i_t - D^i_{t-}$ and is then available for trade at price S^i_t.

The absence of arbitrage is essentially equivalent to the existence of a probability measure $\mathbb{Q}^*$ equivalent to $\mathbb{P}$ under which the gain from holding a security is a $\mathbb{Q}^*$-martingale after deflation by the money market account (e.g., Duffie, 2001). If[1] $G^i_t \doteq S^i_t + \int_0^t \mathrm{d}D^i_u$ denotes the gain from holding security i from time 0 to time t, then by no-arbitrage the following risk-neutral valuation formula applies

$$\frac{S^i_t}{S^0_t} = E^{\mathbb{Q}^*}\left[\frac{S^i_v}{S^0_v} + \int_t^v \frac{\mathrm{d}D^i_u}{S^0_u}\middle| \mathcal{F}_t\right] \tag{2.1}$$

for all $v \geq t \geq 0$ and each $i = 1, \ldots, d$, where we assume that the price of any security is 0 at a given time t if no dividends are paid thereafter. Deflation could be performed by using any security with strictly positive price process: while this may be preferable in some applications (e.g., Bacinello and Ortu, 1994; Biffis and Millossovich, 2006), the use of S^0 helps economic intuition when extending the market to include insurance securities (see Section 2.3).

2.2. **Demographic Uncertainty.** Let us consider an individual aged x years at a reference time 0. We denote by τ her random residual lifetime and denote by $\mathbb{H}$ the filtration generated by the process $N_t \doteq 1_{\tau\leq t}$, which equals zero as long as the individual is alive and jumps to one at death. We enlarge the filtration $\mathbb{F}$ of previous section to include $\mathbb{H}$ and set $\mathbb{G} \doteq \mathbb{F} \vee \mathbb{H}$, with $\mathcal{G}_0$ trivial. We see that τ is a $\mathbb{G}$-stopping time, since at each time t the information carried by $\mathcal{G}_t$ allows us to tell whether death has occurred or not by t. We then consider an enlargement $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$ of the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}^*)$ defined in the previous section and assume that the arbitrage free financial market introduced above preserves its structure after the enlargement. We also take $\mathbb{G}$ strictly larger than $\mathbb{F}$, meaning that knowledge of $\mathbb{F}$ (e.g., observation of security prices) does not yield knowledge of occurrence of τ. The following results hold in the present setup and will be used later: every $\mathbb{G}$-predictable process Y coincides with an $\mathbb{F}$-predictable process $\overline{Y}$ on $\{\tau > t\}$; moreover, every $\mathbb{G}$-stopping time θ coincides with an $\mathbb{F}$-stopping time $\bar{\theta}$ on $\{\tau > t\}$ (see Protter, 2004, p. 370).

[1] Here and in the sequel, $\int_a^b$ denotes integration over $(a, b]$.

The setup can be specialized by making additional assumptions. For instance, τ could be defined by

$$\tau \doteq \inf\left\{t \middle| \int_0^t \mu_u \mathrm{d}u > \xi\right\}, \tag{2.2}$$

with μ a nonnegative $\mathbb{F}$-predictable nonnegative process and ξ a unit exponential random variable independent of $\mathcal{F}_\infty$. This construction is equivalent to the so-called conditionally Poisson (equivalently, Cox or doubly stochastic) setup, meaning that, under $\mathbb{Q}$ and conditional on $\mathcal{F}_\infty$, the random time τ is the first jump time of a Poisson inhomogeneous process with intensity $(\mu_t)_{t\geq 0}$. One of the appealing consequences of (2.2) is that the arbitrage free financial market introduced in Section 2.1 automatically preserves its structure after the enlargement (see Bielecki and Rutkowski, 2001). More prosaically, the setup is appealing because survival and death probabilities resemble stochastic counterparts of formulas traditionally employed by actuaries and demographers (e.g., Biffis, 2005).

2.3. **Insurance Contracts.** It is now natural to extend the financial market by working on the probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$ and introducing a life insurance contract issued to the individual described above. We denote by V the price process of a life policy and by D its cumulated dividend. As opposed to Section 2.1, D is now adapted to the larger filtration $\mathbb{G}$, which means it may depend on the individual's time of death. We let $D = D^d + D^s$, where D^d and D^s represent cumulated benefits contingent on death and survival, defined as

$$\begin{aligned} D^d_t &= \int_0^t B^d_{u-} \mathrm{d}N_u = B^d_{\tau-} 1_{\tau\leq t} \\ D^s_t &= \int_0^t (1-N_u)\mathrm{d}B^s_u = B^s_{\tau-} 1_{\tau\leq t} + B^s_t 1_{\tau>t} \end{aligned} \tag{2.3}$$

for some $\mathbb{F}$-adapted processes B^d and B^s, with B^s of bounded variation. While B^d_u represents a lump sum payable in case of death at time u, B^s_u denotes cumulated benefits paid in case of survival up to time u. The above formulation includes several types of life insurance policies such as endowments, pure endowments, (deferred) annuities, term and whole life assurances: we just need to suitably specify the quantities B^d and B^s. For example, we may represent:

- ⋆ a single benefit $b \in \mathcal{F}_T$ payable in case of survival at a fixed maturity $T > 0$, by setting $B^s_t = 1_{t\geq T} b$;
- ⋆ an $\mathbb{F}$-adapted benefit stream $(b_t)_{t\geq 0}$ payable from time T until death (deferred annuity), by setting $\mathrm{d}B^s_t = 1_{t\geq T} b_t \mathrm{d}t$;
- ⋆ a discrete sequence of lump sum payments $b_1, b_2, \ldots$ at times $T_1, T_2, \ldots$, by letting $B^s_t = \sum_i b_i 1_{t\geq T_i}$ with $b_i \in \mathcal{F}_{T_i}$ for each i.

We could also include in B^s possible annual premiums paid by the policyholder (see Biffis, 2005, for additional examples).

Under no-arbitrage, we can rewrite (2.1) for the extended market as

$$V_t = S^0_t \; E^{\mathbb{Q}}\left[\frac{V_v}{S^0_v} + \int_t^v \frac{\mathrm{d}D_u}{S^0_u} \middle| \mathcal{G}_t\right] \tag{2.4}$$

for all $v \geq t$. For convenience, we let $\overline{V}$ denote the $\mathbb{F}$-predictable *pre-death price* of the security, in the sense that $V_v = 1_{\tau>v}\overline{V}_v$. When τ is defined by (2.2), we obtain

(e.g., Duffie, Schroder and Skiadas, 1996)

$$V_t = 1_{\tau>t}\widehat{S}^0_t \; E^{\mathbb{Q}}\left[\frac{\overline{V}_v}{\widehat{S}^0_v} + \int_t^v \frac{B^d_u}{\widehat{S}^0_u}\,\mu_u \mathrm{d}u + \int_t^v \frac{\mathrm{d}B^s_u}{\widehat{S}^0_u}\middle|\mathcal{F}_t\right], \tag{2.5}$$

where $\widehat{S}^0_\cdot = \exp(\int_0^\cdot (r_s+\mu_s)\mathrm{d}s)$ represents a 'mortality risk-adjusted money market account'. Expression (2.5) shows that the standard risk-neutral machinery passes over quite simply to the mortality-contingent setting, provided we consider fictitious securities paying a fictitious instantaneous dividend $B^d_u\mu_u\mathrm{d}u+\mathrm{d}B^s_u$ under a fictitious short rate $r+\mu$. Indeed, by (2.5) the pre-death gain from holding the security, $\overline{G}_t \doteq \overline{V}_t + \int_0^t (B^d_u\mu_u\mathrm{d}u + \mathrm{d}B^s_u)$, is an $\mathbb{F}$-martingale under $\mathbb{Q}$, after deflation by $\widehat{S}^0$. Formula (2.4) is more general than (2.5), as no doubly stochastic assumption on τ is required (actually, it extends well beyond the information structure introduced in Section 2.2). This has also computational consequences, since any simulation algorithm to compute the expectation in (2.4) will make explicit reference to $\mathbb{G}$, as opposed to (2.5), where only $\mathbb{F}$ is explicitly considered.

3. Insurance contracts embedding early exercise features

We now embed a surrender option in the life contract introduced above, allowing the policyholder to withdraw from the contract at any time before maturity receiving a lump sum called *surrender value*.[2] Let B^w (w for 'withdrawal') be an $\mathbb{F}$-adapted process: we say that the policyholder receives a surrender benefit B^w_θ if she surrenders the contract at time θ. We take θ to be a $\mathbb{G}$-stopping time and call it an exercise policy. If the option is exercised at θ, the cumulated dividend process generated by the contract is $D^\theta + D^w(\theta)$, where D^θ represents the cumulated dividends (2.3) stopped at θ (i.e., $D^\theta_t \doteq D_{t\wedge\theta}$ for all t) and $D^w(\theta)$ is given by

$$D^w_t(\theta) = \int_0^t (1-N_u)B^w_u \mathrm{d}L_u(\theta) = B^w_\theta 1_{\theta\le t, \theta<\tau} \tag{3.1}$$

with $L_u(\theta) \doteq 1_{\theta\le u}$. The case of no surrender is covered by setting $\theta=\tau$, which yields $D^w(\theta)=0$. Some policies may allow surrender only within a time-window $[\underline{t}, T]$, for example in order to recoup the expenses incurred to issue the contract. If that is the case, we just need to set $B^w_u = 0$ for $u\in[0,\underline{t})$.

Let $V^w(\theta)$ denote the price process of the contract when the surrender option is exercised at time θ. By (2.4) we have, on $\{\theta>t\}$:

$$V^w_t(\theta) = S^0_t E^{\mathbb{Q}}\left[\int_t^\theta \frac{\mathrm{d}(D_u + D^w_u(\theta))}{S^0_u}\middle|\mathcal{G}_t\right]. \tag{3.2}$$

Denoting by $\mathcal{T}_{\mathbb{G}}$ the set of finite valued $\mathbb{G}$-stopping times, the price of our contract is then given by the solution of the optimal stopping problem

$$V^{w*}_0 = \sup_{\theta\in\mathcal{T}_{\mathbb{G}}} V^w_0(\theta) = \sup_{\theta\in\mathcal{T}_{\mathbb{G}},\,\theta\le\tau} V^w_0(\theta), \tag{3.3}$$

where we have used the fact that $V^w_0(\theta) = V^w_0(\theta\wedge\tau)$ by (3.1)-(3.2). A solution to (3.3) is called a rational exercise policy, in the sense that it maximizes the initial arbitrage-free value of the resulting claim. While this can be justified by replication arguments when markets are complete, the case of incomplete markets is more

[2]In practice, surrender is usually allowed if the contract provides benefits both in case of death and survival, to avoid antiselection.

delicate (e.g., Duffie, 2001). We do not expand on this issue here and simply employ (3.3) under a given risk-neutral measure $\mathbb{Q}$.

We can now take advantage of the structure of $\mathbb{G}$ to replace θ with an $\mathbb{F}$-stopping time $\bar{\theta}$ coinciding with θ up to time τ and rewrite expression (3.2) on $\{\bar{\theta} > t\}$ as follows (e.g., Bielecki and Rutkowski, 2001)

$$V_t^w(\theta) = \frac{1_{\tau>t} S_t^0}{\mathbb{Q}(\tau > t|\mathcal{F}_t)} E^{\mathbb{Q}} \left[\int_t^{\bar{\theta}} 1_{\tau>t} \frac{\mathrm{d}(D_u + D_u^w(\bar{\theta}))}{S_u^0} \middle| \mathcal{F}_t \right]. \tag{3.4}$$

We can therefore rewrite the optimization problem (3.3) as

$$V_0^{w*} = \sup_{\theta \in \mathcal{T}_{\mathbb{G}}} V_0^w(\theta) = \sup_{\bar{\theta} \in \mathcal{T}_{\mathbb{F}}} V_0^w(\bar{\theta}) = \sup_{\bar{\theta} \in \mathcal{T}_{\mathbb{F}},\, \bar{\theta} \leq \tau} V_0^w(\bar{\theta}), \tag{3.5}$$

with $\mathcal{T}_{\mathbb{F}}$ the set of finite-valued $\mathbb{F}$-stopping times. When the stopping time τ is doubly stochastic, formula (3.4) can be finally rewritten on $\{\tau \wedge \bar{\theta} > t\}$ as

$$V_t^w(\theta) = \widehat{S}_t^0 E^{\mathbb{Q}} \left[\int_t^{\bar{\theta}} \frac{\mathrm{d}(\widehat{D}_u + \widehat{D}_u^w(\bar{\theta}))}{\widehat{S}_u^0} \middle| \mathcal{F}_t \right], \tag{3.6}$$

with $\mathrm{d}\widehat{D}_u = \mathrm{d}B_u^s + B_u^d \mu_u \mathrm{d}u$ and $\mathrm{d}\widehat{D}_u^w(\bar{\theta}) = B_u^w \mathrm{d}L_u(\bar{\theta})$. The value of the contract is then obtained by taking the supremum of the last expression over $\mathbb{F}$-stopping times, as in (3.5).

3.1. **Examples of surrender guarantees.** Surrender guarantees are provided by a number of insurance contracts. A few relevant examples for single-premium policies are provided below. We refer the reader to Bacinello (2005) for considerations on surrender penalties, which are not discussed here.

3.1.1. *Equity-linked endowments.* Endowments provide a lump sum payment at maturity T in case of survival, or a payment at the time of death if it occurs before T. In the equity-linked case, payment amounts depend on the market value of a reference fund and usually embed minimum guarantees. A typical example is represented by benefits of the following form:

$$B_t^s = F_T^s 1_{t \geq T} \quad B_t^d = F_t^d 1_{t<T} \quad B_t^w = F_t^w 1_{t<T}, \tag{3.7}$$

with *terminal guarantees* of the type

$$F_t^e = F_0 \max\left(\frac{S_t}{S_0}, \exp(\kappa_e t) \right), \tag{3.8}$$

or with *cliquet guarantees*

$$F_t^e = F_0 \prod_{u=1}^{\lfloor t \rfloor} \max\left(1 + \eta \left(\frac{S_u}{S_{u-1}} - 1 \right), \exp(\kappa_e) \right), \tag{3.9}$$

where $e = s, d, w$ and $\lfloor t \rfloor$ denotes the integer part of t. In the above expressions F_0 is the initial value of the reference fund, S is the price process of each fund unit, κ_e is the minimum interest guaranteed on different causes of exit (survival at maturity, death or withdrawal). In (3.8), benefits depend only on the current value of the units, while in (3.9) path dependency is introduced by the periodic resettlements of the reference fund. As common in practice, relation (3.9) implicitly assumes yearly resettlements, but of course a different frequency could be considered. With particular reference to the cliquet guarantee, a crucial role is played by the rate

η identifying the portion of performance recognized to the policyholder. Typically one has $\eta \in (0,1]$: when $\eta = 1$, the whole cost of the guarantee is paid at inception; when $\eta < 1$ instead, the cost is (partially) recovered by the insurer when returns on the reference portfolio exceed the minimum guarantee.

3.1.2. *Participating endowments.* In participating contracts the insurer shares profits with policyholders in different ways. As an example, we consider here the 'reversionary bonus' method, according to which shared profits are credited as bonuses to the policy reserves at the end of each year. The crediting mechanism generates a regular adjustment of benefits (including surrender values) that typically allows for some minimum guarantee. As for equity-linked endowments, benefits could still be expressed by (3.7), with

$$F_t^e = F_0 \prod_{u=1}^{\lfloor t \rfloor} \max\left(1 + \eta\left(\frac{S_u}{S_{u-1}} - 1\right), \exp(\kappa_e)\right) \tag{3.10}$$

in the case of unsmoothed profit-sharing, or by

$$F_t^e = F_0 \prod_{u=1}^{\lfloor t \rfloor} \max\left(1 + \frac{\eta}{u \wedge y} \sum_{j=1}^{u \wedge y} \left(\frac{S_{u-j+1}}{S_{u-j}} - 1\right), \exp(\kappa_e)\right)$$

in the case of *smoothed profit sharing*, where smoothing occurs over y years and, as before, $e = s, d, w$. The first case is formally identical to the case of equity-linked endowments with cliquet guarantees, but now F_0 denotes the policy value at inception and S a sort of representative index that synthesizes the performance of the insurer's portfolio. In the smoothed profit sharing case, the credited bonuses depend not only on the most recent performance of the insurer's portfolio, but also on the average performance over the last y years of contract. We note that in both the unsmoothed and smoothed case, absence of arbitrage imposes constraints on the choice of parameters η and $\{\kappa_e\}_{e=s,d,w}$ if the initial policy value F_0 coincides with the single premium (see Bacinello, 2001).

3.1.3. *Whole life assurances.* Whole life assurances provide lump sum payments upon death of the insured. Benefits can in this case be expressed as

$$B_t^s = 0 \quad B_t^d = F_t^d \quad B_t^w = F_t^w, \tag{3.11}$$

with $\{F_t^e\}_{e=d,w}$ defined as in the previous examples, depending on whether the contract is equity-linked or with-profit. Note that (3.11) can be obtained as a particular case of (3.7) by setting $T = \infty$ and $F_T^s = 0$.

3.1.4. *Deferred annuities with death benefit.* Deferred annuities provide payments upon survival at dates $T_0, T_1, \ldots$, with T_0 denoting the end of the deferment period. When combined with a term assurance with maturity T_0 (equivalently, when a refund guarantee is provided on the premiums contributed during the deferment period), the contract allows for surrender before time T_0. Examples are obtained by setting

$$B_t^s = \sum_i b_i 1_{t \geq T_i} \quad B_t^d = F_t^d 1_{t<T_0} \quad B_t^w = F_t^w 1_{t<T_0}, \tag{3.12}$$

with

$$b_i = \frac{F_{T_0}^s}{a_{T_0}} \max\left(\frac{S_{T_i}}{S_{T_0}}, \exp(\kappa_a(T_i - T_0))\right),$$

where $\{F_t^e\}_{e=s,d,w}$ is defined as in the case of equity-linked endowments with terminal or cliquet guarantees, $\kappa_a \geq 0$ is a minimum interest rate guaranteed after the deferment period, and a_{T_0} is a rate of conversion into a life annuity. If the rate of conversion is not fixed at inception, it depends on market (and demographic) conditions that will prevail at time T_0: this is an example where stochastic mortality models play a crucial role (see Biffis and Millossovich, 2006).

To conclude, an example of contract with both a death benefit and survival benefits payable until death is represented by a single premium annuity combined with a whole life assurance with death benefits decreasing over time (representing partial refund of the single premium in case of early death). Similarly, a common case is that of an annuity-certain of (say) k installments, combined with a deferred annuity with payments starting in the $(k+1)$-th period: if death occurs when only $h < k$ installments have been paid, beneficiaries receive a lump sum representing the present value of the residual $(k-h)$ payments.

4. Implementation of the LSMC Approach

The LSMC approach relies on the combination of Monte Carlo simulation and Least Squares regression in an environment where randomness is generated by a multidimensional Markov process X. The method involves three main approximations. A first approximation is represented by discretization of the time dimension, which has the effect of replacing the American claim with a Bermudan claim. Without loss of generality, we consider a unitary discretization step (where the time unit of measure is arbitrary) and set $\mathbb{T} = \{0, 1, \ldots, n\}$ for suitable integer n. The original optimal stopping problem (3.3) is then replaced by its discretized version along the time grid $\mathbb{T}$,

$$\sup_{\theta \in \mathcal{T}_{\mathbb{G},\mathbb{T}}} E^{\mathbb{Q}}\left[g_\theta\right], \tag{4.1}$$

with $\mathcal{T}_{\mathbb{G},\mathbb{T}}$ the family of $\mathbb{T}$-valued $\mathbb{G}$-stopping times and g the square-integrable $\mathbb{G}$-adapted process of discounted future dividends originated by the contract. Using the notation of Section 2, we have $g_t = \int_0^t \mathrm{d}G_u/S_u^0$.

As common when dealing with American options (see Duffie, 2001, for example), one can introduce the Snell envelope of g and apply the dynamic programming principle to develop a backward procedure involving a comparison, at each time step, between the option payoff and the reward from not exercising (continuation value). It is characteristic of the LSMC method to look at such procedure in terms of optimal stopping times. An optimal policy $\theta^* = \theta_0^*$ is computed according to the backward algorithm

$$\begin{cases} \theta_n^* = n \\ \theta_j^* = j 1_{g_j > U_j} + \theta_{j+1}^* 1_{g_j \leq U_j} \qquad \text{for } j = n-1, \ldots, 0, \end{cases}$$

where $U_j = E^{\mathbb{Q}}[g_{\theta_{j+1}^*} | \mathcal{G}_j]$. Since we work in a Markovian environment, we have $U_j = E^{\mathbb{Q}}[g_{\theta_{j+1}^*} | X_j]$ and can write $U_j = u(j, X_j)$ for some Borel functions $u(j, \cdot)$, $j \in \mathbb{T}$. A second approximation is now introduced by replacing each $u(j, X_j)$ with the orthogonal projection from $L^2(\Omega)$ onto the vector space generated by a finite set of functions $\{e_1(X), \ldots, e_H(X)\}$ taken from a suitable basis. For fixed H and each j, we denote by $\widetilde{u}(j, X_j)$ such projection and set $\widetilde{u}(j, X_j) = \beta_j \cdot e(X_j)$, with e the vector-valued function $(e_1, \ldots, e_H)'$ and β_n a suitable coefficient vector $(\beta_j^1, \ldots, \beta_j^H)'$.

A third approximation is then introduced by simulating the state variable process X over the time grid $\mathbb{T}$ (or over a finer grid), in order to employ least squares regression to compute the projections $(\widetilde{u}(j, X_j))_{j\in\mathbb{T}}$. If M is the number of simulations, and X_j^m and g_j^m (with $m = 1, \dots, M$) denote the simulated values of X_j and g_j in the m-th simulation, we set $\widetilde{u}(j, X_j^m) = \beta_j^* \cdot e(X_j^m)$ with β_j^* the least square estimator obtained by solving

$$\beta_j^* = \arg \min_{\beta_j \in \mathbb{R}^H} \sum_{m=1}^{M} \left(g^m_{\theta^*_{j+1}} - \beta_j \cdot e(X_j^m) \right)^2 . \tag{4.2}$$

Clément, Lamberton and Protter (2002) show that, as H goes to infinity, the value function of problem (4.1) with U_j replaced by $\widetilde{u}(j, X_j)$ approaches the value function of the original problem. They also prove almost sure convergence of the Monte Carlo procedure, for fixed H, and provide the asymptotic error distribution. The joint effect of H and M on convergence is less clear: some non-asymptotic results can be found in Gobet, Lemor and Warin (2005), while interesting numerical investigations are reported in Moreno and Navas (2003) and Stentoft (2004).

We now describe the implementation of the general procedure with reference to our valuation setup. We propose two algorithms based on the increasingly restrictive assumptions introduced in Section 2.2.

4.1. **Algorithm 1.** With reference to the generic m-th iteration ($m = 1, \dots, M$), we introduce the following notation:

- τ^m: simulated time of death.
- X_t^m: simulated vector of state variables at time $t \in \mathbb{T}_+ = \{1, \dots, n\}$.
- P_t^m: simulated payoff from the contract at time $t \in \mathbb{T}_+$. Depending on the contract considered, it may involve: a death benefit $B^{d,m}_{\tau^m}$ when $t = \tau^m$; a surrender benefit $B_t^{w,m}$ payable upon survival and surrender at time t; a survival benefit, with a slight abuse of notation denoted by $B_t^{s,m}$, which represents the simulated value of
$$\int_{t-1}^{t} \frac{S_t^0}{S_u^0} 1_{\{\tau > u\}} \mathrm{d}B_u^s.$$
- $v_{t,u}^m$: discount factor for the period $[t, u]$, with $t, u \in \mathbb{T}$ and $t < u$. More explicitly, we have $v_{t,u}^m = S_t^{0,m}/S_u^{0,m}$, with $(S_t^{0,m})_{t\in\mathbb{T}}$ the simulated money market account in the m-th iteration.

The valuation algorithm requires the execution of the following steps:

STEP 0: (*simulation*) Simulate M paths of X over the time grid $\mathbb{T}$, with $n = \lceil \max_m \tau^m \rceil$, where $\lceil t \rceil$ denotes the least integer greater than or equal to t.[3]

STEP 1: (*initialization*) Set $\theta^{*,m} = \lceil \tau^m \rceil$ and $P^m_{\theta^{*,m}} = B^{d,m}_{\theta^{*,m}}$ for $m = 1, \dots, M$.

STEP 2: (*backward iteration*) For $j = n-1, n-2, \dots, 1$:

(1) (*continuation values*) Set $I_j = \{1 \le m \le M : \tau^m > j\}$ and, for $m \in I_j$, set $C_j^m = \sum_{h=j+1}^{\theta^{*,m}} P_h^m \, v_{j,h}^m$.

(2) (*regression*) Regress the continuation values $(C_j^m)_{m\in I_j}$ against $(e(X_j^m))_{m\in I_j}$ to obtain $\widetilde{C}_j^m = \beta_j^* \cdot e(X_j^m)$ for $m \in I_j$. If $B_j^{w,m} > \widetilde{C}_j^m$ then set $\theta^{*,m} = j$ and $P_j^m = B_j^{w,m} + B_j^{s,m}$, otherwise set $P_j^m = B_j^{s,m}$.

[3]Clearly, it may be convenient to simulate all processes over a finer grid.

STEP 3: (*initial value*) Compute the single premium of the contract

$$V_0^* = \frac{1}{M} \sum_{m=1}^{M} \sum_{j=1}^{\theta^{*,m}} P_j^m \, v_{0,j}^m.$$

4.2. **Algorithm 2.** As described in Section 2.3, we can price contracts by using (2.5) and (3.6) rather than (2.4) and (3.2). Even if the underlying assumption is that τ is doubly stochastic in both cases, expression (2.5) reduces valuation to computation of conditional expectations with respect to the smaller filtration $\mathbb{F}$. The algorithm needs to be changed accordingly. We first fix n large enough to represent the maximum residual lifetime (in the grid unit of measure) of our reference insured. The backward procedure will be started from n in every simulation, thus making unnecessary the simulated values τ^m. The notation for the other simulated quantities is as before, except for the following:

- $B_t^{s,m}$: simulated value of

$$\int_{t-1}^{t} \frac{\widehat{S}_t^0}{\widehat{S}_u^0} (\mathrm{d}B_u^s + B_u^d \mu_u \mathrm{d}u). \tag{4.3}$$

- $\widehat{v}_{t,u}^m$: risk-ajusted discount factor for the period $[t,u]$, with $t,u \in \mathbb{T}$ and $t < u$. More explicitly, we have $\widehat{v}_{t,u}^m = \widehat{S}_t^{0,m} / \widehat{S}_u^{0,m}$, with $(\widehat{S}_t^{0,m})_{t \in \mathbb{T}}$ denoting the simulated mortality risk-adjusted money market account introduced in Section 2.3.

The valuation algorithm is modified as follows:

STEP 0: (*simulation*) Simulate M paths of X over the time grid $\mathbb{T}$.
STEP 1: (*initialization*) Set $P_n^m = B_n^{s,m}$, $\bar{\theta}^{*,m} = n$, for $m = 1, \dots, M$.
STEP 2: (*backward iteration*) For $j = n-1, n-2, \dots, 1$:
(1) (*continuation values*) For $m = 1, \dots, M$ let $C_j^m = \sum_{h=j+1}^{\bar{\theta}^{*,m}} P_h^m \, \widehat{v}_{j,h}^m$.
(2) (*regression*) Regress the continuation values $(C_j^m)_{m=1,\dots,M}$ against $(e(X_j^m))_{m=1,\dots,M}$ to obtain $\widetilde{C}_j^m = \beta_j^* \cdot e(X_j^m)$ for every simulated path. If $B_j^{w,m} > \widetilde{C}_j^m$ set $\bar{\theta}^{*,m} = j$ and $P_j^m = B_j^{w,m} + B_j^{s,m}$, otherwise set $P_j^m = B_j^{s,m}$.
STEP 3: (*initial value*) Compute the single premium of the contract

$$V_0^* = \frac{1}{M} \sum_{m=1}^{M} \sum_{j=1}^{\bar{\theta}^{*,m}} P_j^m \, \widehat{v}_{0,j}^m.$$

We note that in the above algorithm we have considered the $\mathbb{F}$-stopping time $\bar{\theta}$ coinciding with θ up to τ (see Section 2.2).

4.3. **Computing the option price.** The simplest way to obtain the time-0 value of the surrender option is to compute, along with V_0^*, the initial value of the European version of the contract V_0, and then find the option price by subtracting V_0 from V_0^*. If V_0 cannot be expressed in closed form, it can be computed by executing the previous algorithms with STEP 2 replaced by

STEP 2: (*backward iteration*) For $j = n-1, n-2, \dots, 1$ set $I_j = \{1 \le m \le M : \tau^m > j\}$ and, for $m \in I_j$, let $P_j^m = B_j^{s,m}$

in Algorithm 1, and by

STEP 2: (*backward iteration*) For $j = n-1, n-2, \ldots, 1$ and for $m = 1, \ldots, M$ let $P_j^m = B_j^{s,m}$

in Algorithm 2.

4.4. **Comparison.** The key difference between the two algorithms is that the second one avoids the simulation of the time of death at the cost of simulating all relevant risk factors up to an arbitrary maximum time n. As a result, depending on the contracts considered, either algorithm may prove more efficient. For example, the first method may be preferable for whole life or annuity contracts, where maturities coincide with death times. On the other hand, the advantage may reduce considerably for contracts with fixed maturity and low terminal age, where only few simulated paths are likely to be shortened by death occurring before maturity. Still, in the numerical examples of Section 5 we find that the first algorithm outperforms the second in computational speed by at least 15%. This may be also due to the following reasons. In the first algorithm computations can be simplified by considering in the regression step, at each exercise date, only trajectories in which the insured is alive, since the continuation value in the remaining trajectories is zero and needs not be estimated. Put another way, the death indicator process N can be excluded from the set of state variables. This may be relevant depending on the relative importance of 'no survival paths' with respect to 'survival paths' at exercise dates. Finally, the second algorithm requires an additional approximation in computing the integral in (4.3). For contracts providing death benefits this may result in additional approximation errors (note that the integrand needs to be evaluated numerically as well), or additional computational effort to contain these errors. On the other hand, the first algorithm is unaffected by this drawback as long as the contract provides lump sum benefits only.

4.5. **Realistic valuations.** So far, we have fixed *an* equivalent martingale measure $\mathbb{Q}$ reflecting the insurer's preferences towards risk. Since the insurance company operates at portfolio level, $\mathbb{Q}$ may reflect diversification effects that the single policyholder cannot enjoy. More generally, our representative policyholder may have different risk preferences and decide whether or not to terminate the contract on the basis of a different probability measure, say $\mathbb{Q}^\sharp$. For tractability, we assume that $\mathbb{Q}^\sharp$ is an equivalent martingale measure preserving the structure of Section 2.2 (precise conditions are given in Biffis, Denuit and Devolder, 2005). We can then let $\theta^{\sharp,*}$ denote the stopping time solving problem (4.1) under $\mathbb{Q}^\sharp$, i.e.

$$\sup_{\theta \in \mathcal{T}_{\mathbb{G},\mathbb{T}}} E^{\mathbb{Q}^\sharp}\left[g_\theta\right].$$

Given the policyholder's optimal policy $\theta^{\sharp,*}$, the insurance company can then value any stream of cashflows dependent on $\theta^{\sharp,*}$ by applying risk-neutral valuation under the 'pricing' measure $\mathbb{Q}$. All life policies considered so far could then be valued by computing $E^{\mathbb{Q}}\left[g_{\theta^{\sharp,*}}\right]$. Now, the inequality

$$V_0^{\sharp,*} \doteq E^{\mathbb{Q}}\left[g_{\theta^{\sharp,*}}\right] \leq \sup_{\theta \in \mathcal{T}_{\mathbb{G},\mathbb{T}}} E^{\mathbb{Q}}\left[g_\theta\right]$$

shows that direct valuation of the contract under $\mathbb{Q}$ would be prudential from the insurer's viewpoint, as discussed in Bacinello (2005). On the other hand, the approach described here could produce a value for the American contract, $V_0^{\sharp,*}$, lower than the one of the corresponding European contract, V_0, again computed under $\mathbb{Q}$.

In this case the surrender option would represent an asset (rather than a liability) for the insurer. Although regulatory and accounting rules may not allow its recognition on the balance sheet, a proper quantification of $V_0^{\sharp,*} - V_0$ is certainly useful for realistic valuations. To do so, we can easily adapt the previous algorithms to the current situation. Since Algorithm 1 and Algorithm 2 can be modified along the same lines, we describe only the first one as an example. The notation is as in the original algorithm for realizations under $\mathbb{Q}$, while we add the superscript $\sharp$ to the corresponding realizations under $\mathbb{Q}^\sharp$.

Algorithm $1^\sharp$.

STEP 0: Construct M paths of X (under $\mathbb{Q}$) and, correspondingly, M paths of $X^\sharp$ (under $\mathbb{Q}^\sharp$), over the time grid $\mathbb{T}$, with $n = \lceil \max_m \tau^m \rceil \vee \lceil \max_m \tau^{\sharp,m} \rceil$.

STEP 1: (*initialization*) For $m = 1, \dots, M$ set $\theta^{\sharp,m} = \lceil \tau^{\sharp,m} \rceil$, $P^{\sharp,m}_{\theta^{\sharp,m}} = B^{\sharp,d,m}_{\theta^{\sharp,m}}$, $\theta^m = \lceil \tau^m \rceil$, $P^m_{\theta^m} = B^{d,m}_{\theta^m}$.

STEP 2: (*backward iteration*) For $j = n-1, n-2, \dots, 1$:

(1) (*continuation values*) Set $I^\sharp_j = \{1 \le m \le M : \ \tau^{\sharp,m} > j\}$ and, for $m \in I^\sharp_j$, let $C^{\sharp,m}_j = \sum_{h=j+1}^{\theta^{\sharp,m}} P^{\sharp,m}_h \, v^{\sharp,m}_{j,h}$; set $I_j = \{1 \le m \le M : \ \tau^m > j\}$ and, for $m \in I_j$, let $P^m_j = B^{s,m}_j$.

(2) (*regression*) Regress the continuation values $(C^{\sharp,m}_j)_{m \in I^\sharp_j}$ against $(e(X^{\sharp,m}_j))_{m \in I^\sharp_j}$ to obtain $\widetilde{C}^{\sharp,m}_j = \beta^\sharp_j \cdot e(X^{\sharp,m}_j)$ for $m \in I^\sharp_j$. If $B^{\sharp,w,m}_j > \widetilde{C}^{\sharp,m}_j$ then set $\theta^{\sharp,m} = j$ and $P^{\sharp,m}_j = B^{\sharp,w,m}_j + B^{\sharp,s,m}_j$, otherwise set $P^{\sharp,m}_j = B^{\sharp,s,m}_j$.

STEP 3: (*initial value*) For $m = 1, \dots, M$, if $\theta^{\sharp,m} < \tau^{\sharp,m} \wedge \tau^m$ then set $P^m_{\theta^{\sharp,m}} = P^m_{\theta^{\sharp,m}} + B^{w,m}_{\theta^{\sharp,m}}$, otherwise set $\theta^{\sharp,m} = \theta^m$. Finally, compute the single premium of the contract as

$$V_0^{\sharp,*} = \frac{1}{M} \sum_{m=1}^{M} \sum_{j=1}^{\theta^{\sharp,m}} P^m_j \, v^m_{0,j}.$$

We note that construction of the paths in Step 0 can be more conveniently performed by simulating all relevant processes (including the Radon-Nykodim density $\mathrm{d}\mathbb{Q}^\sharp/\mathrm{d}\mathbb{Q}|_{\mathcal{G}_t}$) under one measure and then obtaining the corresponding realizations under the alternative measure by applying the Girsanov-Meyer Theorem (Protter, 2004, p. 132). Finally, the value of the European contract V_0 can be computed exactly as described in Section 4.3, i.e. by using the insurance company's equivalent martingale measure $\mathbb{Q}$.

5. Applications

In this section we apply the valuation framework to equity-linked or participating endowments of the type described in Section 3.1.1. As an example, we consider a model of financial and demographic risk factors that is a special case of those described by Biffis and Millossovich (2006): it includes random interest rates and mortality as well as jumps and stochastic volatility in the reference fund dynamics. The case of equity-linked endowments with terminal guarantees was examined in Bacinello, Biffis and Millossovich (2007) by using a particular version of Algorithm 1. However, the only benchmark available was offered by the use of affine transform

methods for the European contract. Here, we validate their results in the American case by using Algorithm 2.

5.1. **Financial and demographic risk factors.** Consider a state variable process $X = (r, Y, K, \mu, N)$ taking values in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \{0, 1\}$. The first three components represent financial risk factors: r is the short rate, Y the log-price process of a reference fund $S = e^Y$, K the square of the instantaneous non-jump volatility of S. The process μ represents the force of mortality of the reference policyholder, while N is the doubly stochastic death indicator. The processes r, Y and K evolve under $\mathbb{Q}$ according to the following stochastic differential equations:

$$\begin{aligned}
\mathrm{d}r_t =& \zeta_r(\delta_r - r_t)\mathrm{d}t + \sigma_r\sqrt{r_t}\mathrm{d}Z_t^r \\
\mathrm{d}Y_t =& \left(r_t - \frac{1}{2}K_t - \lambda_Y\mu_Y\right)\mathrm{d}t + \sqrt{K_t}\Big(\rho_{SK}\mathrm{d}Z_t^K + \rho_{Sr}\mathrm{d}Z_t^r + \\
&+ \sqrt{1-\rho_{SK}^2-\rho_{Sr}^2}\mathrm{d}Z_t^S\Big) + \mathrm{d}J_t^Y \\
\mathrm{d}K_t =& \zeta_K(\delta_K - K_t)\mathrm{d}t + \sigma_K\sqrt{K_t}\mathrm{d}Z_t^K,
\end{aligned}$$

where the processes Z^r, Z^S and Z^K are independent standard Brownian motions, J^Y is a compound Poisson process with jump arrival rate $\lambda_Y > 0$ and lognormally distributed jumps with mean μ_Y and standard deviation $\sigma_Y > 0$. J^Y is assumed to be independent of the vector (Z^r, Z^S, Z^K). The parameters $\zeta_r, \zeta_K, \delta_r, \delta_K, \sigma_r$ and σ_K are all strictly positive, while the correlation coefficients ρ_{SK} and ρ_{Sr} satisfy $\rho_{SK}^2 + \rho_{Sr}^2 \leq 1$. For the intensity of mortality, we take the left continuous version of the process

$$\mathrm{d}\mu_t = \zeta_\mu(m(t) - \mu_t)\mathrm{d}t + \sigma_\mu\sqrt{\mu_t}\mathrm{d}Z_t^\mu + \mathrm{d}J_t^\mu, \tag{5.1}$$

where Z^μ is a standard Brownian motion and J^μ is a compound Poisson process independent of Z^μ, with jump arrival rate $\lambda_\mu \geq 0$ and exponential jumps of mean $\gamma_\mu > 0$. The couple (Z^μ, J^μ) is assumed to be independent of (Z^r, Z^S, Z^K, J^Y). The parameters ζ_μ, σ_μ and the function m are strictly positive. It is easily seen that the vector-valued process X is affine, so that some prices can be computed analytically (see Biffis and Millossovich, 2006) to cross check the performance of the LSMC approach in the absence of early exercise features. We combined analytical results with our Monte Carlo algorithms to choose a satisfactory number of basis functions for the examples below.

5.2. **Numerical examples.** We consider the single premium unit-linked or participating endowments described in Section 3.1.1. The reference insured is a male aged $x = 40$ at time 0. The contract has maturity $T = 15$ years and provides either terminal (see (3.8)) or cliquet guarantees (see (3.9) or (3.10)) on survival, death and surrender benefits. We apply both algorithms described in Section 3 with polynomial basis functions of order 3. The first approximation introduced is the discretization of the time dimension, which has the effect of replacing the American claim with a Bermudan claim: we call Backward Discretization Step (BDS) the length in years of each time interval arising from this discretization. To simulate the state variable process X, we employ a time grid finer than $\mathbb{T}$ and call Forward Discretization Step (FDS) the length in years of each time interval in the finer grid. The parameters used for our simulations are reported in Table 1. With regard to mortality dynamics, we note that the function m in (5.1) is obtained by fitting

a Weibull intensity, given by $m(t) = c_1^{-c_2} c_2 (x+t)^{c_2-1}$ (with $c_1 > 0, c_2 > 1$), to the survival probabilities implied by table SIM2001, commonly used in the Italian endowments market.

< Table 1 about here >

Results for the case of terminal guarantees are reported in Table 2, those for cliquet guarantees in Table 3. Column A_1 (A_2) reports the time-0 values of American contracts obtained by using Algorithm 1 (Algorithm 2), while column E_1 (E_2) reports the values of the underlying European contract computed by using the modified Algorithm 1 (Algorithm 2), as illustrated in Section 4.3. Surrender options values, O_1 and O_2, are obtained by subtracting columns E_1 and E_2 from columns A_1 and A_2. The corresponding standard errors are reported in parenthesis. We ran 19000 simulations with 140 different seeds for terminal guarantees, 30000 simulations with 100 different seeds for cliquet guarantees. In both cases, we used antithetic variables to reduce variance. We found that Algorithm 1 is faster than Algorithm 2 by 15% to 20% in the two cases: this is remarkable, since the contract considered represents an example for which we expect small differences in performance (see Section 4.4).

< Table 2 about here >

< Table 3 about here >

In the case of terminal guarantees we have considered different values for minimum rates guaranteed upon death or survival ($\kappa \doteq \kappa_d = \kappa_s$) and surrender ($\kappa_w$). Of course, the price of the European contract does not depend on κ_w. In the case of cliquet guarantees we have set $\kappa_e = \kappa$ for $e = d, s, w$ and let κ change with the participation coefficient η. From Table 2 one can see that the results obtained with the first and second algorithm are very close. As expected, the value of the European contract is increasing with the minimum interest rate guaranteed upon death or survival, κ, while the value of the American contract is increasing with both κ and the minimum interest rate guaranteed upon surrender, κ_w (see Figure 1). The value of the surrender option, instead, increases with κ_w and decreases with κ. The option becomes worthless for κ high with respect to κ_w, because exiting the contract is then less attractive.

< Figure 1 about here >

In the case of cliquet guarantees, the state variables vector X must be augmented to include the value of the guarantee F^e (see expression (3.9)), since at each time t the value of F_t^e cannot be inferred from X_t. From Table 3 we can see that surrender option values decrease with both the participation rate η and the minimum guaranteed interest κ. They become negligible as soon as η is 60% or greater. This is due to the fact that provision of high participation rates together with minimum guarantees induce the policyholder to stay in the contract. At the opposite, the surrender option is significantly valuable when both η and κ are low. The value of the American contract for different values of the couple (η, κ) is plotted in Figure 2.

We remark that when the value of the European contract is below $S_0 = 100$, neither the price of the minimum guarantee nor S_0 are covered by the premium paid at inception: they are both financed by the annual returns in excess of κ retained by the insurer. This is why the surrender option value becomes very significant and drives the price of the American contract close to S_0.

< Figure 2 about here >

6. Conclusions

In this paper we have presented a general framework for pricing life insurance contracts embedding surrender options. We have introduced two numerical schemes based on the Least Squares Monte Carlo method and described their flexibility in the context of jump-diffusion models for financial and demographic risk factors. As a practical example, we have implemented the schemes for pricing equity-linked and participating endowments providing terminal as well as cliquet guarantees at death, survival and surrender. Future research includes the use of the LSMC approach for risk-management purposes, with a detailed analysis of the implications of market incompleteness on surrender options prices and capital reuirements.

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7. Tables

	r	K	S	μ
BDS $= 0.50$	$r_0 = 0.05$	$K_0 = 0.04$	$S_0 = 100.00$	$\mu_0 = m(0)$
FDS $= 0.01$	$\zeta_r = 0.60$	$\zeta_K = 1.50$	$\rho_{SK} = -0.70$	$\zeta_\mu = 0.50$
	$\delta_r = 0.05$	$\delta_K = 0.04$	$\rho_{Sr} = 0.00$	$\sigma_\mu = 0.03$
	$\sigma_r = 0.03$	$\sigma_K = 0.40$	$\lambda_Y = 0.50$	$\lambda_\mu = 0.10$
			$\mu_Y = 0.00$	$\gamma_\mu = 0.01$
			$\sigma_Y = 0.07$	$c_1 = 83.70$
				$c_2 = 8.30$
				$x = 40$

Table 1. Parameters used in the simulation.

κ	κ_w	E1 (s.e.)	A1 (s.e.)	O1	E2 (s.e.)	A2 (s.e.)	O2
0%	0%	107.185 (0.047)	113.556 (0.031)	6.372	107.224 (0.046)	113.577 (0.030)	6.353
	2%		117.223 (0.031)	10.038		117.237 (0.031)	10.013
	4%		123.687 (0.031)	16.503		123.696 (0.031)	16.472
	6%		137.130 (0.031)	29.945		137.262 (0.030)	30.038
2%	0%	112.675 (0.045)	115.381 (0.033)	2.706	112.698 (0.044)	115.324 (0.033)	2.626
	2%		117.551 (0.031)	4.876		117.524 (0.031)	4.825
	4%		123.727 (0.031)	11.052		123.738 (0.031)	11.040
	6%		137.327 (0.030)	24.652		137.404 (0.030)	24.706
4%	0%	122.901 (0.041)	123.087 (0.033)	0.186	122.904 (0.040)	122.904 (0.033)	0.000
	2%		123.291 (0.033)	0.390		123.130 (0.033)	0.226
	4%		124.507 (0.032)	1.606		124.418 (0.032)	1.514
	6%		137.710 (0.030)	14.809		137.630 (0.030)	14.726

Table 2. Terminal guarantee.

κ	η	E1 (s.e.)	A1 (s.e.)	O1	E2 (s.e.)	A2 (s.e.)	O2
0%	20%	65.938 (0.004)	97.205 (0.002)	31.267	66.239 (0.004)	97.216 (0.001)	30.977
	40%	89.945 (0.009)	99.496 (0.003)	9.551	90.222 (0.009)	99.651 (0.004)	9.429
	60%	121.902 (0.019)	122.067 (0.018)	0.165	122.136 (0.019)	122.251 (0.019)	0.115
2%	20%	69.915 (0.004)	97.598 (0.001)	27.683	70.213 (0.004)	97.609 (0.001)	27.396
	40%	94.938 (0.009)	100.080 (0.004)	5.142	95.210 (0.009)	100.585 (0.005)	5.376
	60%	128.411 (0.019)	128.470 (0.018)	0.059	128.636 (0.019)	128.698 (0.019)	0.062
4%	20%	75.343 (0.004)	98.096 (0.001)	22.753	75.635 (0.004)	98.109 (0.001)	22.474
	40%	100.982 (0.009)	101.959 (0.007)	0.977	101.246 (0.009)	103.107 (0.008)	1.861
	60%	135.952 (0.019)	135.952 (0.019)	0.000	136.165 (0.019)	136.198 (0.019)	0.033

Table 3. Cliquet guarantee.

8. Figures

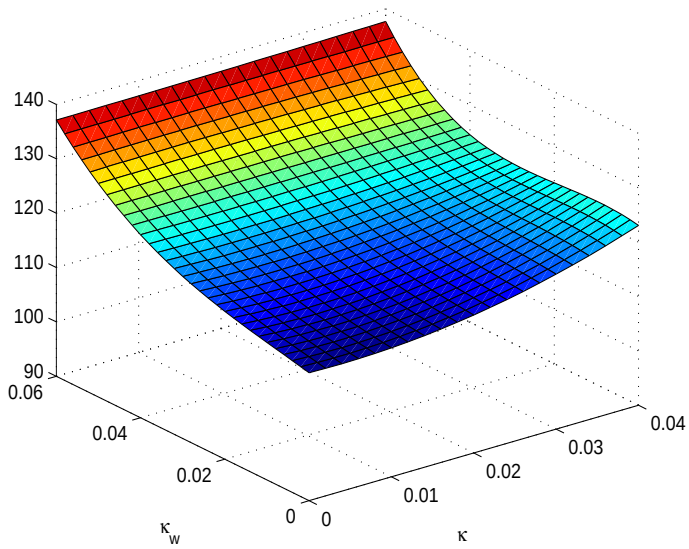

Figure 1. Terminal guarantees, Algorithm 1: value of the American contract.

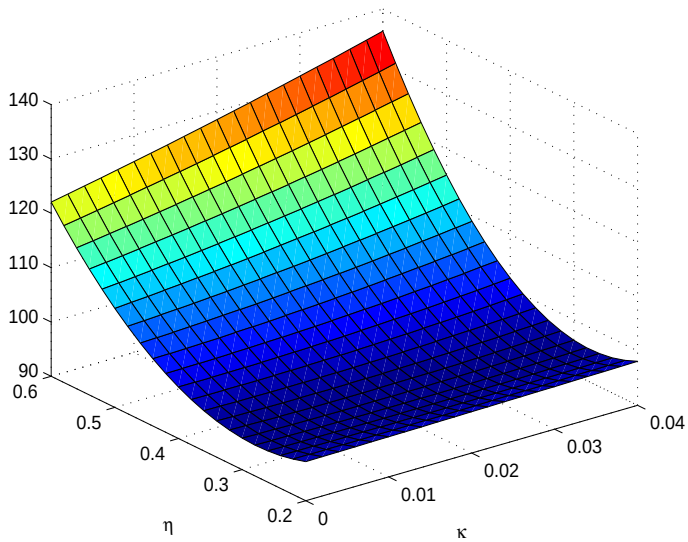

Figure 2. Cliquet guarantees, Algorithm 1: value of the American contract.