Common Pool Resource Game and Coalition Formation

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Abstract

In this paper, we examine the question of which coalition structures are formed in cooperative games with common pool resource games. We introduce a stability concept for a coalition structure called a *sequentially stable* coalition structure by extending the concept of an equilibrium binding agreement (EBA) due to Ray and Vohra (1997). In an EBA, coalitions can only break up into smaller sizes of coalitions, but not merge into larger sizes of coalitions. On the other hand, in our concept of sequential stability, both breaking up and merging are allowed for coalitions. We also use a "step-by-step" approach to describe negotiation steps concretely by restricting how coalition structures can change: when one coalition structure is changed to another one, either (i) only one merging of two separate coalitions into a coalition occurs, or (ii) only one breaking up of a coalition into two separate coalitions happens. As an application of our stability notion, we show that the coalition structure consisting of only the grand coalition structure can be sequentially stable in common pool resource games.

1 Common Pool Resource Games

Let the game of an economy with a common pool resource, in short CPR game, be described by a set of players $N := \{1, 2, \ldots, n\}$. For any player $i \in N$, let $x_i \geq 0$ represent the amount of labour input of i. Clearly, the overall amount of labour is given by $\sum_{j\in N} x_j$. The technology that determines the amount of product is considered to be a joint production function of the overall amount of labour $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0, \lim_{x\to\infty} f'(x) = 0, f'(x) > 0$ and $f''(x) < 0$ for $x > 0$. The distribution of the product is supposed to be proportional to the amount of labour expended by players. In other words, the amount of the product assigned to player i is given by $\frac{x_i}{\sum_{j\in N} x_j} \cdot f(\sum_{j\in N} x_j)$. The price of the product is normalized to be one unit of money and let q be a cost of labor per unit, and we suppose $0 < q < f'(0)$.

Then individual i's income is denoted by

$$m_i(x_1, x_2, ..., x_n) = \frac{x_i}{x_N} f(x_N) - qx_i.$$

Coalition S's total income is denoted by

$$m_S \equiv \sum_{i\in S} m_i = \frac{x_S}{x_N} f(x_N) - qx_S,$$

where $x_S \equiv \sum_{i\in S} x_i$. We consider a game where each coalition is a player. It chooses its total labor input and its payoff is given by the sum of its members' incomes. Naturally we can define a Nash equilibrium of that game.

Definition 1. $(x^*_{S_1}, x^*_{S_2}, ..., x^*_{S_k})$ is an *equilibrium under* $\mathcal{P}$ iff

$$m_{S_j}(x^*_{S_j}, x^*_{S_{-j}}) \geq m_{S_j}(x_{S_j}, x^*_{S_{-j}}), \quad \forall j, \quad \forall x_{S_j} \in \mathbb{R}_+.$$

Proposition 1 (Funaki and Yamato(1999)). *For any* $\mathcal{P} = \{S_1, S_2, ..., S_k\}$*, there exists unique equilibrium* $(x^*_{S_1}, x^*_{S_2}, ..., x^*_{S_n})$ *under* $\mathcal{P}$ *which satisfies*

$$f'(x^*_N) + \frac{(k-1)f(x^*_N)}{x^*_N} = kq, \quad x^*_{S_j} = \frac{x^*_N}{k} \ \ \forall j, \quad x^*_{S_j} > 0 \ \ \forall j,$$

where $x^*_N = \sum_{j=1}^k x^*_{S_j}$.

Given a coalition structure $\mathcal{P} = \{S_1, ..., S_k\}$, let $(x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P})$ be a unique equilibrium under $\mathcal{P}$ and let $x^*_N(\mathcal{P}) = \sum_{i=1}^k x^*_{S_i}(\mathcal{P})$. Moreover, let $m^*_{S_i}(\mathcal{P}) = m_{S_i}(x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))$ be the equilibrium income of coalition S_i for $i = 1, .., k$ and therefore $m^*_N(\mathcal{P}) = \sum_{i=1}^k m_{S_i}(x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P})$.

Proposition 2 (Funaki and Yamato(1999)). *For two coalition structures* $\mathcal{P}_k = \{S_1, S_2, ..., S_k\}$ *and* $\mathcal{P}'_{k'} = \{S'_1, S'_2, ..., S'_{k'}\}$ *with* $k < k'$*,*

$$x^*_N(\mathcal{P}_k) < x^*_N(\mathcal{P}'_{k'}), \quad \frac{m^*_N(\mathcal{P}_k)}{n} > \frac{m^*_N(\mathcal{P}'_{k'})}{n},$$

$$S \in \mathcal{P}_k \ \text{ and } \ S \in \mathcal{P}'_{k'} \implies m^*_S(\mathcal{P}_k) > m^*_S(\mathcal{P}'_{k'}).$$

2 The Core of a Game in Partition Function Form

A *n-person game in Partition Function Form induced by CPR game* is defined by a triple $(N, \Pi(N), v)$. Here N is a plyer set, $\Pi(N)$ is the set of all coalition structures $\mathcal{P}$ of N, and v is a partition function that associates with each admissible coalition S in $\mathcal{P}$ a real number $v(S|\mathcal{P})$. In our model, the value $v(S_i|\mathcal{P})$ under a coalition structure $\mathcal{P}$ is given by

$$v(S_i|\mathcal{P}) \equiv \sum_{j \in S_i} m_j(x^*_{S_1}(\mathcal{P}), x^*_{S_2}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P})),$$

where $(x^*_{S_1}(\mathcal{P}), x^*_{S_2}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))$ is an equilibrium vector under $\mathcal{P}$.

Given a coalition structure $\mathcal{P} = (S_1, S_2, ..., S_k)$, a payoff vector $z \in \mathcal{R}^N$ is *feasible under* $\mathcal{P}$ iff $\sum_{j \in S_i} z_j \leq v(S_i|\mathcal{P}) \ \ \forall i$

For a CPR game, the set of feasible payoff vectors $\mathcal{F}(\mathcal{P})$ is given by $\{z \in \mathbb{R}^n | \sum_{i \in S_j} z_i \leq m^*_{S_j}(\mathcal{P}) \ \ \forall S_j \in \mathcal{P}\}$ in this section.

We have another definition of the set of feasible payoff vectors as follows: $\mathcal{F}(\mathcal{P}) = \{z \in \mathbb{R}^n | z_i = \frac{m^*_{S_j}(\mathcal{P})}{|S_j|} \ \ \forall i \in S_j, \forall S_j \in \mathcal{P}\}$. It is natural to consider this set because of the symmetry of players. We focus on this type of the set in sections 3, 4 and 5.

We denote a set of feasible payoff vectors by $\mathcal{I}$, where

$$\mathcal{I} \equiv \{z \in \mathcal{R}^N | \exists \mathcal{P} \text{ s.t. } \sum_{j \in S} z_j \leq v(S|\mathcal{P}) \ \ \forall S \in \mathcal{P}\}.$$

We introduce a domination relation for two payoff vectors in $\mathcal{I}$. Consider two payoff vectors z, z' in $\mathcal{I}$ and a coalition S in N. We say *z dominates z' via S* and denote $z \ dom_S \ z'$ if the following two conditions hold:

$$(1) \ \forall \mathcal{P} \ni S, \ \sum_{j \in S} z_j \leq v(S|\mathcal{P}), \quad (2) \ z_j > z'_j \ \ \forall j \in S.$$

In addtion, we simply say *z dominates z'* if there exists $S \subseteq N$ such that $z \ dom_S \ z'$, and denote $z \ dom \ z'$.

The set of feasible payoff vector that is not dominated by any other vectors in $\mathcal{I}$ is called the *Core $\mathcal{C}$ of a partition function form game.* Formally it is given by

$$\mathcal{C} \equiv \{z \in \mathcal{I} | \nexists z' \in \mathcal{I} \text{ such that } z' \ dom \ z\}$$

On the other hand, the *Core C of TU Game* (N, v) is defined by

$$C \equiv \{z \in \mathcal{R}^N | \sum_{j \in N} z_j = v(N), \ \sum_{j \in S} z_j \geq v(S) \ \ \forall S \subset N\}.$$

Then in our model we consider *TU game Core* $C(v)$ given by

$$C(v) = \{z \in \mathcal{I} | \sum_{j \in N} z_j = v_{\mathcal{P}^N}(N), \ \sum_{j \in S} z_j \geq v(S) \ \ \forall S \subset N\}.$$

Proposition 3 (Funaki and Yamato(1999)). If $v_{\mathcal{P}^N}(N) > \sum_{i=1}^k v_{\mathcal{P}}(S_i) \quad \forall \mathcal{P} \neq \mathcal{P}^N$, and $v_{\min}(S) = \min_{\mathcal{P} \ni S} v_{\mathcal{P}}(S)$, then $\mathcal{C} = C(v_{\min})$.

To determine v_{min}(S), we provide the following lemma.

Lemma 1. For any two coalition structures $\mathcal{P}_k = \{S_1, S_2, ..., S_k\}$ and $\mathcal{P}'_{k'} = \{S'_1, S'_2, ..., S'_{k'}\}$ with $k < k'$ and $S \in \mathcal{P}_k \cap \mathcal{P}'_{k'}$, $v(S|\mathcal{P}_k) > v(S|\mathcal{P}'_{k'})$. Moreover

$$v_{\min}(S) = \frac{1}{n - |S| + 1}(f(x^*_N(\bar{\mathcal{P}}^S)) - qx^*_N(\bar{\mathcal{P}}^S)),$$

where $v(S|\bar{\mathcal{P}}^S) = \min_{\mathcal{P} \ni S} v(S|\mathcal{P})$.

The following theorem shows that the core in the case is non empty.

Theorem 1 (Funaki and Yamato(1999)). For TU game $(N v_{min})$,

$$C(v_{\min}) \neq \emptyset \text{ and } (\frac{v_{\min}(N)}{n}, \frac{v_{\min}(N)}{n}, ..., \frac{v_{\min}(N)}{n}) \in C(v_{\min})$$

This result is under pessimistic expectations regarding coalition formation among outsiders, we would therfore expect fihsermen to make an agreement dividing the total incomeequally among all players. In this situation, the tragedy of the commons could be avoided.

Next we consider the opposit case, where fishermen's expectations about outsiders'coalition formation are optimistic. We modify the definition of the domination relkation *dom* and introduce a new domination relation $\underline{dom}$ as follows: Given S in N, and $z, z' \in \mathcal{I}$, $z\ \underline{dom}_S\ z'$ $\Longleftrightarrow$

$$(1)\ \exists \mathcal{P} \ni S, \text{ s.t. } \sum_{j \in S} z_j \leq v_{\mathcal{P}}(S), \quad (2)\ z_j > z'_j \quad \forall j \in S.$$

We also define for $z, z' \in \mathcal{I}$,$z\ \underline{dom}\ z' \iff \exists S \subseteq N$ such that $z\ \underline{dom}_S\ z'$.

We can also define the *Core* $\underline{\mathcal{C}}$ *of a partition function form game under optimistic expectations.* It is gien by

$$\underline{\mathcal{C}} \equiv \{z \in \mathcal{I} | \nexists z' \in \mathcal{I} \text{ such that } z'\ \underline{dom}\ z\}$$

We have the following equivalence of the two cores.

Propostion 4 (Funaki and Yamato(1999)). If$v_{\mathcal{P}^N}(N) \geq \sum_{i=1}^k v_{\mathcal{P}}(S_i)$ and $v_{\max}(S) = \max_{\mathcal{P} \ni S} v_{\mathcal{P}}(S)$, then $\underline{\mathcal{C}} = C(v_{\max})$.

We now show the core given by $\underline{\mathcal{C}} = C(v_{\max})$ is empty, which is opposite to the result obtained under pessimistic expectations.

Theorem 2 (Funaki and Yamato(1999)). Let $n \geq 4$. Then for TU-game (N, v_{max}), $C(v_{max}) = \emptyset$.

This theorem states that the core of any game with 4 or more players is empty if every colation has the optimistic expectations. For 3-person gamesit is possible to find both of the existance aand the non-existence of the core of (N, v_{max}) and other intermidiate cases.

If we define w^r for r $(1 \leq r \leq n-1)$ by

$$w^r(S) = \max_{\mathcal{P} \ni S} v(S|\mathcal{P}) \quad if \ \ |S| = r, \quad w^r(S) = \min_{\mathcal{P} \ni S} v(S|\mathcal{P}) \quad otherwise,$$

then, for any (N, w^r) with $n \geq 4r \geq 4$, $C(w^r) = \emptyset$.

Let $f(x) = \sqrt{x}$. Then, $v(S|\mathcal{P}) = \frac{2|\mathcal{P}|-1}{4|\mathcal{P}|^3 q} \quad \forall S \in \mathcal{P}$. This implies $v_{\min}(N) = v_{\max}(N) = \frac{1}{4q}$, $v_{\min}(S) = \frac{2(n-|S|)+1}{4q(n-|S|+1)^3}$, and $v_{\max}(S) = \frac{3}{32q}$. For $n = 3$, $C(v_{\max}) = \emptyset, \quad C(w^1) = \emptyset, \quad C(w^2) \neq \emptyset$.

If we connsider non-symmetric case, we have the following 3-person game. Let $f(x) = \sqrt{x}$. Take a game $(\{1,2,3\}, u)$ such that

$$u(S) = \max_{\mathcal{P} \ni S} v(S|\mathcal{P}) \quad if \quad S = \{1\}, \{2\} \quad u(S) = \min_{\mathcal{P} \ni S} v(S|\mathcal{P}) \quad otherwise.$$

Then $u(S) = \frac{3}{32q}$ if $S = \{1\}, \{2\}, u(S) = \frac{5}{108q}$ if $S = \{3\}, u(S) = \frac{3}{32q}$ if $|S| = 2, u(S) = \frac{1}{4q}$ if $S = N$. Moreover,$(\frac{3}{32q}, \frac{3}{32q}, \frac{2}{32q}) \in C(u)$.

3 Creadible Coalition Structure

We assume that given any coalition structure $\mathcal{P} \in \Pi(N)$, the feasible payoff vector under $\mathcal{P}$, $u(\mathcal{P}) = (u_1(\mathcal{P}), u_2(\mathcal{P}), ..., u_n(\mathcal{P})) \in \mathbb{R}^n$, is uniquely determined. The triple $(N, \Pi(N), (u_i)_{i \in N})$ is called a game with externalities.

We give two examples of games with externalities.

Example 1. Games in partition function form. Given a game in partition function form $(N, \Pi(N), v)$, the feasible payoff vector under $\mathcal{P}$ is given by $u_i(\mathcal{P}) = \frac{v(S|\mathcal{P})}{|S|}$ $\forall i \in S, \forall S \in \mathcal{P}$. (See Thrall and Lucas(1963)).

Example 2. Hedonic games.

A hedonic game $(N, \{\succ_i\}_{i \in N})$ is defined by a pair of a set of players N and a binary relation $\succ_i$ on $\{S \subset N | S \ni i\}$ for all $i \in N$, which represents i's preference over coalitions that contain i. Consider i's utility function u_i over $\Pi(N)$ defined from $\succ_i$: For $\mathcal{P}$ and $\mathcal{P}' \in \Pi(N)$, we define

$$u_i(\mathcal{P}) > u_i(\mathcal{P}') \iff S \succ_i T,$$

where $i \in S, S \in \mathcal{P}$ and $i \in T, T \in \mathcal{P}'$. Then $(N, \Pi(N), (u_i)_{i \in N})$ becomes a game with externalities. (See, for example, Dreze and Greenberg (1980), Bogomalnia and Jackson(2002), Diamantoudi and Xue (2003).)

We introduce two special types of coalition structures. $\mathcal{P}^N = \{N\}$ is called a *grand* coalition structure, and $\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}$ is called a *singleton* coalition structure. We also say that $\mathcal{P}'$ is a *finer* coalition structure of $\mathcal{P}$ ($\mathcal{P}$ is a *coarser* coalition structure of $\mathcal{P}'$) if the coalition structure $\mathcal{P}'$ is given by re-dividing the coalition structure $\mathcal{P}$, that is, $\forall S' \in \mathcal{P}', \exists S \in \mathcal{P}$ such that $S' \subseteq S$ and $|\mathcal{P}'| > |\mathcal{P}|$.

We introduce several stability concepts for a set of coalition structures. This is an alternative way to define a core of a game with externalities. For this purpose, we define two simple concepts of dominations between two coalition structures.

Definition 2. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ *dominates* $\mathcal{P}'$ if
(1) $\mathcal{P}$ is a finer coalition structure of $\mathcal{P}'$, and
(2) there exists $T \in \mathcal{P}$ such that $T \notin \mathcal{P}'$ and $u_i(\mathcal{P}) > u_i(\mathcal{P}')$ $\forall i \in T$.

Definition 3. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ *directly dominates* $\mathcal{P}'$ if
(1) $\mathcal{P}$ is a finer coalition structure of $\mathcal{P}'$, and $|\mathcal{P}| = |\mathcal{P}'| + 1$,
(2) there exists $T \in \mathcal{P}$ such that $T \notin \mathcal{P}'$ and $u_i(\mathcal{P}) > u_i(\mathcal{P}')$ $\forall i \in T$.

We define credible coalition structures by these definitions of dominations.

The following definition is a natural extension of Ray(1989)'s credible core for TU games to games with externalities.

Definition 4.
(1) $\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}$ is *credible*; and
(2) for k $(k = n-1, n-2, ..., 1)$, $\mathcal{P}$ with $|\mathcal{P}| = k$ is *credible* if $\mathcal{P}$ is not directly dominated by any coalition structure $\mathcal{P}'$ where $\mathcal{P}'$ is credible and $|\mathcal{P}'| = k+1$.
The set of all credible coalition structures is called the *credible core*, and is denoted by CC.

This is a recursive definition. First, according to (1), $\mathcal{P}^I$ is credible. Second, we can check whether or not each of a coalition structure of $(n-1)$ coalitions is credible by using the fact $\mathcal{P}^I$ is credible. Third, we can check whether or not each of a coalition structure of $(n-2)$ coalitions is credible by using the fact obtained in the second step, and so on.

Ray and Vohra (1997) extends the concept of the credible core in a different manner. Their concept is called an "equilibrium binding agreement (EBA)". We express EBA's in the following simple way using a recursive definition although this expression is different from the original one.

Definition 5.
(1) $\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}$ is an *EBA*; and
(2') for k $(k = n-1, n-2, ..., 1)$, $\mathcal{P}$ with $|\mathcal{P}| = k$ is an *EBA* if $\mathcal{P}$ is not dominated by any coalition structure $\mathcal{P}'$ where $\mathcal{P}'$ is an EBA and $|\mathcal{P}'| > k$.
The set of all EBA coalition structures is called the *EBA core*.

The difference between the credible core and the EBA core is as follows: In a credible coalition structure, only the direct domination is considered, but in an EBA coalition structure, every possible domination is considered.

The following example shows that the concept of a credible coalition structure is different from an EBA. It is more difficult to find an EBA.

Example 3. Consider a symmetric 5-person game in partition function form (N, v), where $N = \{1, 2, 3, 4, 5\}$ and

$$v(N|\mathcal{P}^N) = 50. \quad \text{For any } \mathcal{P}_2 \text{ s.t. } |\mathcal{P}_2| = 2 \text{ and for any } S \in \mathcal{P}_2, v(S|\mathcal{P}_2) = 18.$$

$$\text{For any } \mathcal{P}_3 \text{ s.t. } |\mathcal{P}_3| = 3 \text{ and for any } S \in \mathcal{P}_3, v(S|\mathcal{P}_3) = 8.$$

$$\text{For any } \mathcal{P}_4 \text{ s.t. } |\mathcal{P}_4| = 4 \text{ and for any } S \in \mathcal{P}_4, v(S|\mathcal{P}_4) = 5.$$

$$\text{For any } \{i\} \in \mathcal{P}^I, v(\{i\}|\mathcal{P}^I) = 3.$$

Figure 1 shows that all possible coalition structures and the feasible payoff vectors under each coalition structure. Here the circle shows the coalition and the number in the circle indicates the cardinality of the coalition. The vector under each coalition shows the feasible payoffs $u_i(\mathcal{P}) = \frac{v(S|\mathcal{P})}{|S|}$. In this figure, every coalition structure $\mathcal{P}$ except for $\mathcal{P}^I$ is directly dominated by a coalition structure $\mathcal{P}'$ such that $|\mathcal{P}'| = |\mathcal{P}| + 1$ (see Definition 2). Then it is easy to check that $\mathcal{P}^I = \{1; 1; 1; 1; 1\}, \mathcal{P}^N = \{5\}, \{1; 2; 2\}, \{1; 1; 3\}$[1] are credible coalition structures (see Definition 3). On the other hand, it is not hard to see that $\{1; 4\}$, $\{1; 2; 2\}$, and $\mathcal{P}^I$ are EBA's (see Definition 4).

[1]Here $\{1; 2; 2\}$ means every considerable coalition structure with one singleton and two 2-person coalitions and $\{1; 1; 3\}$ means every considerable coalition structure with two singletons and one 3-person coalition.

In both definitions of the credible core and EBA core, coalitions can only break up into smaller sizes of coalitions, but not merge into larger sizes of coalitions. In particular, this means that the singleton coalition structure consisting only of one-person coalitions belongs to both the credible and EBA cores.

We apply the credibility concepts to this common pool resource game.

Example 4. In a common pool resource game, suppose a production function $f(x)$ is given by $f(x) = \sqrt{x}$.

(1) When $n = 4$, the singleton coalition structure $\mathcal{P}^I$ and all coalition structures consisting of two coalitions are both credible and modified credible.

(2) When $n = 5$, all coalition structures consisting of odd number of coalitions are credible. All coalition structures consisting of odd number of coalitions except for $\{\{i\}, \{j\}, T\}$ $(|T| = 3)$ are modified credible.

(3) When $n = 6$, all coalition structures containing even number of coalitions are credible. Only the grand coalition structure $\mathcal{P}^N$, the singleton coalition structure $\mathcal{P}^I$, $\{Q, R\}$ $(|Q| = |R| = 3)$ and $\{\{i\}, \{j\}, T, U\}$ $(|T| = |U| = 2)$ are modified credible.

The following theorem shows that if the number of players is odd, then coalition structures consisting of odd numbers of coalitions are credible, in particular, the grand coalition structure is credible and a credible core allocation exists. If the number of players is even, then coalition structures consisting of even numbers of coalitions are credible. In this case, although the grand coalition structure is not credible, coalition structures consisting of (*n-1*) -person coalition and one-person coalition are credible. This result is rather simple, but for the modified credibility, it is not easy to get a general result.

Theorem 3. *In a common pool resource game, let $n \geq 4$ and $\mathcal{F}(\mathcal{P}) = \{z \in \mathbb{R}^n | z_i = \frac{m^*_{S_j}(\mathcal{P})}{|S_j|} \ \forall i \in S_j, \forall S_j \in \mathcal{P}\}$. If n is odd, $\mathcal{P}$ consisting of odd number of coalitions is credible, and $CC(\mathcal{P}^N) \neq \emptyset$. If n is even, $\mathcal{P}$ consisting of even number of coalitions is credible, and $CC(\mathcal{P}^{N\backslash i}) \neq \emptyset$. Here $\mathcal{P}^{N\backslash i} = \{N \setminus \{i\}, \{i\}\}$*

Unfortunately we cannot find a general property of a modified credible core of a common pool resorce game.

Example 5. In a common pool resource game, let $f(x) = x^\alpha$, and let $n = 8$. When $\alpha = 0.2, 0.5, 0.8$, the grand coalition structure $\mathcal{P}^N$ is both credible and modified credible. When $\alpha = 0.001, 0.9, 0.995$, the grand coalition structure $\mathcal{P}^N$ is not modified credible but credible.

In both definitions of credible cores and modified credible cores, only breaking up is allowed for coalitions. In the next section, we propose another new concept of stability of payoff configurations such that coalitions can both break up and merge into.

4 Sequentially Stable Coalition Structures

In this section, we give our main stability concept called a "sequentially stable coalition structure". First we give a definition of sequential domination, and after that we give a definition of a sequentially stable coalition structure.

Definition 6. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ *sequentially dominates* $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^T$ such that

(1) $\mathcal{P}_T = \mathcal{P}$ and $\mathcal{P}_0 = \mathcal{P}'$,

(2) for all t $(0 \leq t \leq T-1)$, either $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| + 1$, or $\mathcal{P}_{t+1}$ is a coarser coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| - 1$, and

(3) for all t $(0 \leq t \leq T-1)$, for some $S \in \mathcal{P}_{t+1}$ with $S \notin \mathcal{P}_t$,

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \forall i \in S.$$

We use the following notation for this sequence of coalition structures:

$$\mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \ldots \to \mathcal{P}_T.$$

The condition (3) means that if $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$, for any member i in one of the two divided coalitions S and T such that $S, T \in \mathcal{P}_{t+1}$ and $S \cup T \in \mathcal{P}_t$, his payoff $u_i(\mathcal{P}_t)$ is smaller than his terminal payoff $u_i(\mathcal{P}_T)$; and if $\mathcal{P}_{t+1}$ is a coarser coalition structure of $\mathcal{P}_t$, for any member i in two combined coalitions S and T such that $S, T \in \mathcal{P}_t$ and $S \cup T \in \mathcal{P}_{t+1}$, his payoff $u_i(\mathcal{P}_t)$ is smaller than his terminal payoff $u_i(\mathcal{P}_T)$.

Definition 7. We say that $\mathcal{P}^* \in \Pi(N)$ is a *sequentially stable coalition structure* if for all other coalition structures $\mathcal{P} \neq \mathcal{P}^*$, $\mathcal{P}^*$ sequentially dominates $\mathcal{P}$.

We will compare our domination relation with those of Ray and Vohra(1997) and Diamantoudi and Xue (2002). We have a slightly modified domination due to Ray and Vohra called *RV'-domination* by changing the condition (2) in Definition 6 into the following condition (2').

Definition 8. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ *RV'-dominates* $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^T$ such that

(1) $\mathcal{P}_T = \mathcal{P}$ and $\mathcal{P}_0 = \mathcal{P}'$,

(2') for all t $(0 \leq t \leq T-1)$, $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| + 1$.

(3) for all t $(0 \leq t \leq T-1)$, for some $S \in \mathcal{P}_{t+1}$ with $S \notin \mathcal{P}_t$,

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \forall i \in S.$$

Note in condition (2'), only refinement of coalition structures is allowed. The set of EBA coalition structures is defined by the following set $\mathcal{E}$ of coalition structures such that

(a) for any coalition structure $\mathcal{P}' \notin \mathcal{E}$, there exists $\mathcal{P} \in \mathcal{E}$ such that $\mathcal{P}$ RV-dominates $\mathcal{P}'$, and

(b) for any coalition structure $\mathcal{P}' \in \mathcal{E}$, there is no $\mathcal{P} \in \mathcal{E}$ such that $\mathcal{P}$ RV-dominates $\mathcal{P}'$.

Indeed the set $\mathcal{E}$ is the vNM-stable set via RV'-domination (Diamantoudi and Xue (2002)) because condition (b) corresponds to external stability of the vNM-stable set, and condition (c) corresponds to internal stability of the vNM-stable set. For our notion of sequential domination, the singleton set consisting of any sequentially stable coalition structure is also the vNM-stable set via that domination.

In the exact definition of the domination (RV-domination) of Ray and Vohra(1997), they require an additional condition: $\mathcal{P}$ *RV-dominates* $\mathcal{P}'$ iff $\mathcal{P}$ *RV'-dominates* $\mathcal{P}'$ and for any $\hat{\mathcal{P}}$ on any possible sequence from $\mathcal{P}'$ to $\mathcal{P}$, $\mathcal{P}$ *RV'-dominates* $\hat{\mathcal{P}}$. This is the difference of the two concepts. To RV-dominate other coalition structures is more difficult than to RV'-dominate.

If we change the conditions (2) and (3) in Definition 6 into the following conditions (2") and (3'), then we have a domination relation of Diamantoudi and Xue (2002) called *DX-domination.*

Definition 9. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ *DX-dominates* $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^T$ such that

(1) $\mathcal{P}_T = \mathcal{P}$, $\mathcal{P}_0 = \mathcal{P}'$, and

(2") for all t $(0 \leq t \leq T-1)$, $\mathcal{P}_{t+1}$ and $\mathcal{P}_t \equiv \{S_1, S_2, ..., S_k\}$ satisfy the following condition; there exists a coalition $Q(t) \subseteq N$ such that

(i) $Q(t) = Q_1 \cup Q_2 \cup ... \cup Q_l$, $Q_j \in \mathcal{P}_{t+1}$ $\forall j = 1, 2, .., l$ and Q_js are disjoint,

(ii) $\forall j = 1, 2, ..., k, S_j \cap Q(t) \neq \emptyset \Rightarrow S_j \setminus Q(t) \in \mathcal{P}_{t+1}$,

(iii) $\forall j = 1, 2, ..., k, S_j \cap Q(t) = \emptyset \Rightarrow S_j \in \mathcal{P}_{t+1}$.

(3') for all t $(0 \leq t \leq T-1)$,

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \forall i \in Q(t).$$

The element of the vNM-stable set of the coalition structures using DX-domination is called the set of *Extended EBA (EEBA)* coalition structures. Condition (2") in Definition 9 implies that many possibilities of refining and merging are allowed for coalitions. On the other hand, the way of changing coalitions should be step by step and no jump are allowed in Definition 5 of sequential domination. For two coalition structures $\mathcal{P}$ and $\mathcal{P}'$, $\mathcal{P}$ DX-dominates $\mathcal{P}'$ if $\mathcal{P}$ sequentially dominates $\mathcal{P}'$ because (2) in Definition 6 implies (2") in Definition 8. Hence, sequential stability is a refinement of the notion of EEBA's in the sense that if a coalition structure $\mathcal{P}$ is sequentially stable, then the singleton set consisting only of $\mathcal{P}$ is an EEBA. However, the converse is not true: the singleton set consisting of one coalition structure that is not sequentially stable may be an EEBA. Moreover, there is no logical relation between sequential stability and the notion of EBA's. The following example illustrate these facts:

Example 6. Let us consider the 5-person game in Example 3 once more. We will show that the grand coalition structure $\mathcal{P}^N$ is sequentially stable.

The proof consists of 5 steps.

(1) $\{2;3\} \to \mathcal{P}^N$ because every player in $\{2;3\}$ gets more payoff at $\mathcal{P}^N$.

(2) $\{1;2;2\} \to \{2;3\} \to \mathcal{P}^N$ and $\{1;1;3\} \to \{2;3\} \to \mathcal{P}^N$ because every player in $\{1;2;2\}$ and $\{1;1;3\}$ gets more payoff at $\mathcal{P}^N$ and (1) holds.

(3) $\{1;1;1;2\} \to \{1;2;2\} \to \{2;3\} \to \mathcal{P}^N$ because every player in $\{1;1;1;2\}$ gets more payoff at $\mathcal{P}^N$ and (2) holds.

(4) $\{1;1;1;1;1\} \to \{1;1;1;2\} \to \{1;2;2\} \to \{2;3\} \to \mathcal{P}^N$ because every player in $\{1;1;1;1;1\}$ gets more payoff at $\mathcal{P}^N$ and (3) holds.

(5) $\{1;4\} \to \{1;1;3\} \to \{2;3\} \to \mathcal{P}^N$ because the deviation of one person in the 4-person coalition in $\{1;4\}$ increases his payoff at the final coalition structure $\mathcal{P}^N$ and (2) holds.

These observations imply that the grand coalition structure $\mathcal{P}^N$ is sequentially stable.

Moreover, $\mathcal{P}^N$ is only one sequentially stable coalition structure. The reason is as follows: All the coalition structures except for $\mathcal{P}^N$ and $\{1;4\}$ are not sequentially stable because they are Pareto dominated by $\mathcal{P}^N$ and they cannot sequentially dominate $\mathcal{P}^N$. Next consider

$\{1; 4\}$. Though the coalition structure $\{1; 4\}$ is also Pareto efficient, it is not sequentially stable, because $\{1; 4\}$ cannot sequentially dominate $\{1; 1; 1; 2\}$.

On the other hand, $\{1; 4\}$ DX-dominates $\{1; 1; 1; 2\}$. Hence it is easy to check that $\{1; 4\}$ as well as $\mathcal{P}^N$ is an EEBA. Besides, as mentioned in Example 3, $\{1; 4\}$, $\{1; 2; 2\}$, and $\mathcal{P}^I$ are EBA's.

The properties of EEBA's are examined in Diamantoudi and Xue (2002). In their paper, they give the following proposition:

Definition 10. The coalition structure $\mathcal{P} \in \Pi(N)$ is Pareto efficient if there does not exist $\mathcal{P}' \in \Pi(N)$ such that $u_i(\mathcal{P}') > u_i(\mathcal{P})$ for any $i \in N$.

Proposition 5 (Diamantoudi and Xue (2002)). Let $\mathcal{P}^* \in \Pi(N)$ be Pareto efficient. $\mathcal{P}^*$ is an EEBA if

(a) $u_i(\mathcal{P}^*) > u_i(\mathcal{P}^I)\ \forall i \in N$, and

(b) for all $\mathcal{P} \in \Pi(N)$ such that $\mathcal{P} \neq \mathcal{P}^*$ and $\mathcal{P} \neq \mathcal{P}^I$, there is a coalition $S \in \mathcal{P}$ such that $|S| > 1$ and $u_i(\mathcal{P}^*) > u_i(\mathcal{P})$ for some $i \in S$.

The similar proposition holds for our notion of sequential stability.

Proposition 6. Let $\mathcal{P}^* \in \Pi(N)$ be Pareto efficient. $\mathcal{P}^*$ is sequentially stable if

(a) $\mathcal{P}^*$ sequentially dominates $\mathcal{P}^I$, and

(b) for all $\mathcal{P} \in \Pi$ such that $\mathcal{P} \neq \mathcal{P}^*$ and $\mathcal{P} \neq \mathcal{P}^I$, there is a coalition $S \in \mathcal{P}$ such that $|S| > 1$ and for some member $i \in S$, $u_i(\mathcal{P}^*) > u_i(\mathcal{P})$.

We will give a simple condition for which only the grand coalition structure is sequentially stable in a partition function form game.

Proposition 7. Consider an n-person partition function form game which satisfies

$$\frac{v(N|\mathcal{P}^N)}{n} > \frac{v(S|\mathcal{P})}{|S|} \quad \forall \mathcal{P} \in \Pi(N) \ \ \forall S \in \mathcal{P}.$$

Then only $\mathcal{P}^N$ is sequentially stable.

This result says that in a partition function form game, if the per capita value of the grand coalition is larger than that of any other coalition under any coalition structure, then the set of sequential stable coalition structures consists only of the grand coalition. Moreover, it coincides with the set of EEBA's. However it is different from the EBA core because the EBA core contains $\mathcal{P}^I$ also.

5 Sequentially Stable coalition Structure in CPR Games

We will identify a condition for which the grand coalition structure is sequentially stable. We will give two theorems. The basic idea behind the proofs of the theorems are the same.

First consider a case $n = 2^m$ $(m \geq 2)$. We say $\mathcal{P}$ is *a k-th stage coalition structure* if $|\mathcal{P}| = k$.

Theorem 4. Let $n = 2^m (m \geq 2)$. If $B(k) < 1/2^{k-1}$ for all $k (k = 2, ..., m, m+1)$, the grand coalition structure is sequentially stable.

The proof consists of 4 steps. The outline of the proof is as follows:
(1) The grand coalition structure $\mathcal{P}^N$ sequentially dominate some key coalition structure $\mathcal{P}^*$.
(2) Every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| = |\mathcal{P}^*|$ is sequentially dominated by $\mathcal{P}^N$.
(3)Every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| < |\mathcal{P}^*|$ other than $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$.
(4)Every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| > |\mathcal{P}^*|$ is sequentially dominated by $\mathcal{P}^N$.

Next consider a case that $n = 2^m + l \ \ (m \geq 2, 0 \leq l \leq 2^m - 1)$. This theorem is an extension of Theorem 4.

Theorem 5. Let $n = 2^m + l \ \ (m \geq 2, 0 \leq l \leq 2^m - 1)$. If the inequalities

$$B(2^{m-h-1} + 2) < \frac{2^{h-1}}{n} \quad (h = 1, 2, ..., m-1) \qquad (*)$$

and $B(2) < \frac{2^{m-1}}{n}$ hold, and $B(k)$ is monotonically decreasing in k, then the grand coalition structure is sequentially stable.

Notice that if $\mathcal{P} = \mathcal{P}^N$, then $k = 1$ and $r_1 = n$, so that

$$x_N^*(\mathcal{P}^N) = \alpha(x_N^*(\mathcal{P}^N))^{\alpha-1}/q = \left(\frac{\alpha}{q}\right)^{1/(1-\alpha)}.$$

$$f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N) = (1-\alpha)(x_N^*(\mathcal{P}^N))^{\alpha}.$$

This implies

$$B(k) = \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/[k^2\{f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\}]$$

$$= \frac{1}{k^2}\left(\frac{\alpha - 1 + k}{\alpha k}\right)^{\alpha/(1-\alpha)}.$$

Corollary 1. If $f(x) = x^\alpha$, then for some $\alpha \in (0,1)$, the grand coalition structure $\mathcal{P}^N$ is sequentially stable for any number of players $n = |N| \geq 4$.

This corollary says that if we apply our stability concept to a common pool resource game, the grand coalition structure can be sequentially stable for any number of players.

Coalition structures other than the grand coalition structure could be sequentially stable. For example, in a 6-person game with $f(x) = \sqrt{x}$, the coalition structures consisting of $(n-1)$-person coalition and one-person coalition, $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\}$ $(\{i\} \in N)$ are also sequentially stable. However, such a coalition structure is quite unfair in the sense that the payoff of the player in one-person coalition is equal to the sum of all other players' payoffs. We will examine under which condition these undesirable coalition structures are unstable. For $\mathcal{P}$ with $|\mathcal{P}| = k$, let $C(k) \equiv \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/\{f(x_N^*(\mathcal{P}^{N\setminus\{i\}})) - f'(x_N^*(\mathcal{P}^{N\setminus\{i\}}))x_N^*(\mathcal{P}^{N\setminus\{i\}})\}$.

Theorem 6. Let $n \geq 5$. If $C(3) \geq \frac{9}{8}$, then the coalition structures $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\}$, $(\{i\} \in N)$ are not sequentially stable.

By applying this theorem to the case in which the production function is give by $f(x) = x^\alpha \ \ (0 < \alpha < 1)$, we have the following:

Corollary 2. Let $n \geq 5$. If $f(x) = x^\alpha$ and $\alpha \geq 0.583804$, then the coalition structures $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\}$, $(\{i\} \in N)$ are not sequentially stable.

The above result shows that for any number of players, the coalition structures $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\}$ cannot be sequentially stable if α is suitably large.

Remark 1. In our definition of domination, either (i) only two coalitions can merge into one coalition, or (ii) one coalition can break up into two coalitions at each step in a sequence. It is possible to define a slightly different notion of domination such that more than two coalitions are allowed to merge into one coalition at each step in a sequence. Our original concept of sequential stability is a refinement of this alternative notion. For this definition of domination, however, we can prove that the unfair coalition structure $\mathcal{P}^{N\setminus\{i\}}$ sequentially dominates any other coalition structure for a sufficiently large n.

Proposition 8. Suppose we allow that singleton coalition structure $\mathcal{P}^I$ can merge into $\mathcal{P}^{N\setminus\{i\}}$ directly at one step. Given $\alpha \in (0, 1)$, $\mathcal{P}^{N\setminus\{i\}}$ is sequentially stable for a sufficiently large n.

Remark 2. Because our sequential domination implies DX-domination, it follows from Corollary 1 that the grand coalition structure can be an EEBA for any number of players if $|N| \geq 4$. However, a set of EEBA's might contain several other coalition structures. In particular, the unfair coalition structure $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\}$ is an EEBA for a sufficiently large n. In fact, this follows from Proposition 6, because DX-domination is implied by domination under the assumption in Proposition 6 that the singleton coalition structure $\mathcal{P}^I$ can merge into $\mathcal{P}^{N\setminus\{i\}}$ directly at one step. It is difficult to eliminate the possibility that the coalition structures $\mathcal{P}^{N\setminus\{i\}}$ is an EEBA because the singleton player gets the maximal payoff among the payoffs under all coalition structures. (See Diamantoudi and Xue (2002) for a related argument.)

6 Concluding Remarks

We have proposed a sequentially stable coalition structure as a new concept of stability in coalition formation problems. This concept is an extension of EBA coalition structures. We have shown that the grand coalition structure can be sequential stable in common pool resource games.

In this paper, each coalition structure corresponds to one payoff vector. For a more general case in which each coalition structure corresponds to many possible payoff vectors, we have to consider a payoff configuration defined by $(z, \mathcal{P})$, which satisfies $z \in \{z | z \in \mathcal{F}(\mathcal{P})\}$. Here $\mathcal{F}(\mathcal{P})$ is a set of feasible payoff vectors under $\mathcal{P}$. In this case, it is not easy to compare the present payoff configuration to the final payoff configuration because of the multiplicity of the final payoff vectors. Then we should take into account sequential domination between two feasible payoff vectors in the same coalition structure. This topic is left for a future research.

We can apply our stability concept to other economic situations like public goods provision games and Cournot oligopoly games. It is generally difficult to check which coalition structures are EBA's in Cournot oligopoly games (Ray and Vohra (1997)). Examining sequential stability of coalition structures in these economic environments is an open question.

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