# DC Pension Plan Defaults and Individual Welfare*

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**Abstract**

A growing body of research shows that most DC scheme participants simply follow the given defaults, although they have the freedom to choose. As a result, default designs have dramatic impact on individuals' retirement saving behaviors. Given the fact that the default design matters, the goal of this paper is to evaluate and design better default options to help DC plans participants to save and invest wisely. The proposed defaults can be characterized by age-dependent contribution and investment rules. We find potentially large economic welfare gains by following the age-dependent defaults above the current standard default design. Furthermore, we find that the optimal contribution choice is more important than the optimal portfolio choice in welfare terms. As to the methodology, we solve a realistic life cycle model with taxable and tax-deferred accounts for the optimal defaults, while taking housing and medical expenditures into account.

Keywords: Tax-Deferred Account, retirement savings, life cycle, DC plan default, contribution rate, life cycle fund, liquidity constraints

JEL codes: E21, G11

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# 1 Introduction

Individual DC pension schemes offer each participant the freedom to choose and to implement the optimal consumption and investment strategies to their own needs. Life cycle theory has shown us the optimal saving and investing strategies. In reality, however, the early experiences with DC schemes show that most people do not choose the optimal saving and investment strategies. A growing body of research shows that most people simply follow the given defaults (Choi, Laibson, Madrian, Metrick (2004), Choi, Laibson, Madrian (2004), Lusardi and Mitchell (2006), Benartzi, Peleg and Thaler (2007)). They show that the default designs have surprisingly significant impacts on the participation, contribution and investment outcomes. Choi, et al (2004) reported that, until late 90s', non-participation was the standard enrollment default, i.e., the individuals are not enrolled unless one opts in. Under such default, participation rates were low, ranging from 26-43% six month after hiring, and 57-69% three years after hiring. Since late 90s', automatic enrollment started to be implemented in DC plans. Consequently, the reported participation rates exceeded 85% regardless of the tenure of the employee under automatic enrollment regime. Furthermore, 65%-87% of the participants adopted the default contribution rate of 3% or 4% of income, and the default investment in money market accounts. About 45% of the participants still stuck with these defaults three years later.

Given the dramatic impact of defaults, naturally, we want to have the defaults as good as possible. Therefore, we ask the question: is the current popular default design[1] the best possible design in welfare terms? If not, can we design a better default which may achieve nearly optimal welfare outcome? To this end, the goal of this study is to find the optimal age-dependent contribution and investment rules, and to evaluate to what extent these default rules help to improve the individual welfare.

The life cycle theory is a very useful framework for such analysis, and it provides us many insights[2]. There are two controls in the life cycle planning problems, namely the optimal consumption and the optimal portfolio choices. Both strategies may depend on one's age, income, wealth accumulation and other economic state variables, and hence may differ by age and economical circumstances. A better default design thus should have two aspects, namely a better saving default and a better investment default. Several studies propose the so-called life-cycle funds as portfolio

[1] The current defaults refer to automatic enrollment with flat contribution rate of 3% or 4% of income, together with a money market investment.

[2] The life cycle theory has a long line of literature starting from Merton (1969, 1971). Some of the more recent papers are Carroll (1992, 1994, 1997) on precautionary saving, Gourinchas and Parker (2002) on life cycle consumption, Gomes and Michaelides (2005) and Cocco, Gomes and Maenhout (2005) on life cycle portfolio choice, Benzoni, Collin-Dufresne, and Goldstein (2007) on life cycle strategies with cointegrated labor income with market returns, Cocco (2005) on portfolio choice in the presence of housing risk, and Gomes, Kotlikoff and Viceira (2008) on life cycle investing with flexible labor supply.

allocation default. (Bodie, McLeavey and Siegel (2007), Viceira (2007)) The idea of life-cycle funds is to mimic the theoretical life cycle portfolio strategies using an age-dependent portfolio rebalancing rule. In such life-cycle funds (also known as target maturity funds) the portfolio allocation to stock mutual funds declines as one ages, and is replaced gradually by safer assets like bonds and cash. In fact, the life cycle funds implements a simplified version of the optimal portfolio strategy. The life-cycle funds start to be implemented as default by many DC scheme providers recently, and are expected to have large impact on the asset allocation outcome of most DC contributors.

However, only changing the portfolio default alone does not help much if people do not contribute or fail to contribute enough. To address this pressing issue, recently, the United States passed the Pension Protection Act of 2006, which encourages the adoption of several 'autosave' features in the DC plans. These 'autosave' features include automatic enrollment, employer contribution, contribution escalation, and qualified investment default. (see Beshears, Choi, Laibson, Madrian, and Weller (2008)) The contribution escalation is based on an interesting idea called Save More Tomorrow, which dramatically stimulates participants to save more (Thaler and Benartzi (2004)). In such scheme, participants agree to automatically increase their saving rate whenever their salaries get raised. However, as the authors claim, this design is solely based on behavior motivations, but not on financial economical considerations. Are the increased saving rates optimal or nearly optimal over life cycle? These questions are not addressed in the literature.

The life-cycle funds tackled the asset allocation aspect of the life cycle theory but not the consumption aspect. Therefore, the next step to go forward is to extend the fixed contribution rate to include some age- (or wage-) dependent features. The main idea in this paper is to design the simple default rules to mimic the optimal consumption during one's life cycle. In addition, we evaluate to what extent these age- (or wage-) dependent default contribution and investment rules are beneficial to the participants and can be recommended as default options for individual pension plans.

A closely related paper is by Gomes, Michaelides and Polkovnichenko (2006) who study the optimal portfolio choices and optimal saving strategy for rational individuals with a taxable account and a tax-deferred DC account.[3] Since individuals are borrowing constrained and the illiquid DC scheme makes the liquidity constraint more severe.[4] The setting with two accounts is particularly important to capture the liquidity constraint when designing the contribution rules. The classical life cycle models with single account is inadequate for studying contribution rules. This paper

[3]Dammon, Spatt and Zhang (2004) also study the optimal portfolio choices with taxable and tax-deferred accounts. They show that the taxes on dividends, capital gains and interests have an impact on the portfolio choices in taxable and tax-deferred accounts.

[4]Under severe circumstances early withdrawal from DC account is allowed (but subject to certain penalty cost). In the baseline model of this paper, we assume that borrowing or early withdrawal from the DC savings are not allowed, therefore DC account is illiquid during the working period.

adapts a similar modeling setup as Gomes et al (2006), but with a different focus. Our focus here is on the optimal default saving and investing rules.

In this paper, we explicitly model the liquid taxable account and illiquid tax-deferred DC plans. We show the optimal contribution rates and investment policies under this setting. We also show various optimal default designs for the given default specifications, including constant and age-dependent features. Furthermore, we carefully model the income risk, housing and medical expenditures, in order to realistically quantify and evaluate the default designs. We use the dynamic programming and the Endogenous-grid method (Carroll (2007)) to solve the extended life cycle model with two accounts.

Our main findings are the following. First, we find potentially large economic welfare gains by following the smart age-dependent contribution rule above the current standard default design. In terms of certainty equivalent consumption, the age dependent default leads to 6% increase of the certainty equivalent consumption per year, comparing to the current defaults. Using the fully optimal strategies as welfare benchmark, the current default design delivers maximally 91% of welfare relative to the optimal welfare level. Whereas, the age-dependent contribution and investment default design delivers more than 99% relative to the optimal welfare level. Therefore the simple age-dependent contribution rule and appropriate investment strategies can achieve nearly optimal welfare level.

Second, we find that the contribution (or saving) choice has larger impact on welfare than the portfolio choice does. Replacing the optimal fixed asset mix by an optimal life cycle fund, the certainty equivalent consumption gains 0.8% per year. Whereas replacing the constant contribution rate by an age-dependent contribution rate (keep the asset mix fixed at an optimal level) raises the certainty equivalent consumption by 3.8% per year. The life cycle literature has mainly focused on the importance of portfolio choices. Here we show that setting the contribution (or saving) right is more important in welfare terms.

As to the policy implications, the age-dependent contribution and investment rules can be recommended as default. Our finding is consistent with the auto-save features encouraged by the Pension Protection Act of 2006. Retirement saving and investing play an increasingly important role in our ageing societies. We believe our user-friendly and thoughtful default saving rules resolve the worrying concerns of millions (or billions) of people. This paper contributes to the life cycle literature and the DC industry by characterizing the optimal age-dependent contribution and investment rules for DC participants.

The organization of the paper is the follows. Our analysis start from the classical life cycle planning problem of a rational representative agent in Section 2 and 3, in order to describe the optimal life cycle saving and investment strategies in an ideal world. There we see several age related patterns regarding the asset allocation and consumption strategies over life time. In Section 4, we study the age-dependent default designs for passive participants in individual-based DC schemes. We focus on the effect of several designs of default options, and to see how far one can push the welfare closer to the optimal strategies by using the age-dependent policies as default setting. Section 5 discusses the main results, and Section 6 concludes.

# 2 Model Setup

## 2.1 Individual's preference

We assume that all individuals start working at age 25 ($t = 0$) and retire at 65 ($t = R = 40$). For simplicity, we assume that the individuals die at age 85 ($t = 60 = T$). During the working period ($0 \leq t < 40$) the individuals earn stochastic labor income, denoted by $Y_t$. During the retirement period ($R \leq t < T$), the individuals receive no income but consume their accumulated wealth, denoted by $W_t$. Individuals drive utility over single consumption goods (normalized by price inflation). Individual's preference is captured by the constant relative risk aversion utility function as

$$E\left[\int_0^T \beta^{-t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right]$$

where $\gamma$ is the risk aversion parameter and $\beta$ is the subjective discount factor. In the baseline model, we fix $\gamma = 5$, and $\beta = 0.97$, following the life cycle literature.

## 2.2 Return dynamics

There are two financial assets traded in the market, one risk free and one risky (both in real terms). The real risk free asset offers a fixed real interest rate $r$. The real price of the risky stock index, $E_t$, follows geometric Brownian motion with a constant drift. Dividends are reinvested. The aggregate real wage index $G_t$ is stochastic. The uncertain stock returns are potentially correlated with stochastic aggregate wage growth. This contemporaneous correlation is denoted by $\rho_{sg} \equiv corr(dZ_{E,t},\ \ dZ_{g,t})$ where $dZ_{E,t}$ and $dZ_{g,t}$ denote the independent Brownian incremental of stock returns and aggregate wage growth rates. The stock return and wage growth rate dynamics are the following

$$dE_t/E_t = \mu dt + \sigma_E\sqrt{1-\rho^2}dZ_{E,t} + \rho_{sg}\sigma_E dZ_{g,t} \quad (1)$$

$$dG_t/G_t = \overline{g}dt + \sigma_g dZ_{g,t} \quad (2)$$

where $\mu$ is the instantaneous drift, and $\sigma_E$ is the volatility of stock returns. In the baseline model, we assume $\mu = 6\%$, and $\sigma_E = 15\%$, annualized.

### 2.3 Labor income dynamics

Let $t$ denotes calendar year and $t_0$ the year of birth, so that $t - t_0$ is the age of the individual under consideration. The individuals' real labor income $Y_{t-t_0} = G_t N_{t-t_0}$ can be decomposed into two component, an aggregate wage component, $G_t$, and an age-dependent idiosyncratic component $N_{t-t_0}$. The growth rate of aggregate wage component is determined according to eq(2). While the idiosyncratic wage component, $N$, has an age-dependent drift $f(t-t_0)$ to generate the hump-shape of earnings and normally distributed permanent shocks. The real labor income process is specified as follows

$$
\begin{aligned}
Y_{t-t_0} &= G_t N_{t-t_0} & (3)\\
\ln G_t &\equiv \ln G_{t-1} + g_t & (4)\\
\ln N_{t-t_0} &= \ln N_{t-1-t_0} + f(t-t_0) + \sigma_n \eta_t & (5)\\
&= \ln N_{t-1-t_0} + (a_0 + a_1 (t - t_0)) + \sigma_n \eta_t & (6)
\end{aligned}
$$

where $\eta_t \sim i.i.d.N(0,1)$. I assume that all individuals start working at age 25 and retire at age 65. Their starting annual salary $N_{25}$ is normalized to \$20,000. Due to the growth of aggregate wage, each generation has different starting salaries $Y_{25}$. The parameter $a_0$ and $a_1$ are set according to the calibration of Benzoni, Gollin-Dufresne, and Goldstein (2007) for the high education group ($a_0 =$ 0.066 and $a_1$= -0.0024). The figure below show the quantile distribution of $N$ and $G$ over time.

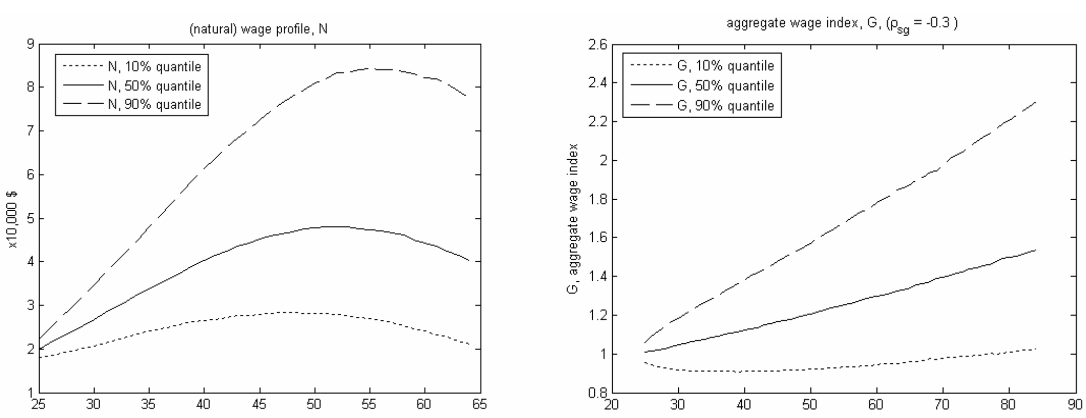


Figure 1: Income profiles. (a) Left panel shows the life cycle profile of the age-dependent idiosyncratic component $N_{t-t_0}$; (b) Right panel shows the aggregate wage component, $G_t$.

The old-age social security benefit is assumed to be a fraction of the final labor income, following the common modeling in the life cycle literature. Let $SS_R$ denote the social security benefit at age 65, amounting to s percentage of the final salary, $SS_R = s * Y_{R-1}$. The social security is indexed with the aggregate wage growth, so that $SS_t = G_t SS_R$, for $t > R$. In the baseline model, we consider $s = 30\%$.[5]

## 2.4 Other background risks

In order to depict the realistic retirement saving and investing strategies, we need a realistic description of the major background risks that households may face. In our model, we take the estimated housing expenditures and out-of-pocket medical expenditures in to account. Both expenditures are modeled as exogenous shocks to the budget process.

We assume that individuals pay off all their mortgages before age 80. The exogenous housing expenditure represents a fraction of labor income during working period, and a fraction of final income during the retirement period. Based on the estimation of Gomes and Michaelides (2005), the ratio of housing expenditure to income has the following age-dependent mean and variances

$$H_t/Y_t = h_t \sim N\left(\overline{h}\left(t\right), \sigma_h^2\left(t\right)\right) \tag{7}$$

where the age-dependent mean $\overline{h}\left(age\right) = h_0 + h_1 * age + h_2 * age^2 + h_3 * age^3$, with $h_0 = 0.71$, $h_1 = -0.035$, $h_2 = 0.00072$, $h_3 = -0.0000049$. The calibration is based on the estimation results of Gomes and Michaelides (2005) using PSID data. Furthermore, uncertainty in the housing expenditure is captured by $\sigma_h = 2\%$, and the correlation between shocks to $h_t$ and income $Y_t$ is set at -0.5. Figure

The ratio of out-of-pocket medical expenditure to income, following the parameterization and estimation results of Scholz, Seshadri and Khitatrakun (JPE 2006), is modeled as follows

$$\ln(M_t/Y_t) = -7.316 + 0.012 * age + 0.00066 * age^2 + \varepsilon_M$$

with $\varepsilon_M \sim N\left(0, \sigma_M^2\right)$, and $\sigma_M = 20\%$.

[5] $s = 30\%$ might overstate the benefit for the high final salary individuals, and understate the benefit for the low final wage individuals.

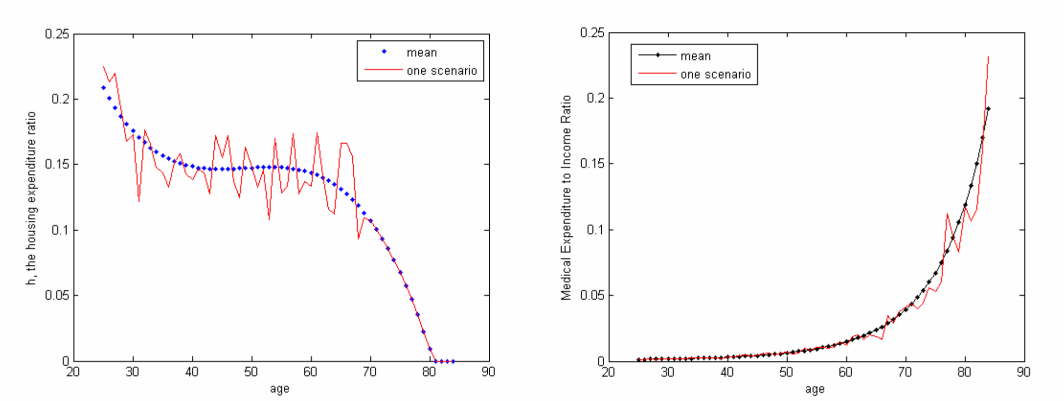


Figure 2: Housing and medical expenditures. (a) Left panel shows the mean and one simulation of the ratio of housing expenditure to income over life cycle; (b) Right panel shows the mean and one simulation of the ratio of out-of-pocket medical expenditure to income over life cycle.

# 3 Optimal life cycle strategies

This section studies the optimal life cycle planning problem of an individual under realistic economic settings, where individual-based DC schemes are provided by employers or financial institutions. These individual DC schemes are called Tax-Deferred Account (TDA), since income tax and dividends tax are exempted or postponed till retirement. The consumers thus may keep their retirement savings in these Tax-Deferred Account, and may also hold other private wealth in Taxable Account. Section 3.1. describes the life cycle model for an individual with tax-deferred DC account and taxable account. Section 3.2. shows the optimal strategies under the tax benefit driven setting (the baseline model). Section 3.3. presents the results with employer matching.

## 3.1 Model setup

In many countries, individuals save into a DC pension account with certain tax advantages (delaying income tax and/or exempt from dividend and capital gain taxes). The DC contributions are exempted from income tax $\tau^y = 30\%$, only the withdrawal from DC account is taxable at a lower income tax rate $\tau^o = 20\%$ when retired. Such tax benefits provide certain incentive for individuals to save in the tax deferred account. Individuals may choose how much and when to contribute into the DC account. Let $m_t$ denote the contribution rates into the DC pension plan. Another important consideration is that one can not withdraw the DC wealth before retirement, that is, $m_t \geq 0$ (the contribution rates must be non-negative). This feature imposes the liquidity constraint on individual. The liquidity issue also distinguishes the optimal strategies with TDA and TA from the strategies with only TA. Furthermore, in order avoid large tax arbitrage, the contribution rate is capped at 20%, so that maximally 20% of gross income can be contributed in DC plan in each year.

Let us introduce more notations. $W_t^{\tau}$ denotes individual's wealth in the taxable account, $W_t^{DC}$ denotes the individual's DC pension saving which is tax-deferred, $\widetilde{R}_{t+1}^{e} = E_{t+1}/E_t$ denotes the total return on equities, $R^f = \exp(r)$ denotes the real risk free rate. The income tax rate for the workers is $\tau^y = 30\%$, and for the retirees is $\tau^o = 20\%$. The asset allocations within the private saving and the DC saving account are denoted as $\omega_t^{\tau}$, and $\omega^{DC}$ respectively. In this version of the paper, we assume the asset allocation in Taxable and Tax-deferred DC accounts are identical. Given the focus of this study on consumption, and the simplification assumption that dividend and capital gain are not taxed, the identical portfolio assumption has negligible impact of the main results of the paper. Individuals optimize life time utility of consumption by choosing consumption, contributions into the illiquid DC pension scheme, and their asset allocations for both the private saving and the DC saving accounts.

$$V = \max_{\{C_t, \omega_t^{\tau}, \omega_t^{DC}, m_t\}_{t=1}^{T}} E\left[\sum_{t=1}^{T} \beta^{t-1} \frac{C_t^{1-\gamma}}{1-\gamma}\right] \tag{8}$$

s.t. the wealth dynamics, before retirement ($t < R$), as

$$\begin{aligned} W_{t+1}^{\tau} &= (W_t^{\tau} - C_t - (1-\tau) m_t Y_t) \left(R^f + \omega_t^{\tau}\left(\widetilde{R}_{t+1}^{e} - R^f\right)\right) + (1-\tau^y) Y_{t+1} - H_{t+1} - M_{t+1} && (9) \\ W_{t+1}^{DC} &= \left(R^f + \omega_t^{DC}\left(\widetilde{R}_{t+1}^{e} - R^f\right)\right) \quad \left[W_t^{DC} + m_t Y_t\right] && (10) \end{aligned}$$

and the no-borrowing constraint

$$\begin{aligned} W_t^{\tau} &\geq 0 && (11) \\ 20\% &\geq m_t \geq 0 && (12) \end{aligned}$$

After retirement ($T \geq t \geq R$), the DC savings become available for consumption. Individual combines the two savings with a total wealth of $W_R = W_R^{\tau} + (1-\tau)\, W_R^{DC}$. This modeling assumption is made based on the observation that many people cash out their DC savings upon retirement. Formally, the wealth dynamics of the savings during the retirement period are as follows.

$$W_{t+1} = \left[\omega_t^{\tau} \widetilde{R}_{t+1}^{e,\tau} + (1-\omega_t^{\tau})\, R_t^f\right] [W_t - C_t] + (1-\tau^o) SS_{t+1} - H_{t+1} - M_{t+1} \tag{13}$$

Since the individuals are always borrowing constrained, the balances of the two savings must always be non-negative. Furthermore the individuals are short-sales constrained as well. These imply that

$$1 \geq \omega_t^{\tau} \geq 0 \ , \ 1 \geq \omega_t^{DC} \geq 0 \tag{14}$$

The problem has no analytical solution. We use the dynamic programming principle together with the Endogenous-grid method (Carroll (2007)) to solve the extended life cycle model with two accounts. See Appendix A for the details of the solution technique.

### 3.2 The life cycle saving and investing profiles

This subsection shows the optimal life cycle profiles of the individuals with both taxable and tax-deferred DC accounts. The distribution of the life cycle profiles are characterized by 5%, 50% 95% quantiles.

Figure 3 (left panel) shows the consumption profile, which slightly increasing over time, due to the increasing wage profile. Figure 3 (right panel) shows wealth accumulation in the TA and TDA accounts. Early in life, young individuals accumulate moderate wealth in taxable account as precautionary savings against various background risks (incl. income uncertainties and expenditure uncertainties). Strikingly, the young individuals make zero contribution to the Taxable DC account for the first ten years of working period. The DC wealth starts to grow rapidly after mid-age, when the individuals are not liquidity constrained, their retirement saving motives get stronger, and tax benefits are much higher. The growth of TDA account is boosted by transferring wealth from TA to TDA account via higher contributions around mid-age.

Figure 4 (left panel) shows the portfolio allocation in stocks over life time. Here we assume the asset allocation in Taxable and Tax-deferred DC accounts are identical. As explained by Campbell and Viceira (2002), due the leverage effect of human capital[6], the portfolio allocation to equities generally decreases over time. Here we confirm this finding under the two accounts setting. Figure 4 (right panel) shows the contribution rate profile, with zero contribution in the first 10 years, and rapidly increasing in the mid-age, and eventually reaching to 20% ceiling around before retirement.

[6]The human capital is the present value of the future disposable incomes (i.e. after-tax income or after-tax social security benefit, subtract from housing and medical expenses).

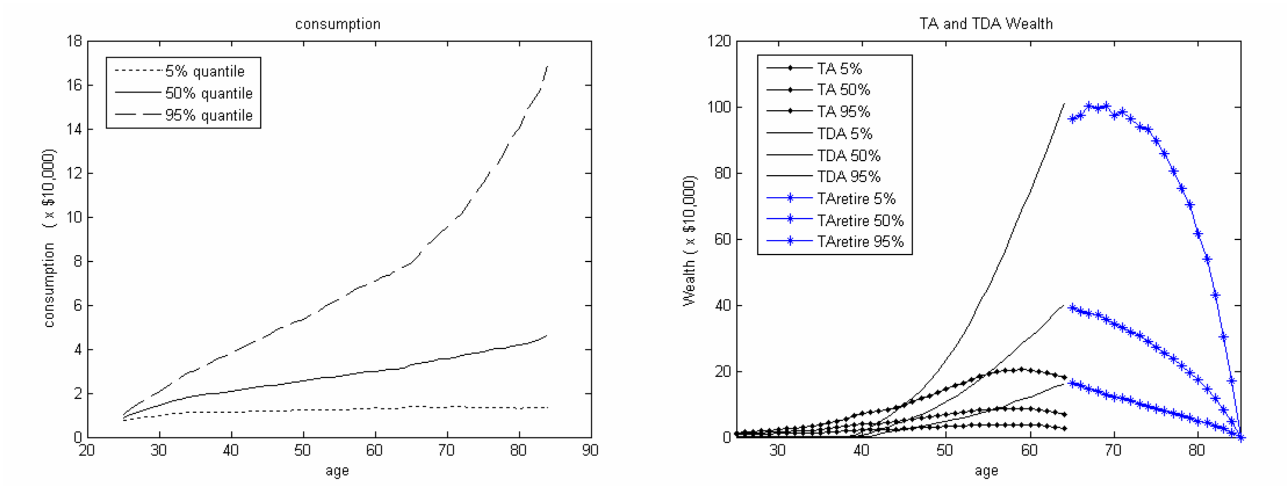


Figure 3: (left) Life-cycle consumption, and (right) asset accumulation in Taxable and Tax-deferred DC accounts (5%, 50% and 95% quantiles).

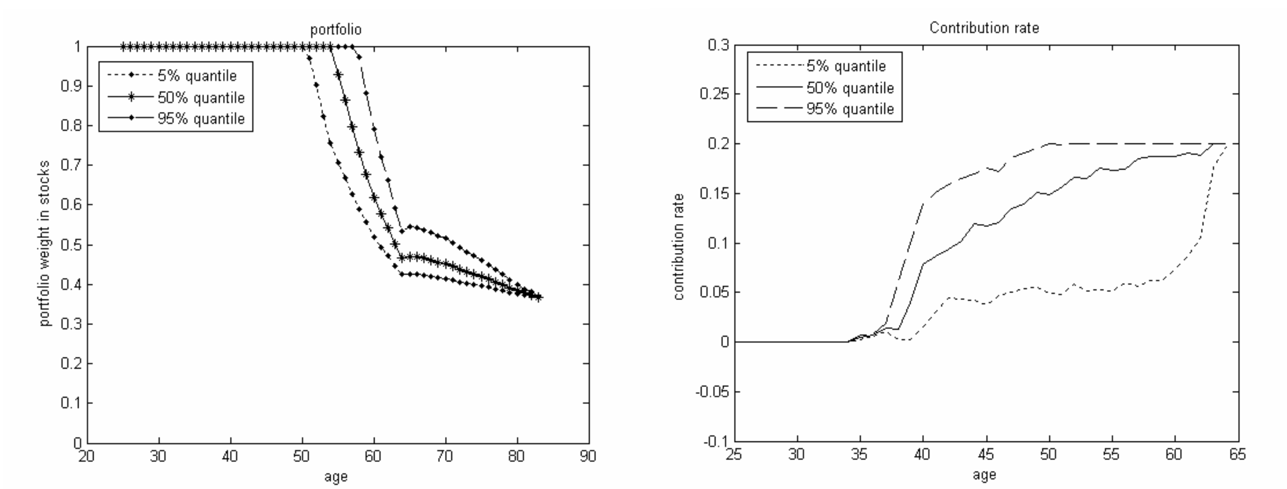


Figure 4: (left panel ) Life-cycle portfolio choice assuming the identical asset allocation in Taxable and Tax-deferred DC accounts, (5%, 50% and 95% quantiles); (right panel) life-cycle contribution rate $m_t$ (5%, 50% and 95% quantiles).

In order to see under which situation the individual should change their optimal contribution, we regress the de-meaned contribution rate on log income and log wealth ratio, as (15). Table 1 reports the median and mean of the contribution rate over life cycle, the regression coefficients and the standard errors, and the resulting $R^2$. Table 1 shows that 10% increase of income tops the contribution rate by about 0.4%, while 10% increase of TDA-to-TA-wealth-ratio is likely to decrease the contribution rate by 0.4% to 3%.

$$m_t - \overline{m}_t = \beta_1 \ln Y_t + \beta_2 \ln(W_t^{DC}/W_t^{\tau}) \tag{15}$$

| age | m median | m mean | coef. lnP | s.e. lnP | coef. ln(TDA/TA) | s.e. ln(TDA/TA) | R sqr |
|---|---|---|---|---|---|---|---|
| 35 | 0,4% | 0,5% | 0,001 | 0,0001 | 0,000 | 0,0000 | 5% |
| 40 | 8% | 7% | 0,038 | 0,0012 | 0,028 | 0,0008 | 18% |
| 45 | 12% | 11% | 0,003 | 0,0001 | -0,315 | 0,0018 | 87% |
| 50 | 15% | 14% | 0,045 | 0,0007 | -0,191 | 0,0029 | 47% |
| 55 | 17% | 15% | 0,041 | 0,0010 | -0,092 | 0,0022 | 26% |
| 60 | 19% | 17% | 0,031 | 0,0008 | -0,045 | 0,0011 | 25% |
| 64 | 20% | 20% | 0,002 | 0,0001 | -0,002 | 0,0001 | 5% |

Table 1: Results from regressing the contribution rate $m_t$ on log income $Y_t$ and log of the TDA-to-TA-wealth-ratio $W_t^{DC}/W_t^{\tau}$. The standard errors are reported in brackets.

### 3.3 With Employer Matching

Here we consider a variant of the baseline model, including the employer matching. In the United States (and some other countries), the employer match and tax saving on dividend and capital gain taxes provide the main incentive to contributing into the tax-deferred account. Individuals may choose how much and when to contribute into the DC account. The employer match may take various forms in practice. Here we consider a common practice, i.e., the employer match 100% with the employee's contribution up to a limit of 6%, as parameterized in GMP2006. However, the total contribution should not exceed 20%. Let $m_t^{DC}$ denote the total contribution rates into the DC pension plan.

$$m_t^{DC} = \min\left(m_t + \min\left(m_t, 6\%\right), 20\%\right) \tag{16}$$

The DC wealth dynamic before retirement ($t < R$) is slightly modified to

$$W_{t+1}^{DC} = \left(R^f + \omega_t^{DC}\left(\widetilde{R}_{t+1}^e - R^f\right)\right) \quad \left[W_t^{DC} + m_t^{DC} Y_t\right] \tag{17}$$

Figure 5 to 6 show the life cycle profiles of consumption, wealth accumulation, portfolio choice and contribution rates, in the optimal scheme with employer matching. Most of the profiles are very similar to those in the baseline setup. Figure 6 (right panel) shows the contribution rates profile. It shows a clear ladder shape, increasing over life cycle. The contribution rate is zero for the first 9 years of working life. The young individuals are liquidity constrained, because they face relatively low incomes but high housing expenditures. They leave the employer match on the table. Between age 35 and 45, the contribution rate increases quickly to 6%, earning the maximal employer matching. After age 45, it increases up to 14%, so that together with the employer matching the total contribution reaches the 20% ceiling.

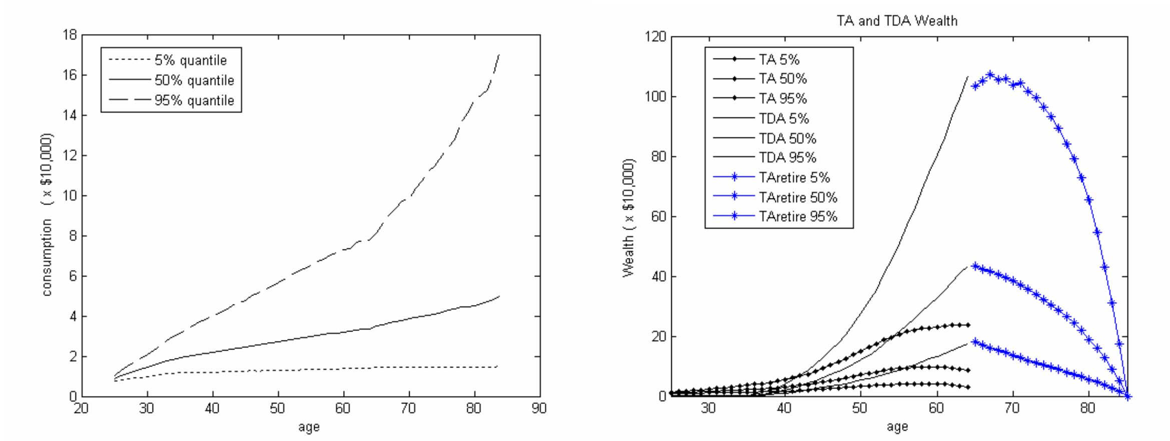


Figure 5: (left) Life-cycle consumption, and (right) asset accumulation in Taxable and Tax-deferred DC accounts with employer match (5%, 50% and 95% quantiles).

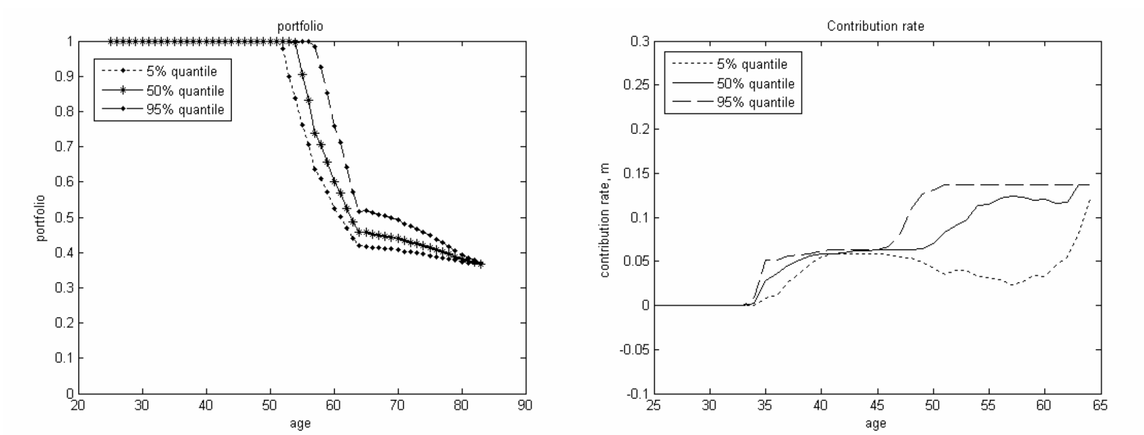


Figure 6: (left panel ) Life-cycle portfolio choice assuming the identical asset allocation in Taxable and Tax-deferred DC accounts, (5%, 50% and 95% quantiles); (right panel) life-cycle contribution rate $m_t$ with employer match (5%, 50% and 95% quantiles).

### 3.4 Welfare evaluation

Let's take the certainty equivalent consumption as the welfare measure, which can be easily backed out from the following equality

$$V = \int_0^T e^{-\delta t} \frac{(CEC)^{1-\gamma}}{1-\gamma} dt \tag{18}$$

For easy interpretation, we divide the CEC by the first labor income $Y_{25}$. The welfare obtained by TA/TDA investors by implementing the optimal strategies are the following. The welfare obtained from our baseline model sets up the benchmark (an upper limit) for the comparison across various defaults.

| | baseline model | variant 1 (with employer match) |
|---|---|---|
| CEC/$Y_{25}$ | 0.688 | 0.702 |

# 4 Default designs

The previous section discusses the optimal strategies. This section discusses the default designs of the DC accounts. Given the dramatic impact of defaults, we ask the question, is the current popular default design the best possible design in welfare terms? If not, can we design a better default which may achieve nearly optimal welfare outcome? For designing the optimal default options, we model the cases where individuals staying with the defaults throughout the life cycle, as if the default DC plan were made mandatory. In section 5, we will compare the welfare cost of different default designs relative to the optimal strategies, and investigating whether age-dependent default is able to help to reduce the welfare cost.

Each subsection describes a default specifications. The main characteristics of default designs are summarized in the following table.

| | Default #1 | Default #2 | Default #3 | Default #4 |
|---|---|---|---|---|
| portfolio | constant | age-dependent | constant | age-dependent |
| contribution | constant | constant | age-dependent | age-dependent |

Under a given default design specification, the individuals optimize the their objective (8), subject to the budget constraints in TA and TDA (9, 10), and no-borrowing constraint (11) and no-short selling constraint (14). Detailed solution methodology is given in Appendix B.

## 4.1 Default #1:

Main features of this default setting include a constant contribution rate, and a constant portfolio choice throughout (working) life. These constant features resemble the current standard default options, which typically fix the contribution rate at 4% and invest in money market accounts, with no further adjustment.

The optimization problem can be reformulated as follows: Individuals optimize utility over life time consumptions, by choosing the consumption level in each period, and a constant saving rate $m$

and a constant portfolio $\omega$ at the beginning of their careers. Effectively, we are replacing the optimal strategies in Section 3 by constants, i.e., $\omega_t^{DC} = \omega_t^{\tau} = \omega$, and $m_t^{DC} = m$ (with $0 \leq m \leq 20\%$). As before, borrowing is not allowed, which implies $0 \leq W_t^{\tau} - C_t - (1 - \tau^y)mY_t$, and short sales is allowed, implying $0 \leq \omega \leq 1$.

Table 2 shows, for a given flat contribution rate, the corresponding optimal fixed portfolio choice and the welfare level under the specification of Default #1. It shows that if contribution rate is fixed at 4%, then the best fixed portfolio is investing 85% in equities throughout life. This gives a certainty equivalent consumption of 0.645 units of first year labor income. When comparing the CEC level across different contribution rate, we find that, for the assumed amount of tax benefit, without employer matching, the zero contribution leads the best outcome. It means that the tax benefit in our baseline model is not large enough to compensate for the liquidity loss when young.

Table 3 shows the results with employer matching. It shows that, with employer matching as (16) in addition to the tax benefit, a small but positive contribution rate of 4% is beneficial for the individual.

In addition to the results shown in Table 2 and 3, we also compute the welfare obtained under the current default setting, with flat contribution rate of 4% of income, together with a money market investment. The current default setting delivers a CEC of 0.625, which is reported in Table 4.

| Contribution rate (m) | The optimal portfolio weight in equities (w) | CEC |
|---|---|---|
| 0% | 80% | 65.8% |
| 1% | 80% | 65.6% |
| 2% | 82.5% | 65.2% |
| 3% | 85% | 65% |
| 4% | 85% | 64.5% |
| 5% | 87.5% | 63.5% |

Table 2: Default #1 (no employer matching) with flat contribution rate and the corresponding optimal fixed portfolio weight in equities.

| Contribution rate (m) | Portfolio weight in equities (w) | CEC |
|---|---|---|
| 0% | 80% | 65.8% |
| 1% | 80% | 65.9% |
| 2% | 82.5% | 66% |
| 3% | 85% | 66.2% |
| 4% | 85% | 66.4% |
| 5% | 87.5% | 66% |

Table 3: Default #1 (with employer matching) with flat contribution rate and the corresponding optimal fixed portfolio weight in equities.

## 4.2 Default #2:

Changing from Default #1, we relax the restriction of a constant portfolio throughout life, but replacing it with an age-dependent strategy. One popular rule of thumb describes a linearly relationship between age (denoted by $t - t_0$) and the share of risky assets as

$$\omega_t = \max[0, \quad \min[f_0 + f_1 (t - t_0), 1]] \tag{19}$$

The individual follows the the age-dependent allocation rule. Individual optimizes his life time utility of consumption by choosing consumption in each period, and choosing the constant saving rate $m$ and the parameter $(f_0, f_1)$ at the beginning of his career. As before, borrowing or short sales are not allowed, which implies $0 \leq \omega_t \leq 1$ and $0 \leq m \leq 20\%$.

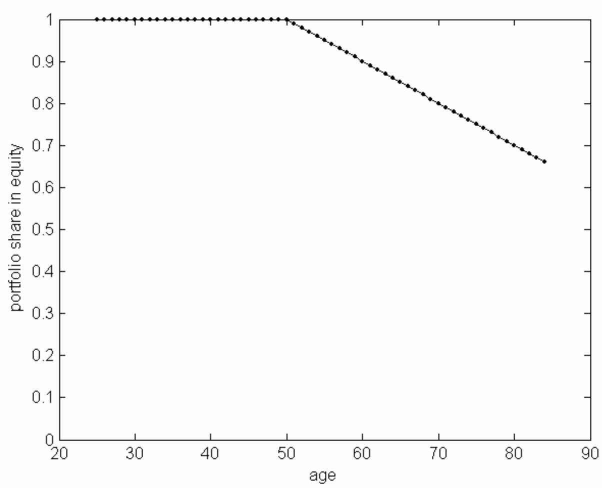

Figure 7: The optimized age-dependent portfolio rule for given contribution rate of 4%, as specified in Default #2.

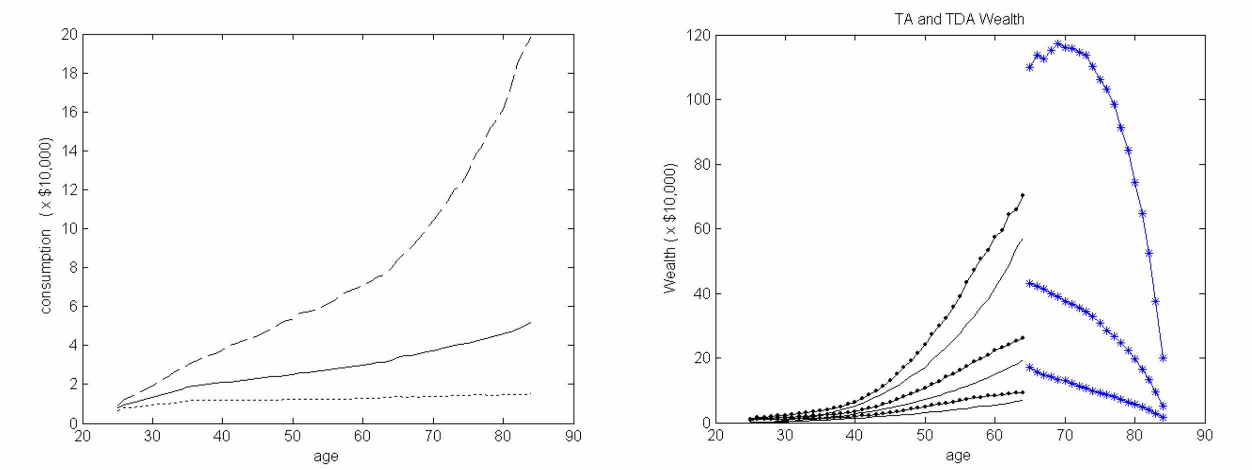


Figure 8: (left) Life-cycle consumption, and (right) asset accumulation in Taxable and Tax-deferred DC accounts, as specified in Default #2 (5%, 50% and 95% quantiles).

Figure 7 shows the optimized age-dependent portfolio rule corresponding to the given contribution rate of 4%. This life cycle fund resembles the age profile of the optimal portfolio choices shown in Figure 4. Figure 8 shows the resulting life cycle quantile profiles of consumption (left panel) and the wealth accumulation is both TA and TDA accounts (right panel). Comparing to the optimal strategies in Section 3, the consumption in this default is slightly lowered for the young, and slightly higher for the retirees, due to constant contribution rate of 4%. Since the contribution rate is capped at 4%, the accumulated DC wealth at the end of the working period is substantially smaller than the one in the optimal situation.

### 4.3 Default #3:

Changing from Default #1, we relax the restriction of a constant contribution rate throughout life, but still keep the constant portfolio in both TA and TDA. Therefore the main features of the TDA are the age-dependent contribution rate and a constant portfolio choice. A simple age-dependent contribution rate may be linearly depending on age as follows:[7]

$$m_t = \max[0, \quad \min[d_0 + d_1(t - t_0), \quad 20\%]] \tag{21}$$

[7]An alternative modeling of the age-dependent contribution rate may include the quadratic term in age. For example

Suppose the individual follows the age-dependent contribution rule. The individual optimizes his life time utility of consumption by choosing consumption in each period, and choosing the constant portfolio $\omega$ and the parameter $(d_0, d_1)$ for the contribution schedule at the beginning of his career. As before, borrowing or short sales are not allowed, which implies $0 \leq \omega \leq 1$ and $0 \leq m_t \leq 20\%$.

Figure 9 shows the optimized age-dependent contribution rule for the jointly optimized portfolio choice of 85%, as specified in Default #3. The age-dependent contribution rule resembles the life cycle profile of the optimal contribution rates in Section 3. For the first 12 years of working period, the individuals do not contribute to the DC plan, because of the liquidity constraints. Between age 38 and 60, the contribution rates are non-zero and increase linearly with age, by about 0.8% per year. Between age 60 and 64, the individuals are not liquidity constrained any longer, therefore the maximum contributions are optimally chosen, due to the tax benefits. Figure 10 shows the resulting life cycle quantile profiles of consumption (left panel) and the wealth accumulation is both TA and TDA accounts (right panel). We see that DC wealth is accumulated rapidly after mid-age.

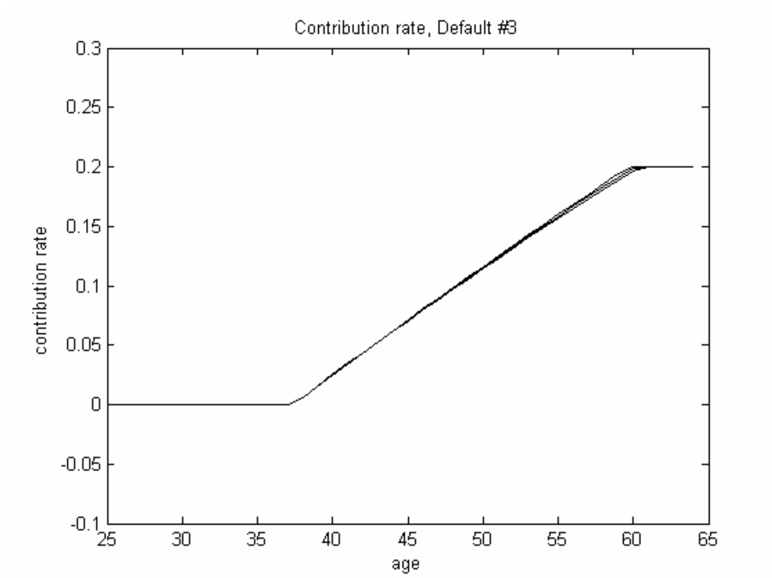


Figure 9: The optimized age-dependent contribution rule for given portfolio choice of 85%, as specified in Default #3.

---

$$m_t = d_0 + d_1 * (t - t_0) + d_2 * (t - t_0)^2 \tag{20}$$

This alternative modeling is able to capture the possible hump shape of the optimal contribution rates as seen in Section 3. This specification will be investigated in the future research.

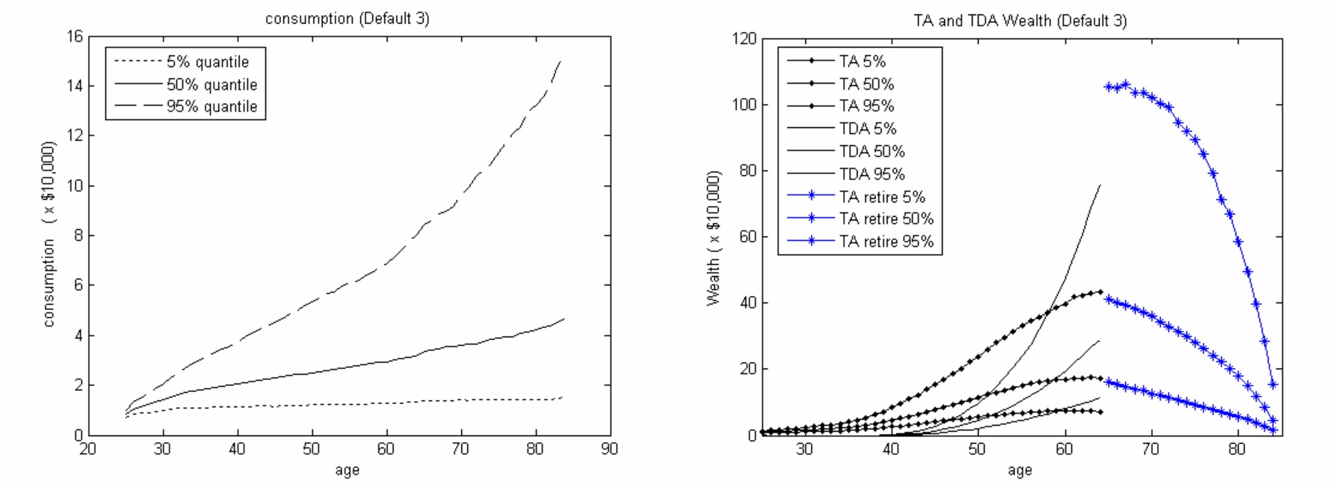


Figure 10: (left) Life-cycle consumption, and (right) asset accumulation in Taxable and Tax-deferred DC accounts, as specified in Default #3 (5%, 50% and 95% quantiles).

### 4.4 Default #4:

Default #4 is a combination of Default #2 and #3. It specifies an age-dependent contribution rule and an age-dependent portfolio rule, i.e.,

$$m_t = \max[0, \quad \min[d_0 + d_1(t - t_0), \quad 20\%]] \tag{22}$$

$$\omega_t = \max[0, \quad \min[f_0 + f_1 (t - t_0), 1]] \tag{23}$$

## 5 Main Results

Table 4 summarizes the main findings of this paper. Table 4 compares the optimal strategies with the current default design, and the step by step improvements above it. The second row ("TDA / TA") reports the certainty equivalent consumption of the optimal strategies, which sets an upper limit of the welfare level. The third row ("Current") reports the CEC obtained under the current default setting of 4% contribution rate and zero equity exposure. Starting from here, we first improve the asset allocation by setting it to an optimal level of 85% in equities, while keeping the contribution rate at 4%. This step improves the CEC by 2% per year from 0.625 to 0.645, as reported in the forth row. Then, we further improve the asset allocation by choosing an optimal life cycle fund, using the specification of Default #2. This step further improves the CEC by 0.8%

per year from 0.645 to 0.653, as reported in the fifth row. In the last step, in stead of choosing a life cycle fund, we replace the flat contribution rate by an optimal age-dependent contribution rate, using the specification of Default #3. This step further improves the CEC by 3.8% per year from 0.645 to 0.683. Accumulatively, the welfare gain amounts to 2.3 times of first year income. Furthermore, the age-dependent contribution and investment default design delivers more than 99% relative to the optimal welfare level.

Our main findings are the following. First, the simple age-dependent contribution rule and appropriate investment strategies can achieve nearly optimal welfare level. We find potentially large economic welfare gains by following the smart age-dependent contribution rule above the current standard default design. Using the fully optimal strategies as welfare benchmark, the current default design delivers maximally 91% of welfare relative to the optimal welfare level. Whereas, the age-dependent contribution and investment default design delivers more than 99% relative to the optimal welfare level. In terms of certainty equivalent consumption, the age dependent default leads to 6% increase in annualized certainty equivalent consumption per year. Over a life time, the welfare gain amounts to 3.6 times of first year income.

Second, we find that the contribution (or saving) choice has larger impact on welfare than the portfolio choice does. Replacing the optimal fixed asset mix by an optimal life cycle fund, the certainty equivalent consumption gains 0.8%. Whereas replacing the constant contribution rate by an age-dependent contribution rate (keep the asset mix fixed at an optimal level) raises the certainty equivalent consumption by 3.8%. The life cycle literature has mainly focused on the importance of portfolio choices. Here we show that setting the contribution (or saving) right is more important in welfare terms.

| Designs | CEC (% of 1st y income) | Relative to $CEC_{TDA/TA}$ |
|---|---|---|
| TDA / TA | 68.8% | 100% |
| Current (w=0, m=4%) | 62.5% | 91% |
| w*=85%, m=4% | 64.5% | 93.7% |
| Life cycle fund*, m=4% | 65.3% | 95% |
| **w*=85%, m is age-dep.** | **68.3%** | **99.2%** |

Table 4: Welfare comparison across default designs

# 6 Conclusions

Given the dramatic impact of defaults in individual DC schemes, we investigate whether and how much we can improve the welfare by changing the default design. In this paper, we explicitly model the liquid taxable account and illiquid tax-deferred DC plans. Furthermore, we carefully model the income risk, housing and medical expenditures, in order to realistically quantify and evaluate the default designs. We use the dynamic programming and the Endogenous-grid method (Carroll (2007)) to solve the extended life cycle model with two accounts.

We find potentially large economic welfare gains by following the smart age-dependent contribution rule above the current standard default design. Using the fully optimal strategies as welfare benchmark, the current default design delivers maximally 91% of welfare relative to the optimal welfare level. Whereas, the age-dependent contribution and investment default design delivers more than 99% relative to the optimal welfare level. In terms of certainty equivalent consumption, the age dependent default leads to 6% increase in annual consumption. Therefore the simple age-dependent contribution rule and appropriate investment strategies can achieve nearly optimal welfare level. Second, we find that the contribution (or saving) choice has larger impact on welfare than the portfolio choice does. The life cycle literature has mainly focused on the portfolio choices. However we show that contribution (or saving) choice is more important in welfare terms.

As to the policy implications, the age-dependent contribution and investment rules can be recommended as default. Our finding is consistent with the auto-save features encouraged by the Pension Protection Act of 2006. Retirement saving and investing play an increasingly important role in our ageing societies. We believe our user-friendly and thoughtful default saving rules resolve the worrying concerns of millions (or billions) of people. This paper contributes to the life cycle literature and the DC industry by characterizing the optimal age-or wage-dependent contribution and investment rules for DC participants.

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# A Solution method of Section 3, with TA and TDA

## A.1 Solving the retirement period

First, we solve the retirement periods ($R \leq t \leq T$). Upon retirement, the taxable savings and tax-deferred DC savings are combined into one savings, because both are freely accessible for consumption. The combined wealth is denoted by $W_R = W_R^\tau + (1-\tau^o) W_R^{DC}$. A lower tax rate $\tau^o$ is applied to the DC wealth at the retirement date. Formally, the normalized objective function and the normalized budget constraint, for the retirement period $R \leq t \leq T$, are

$$\begin{aligned} v(w_t) &= \max \frac{c_t^{1-\gamma}}{1-\gamma} + \beta E_t \left[ \left( R_{t+1}^G R_{t+1}^N \right)^{1-\gamma} v(w_{t+1}) \right] && (24) \\ s.t.\ w_{t+1} &= (w_t - c_t) \left[ R^f + \omega_t \left( \widetilde{R}_{t+1}^e - R^f \right) \right] \left( R_{t+1}^G \right)^{-1} + (1-\tau^o) ss - h_{t+1} - med_{t+1} && (25) \\ w_t^\tau &\geq c_t; \quad 0 \leq \omega_t \leq 1 && (26) \end{aligned}$$

The optimal consumption and portfolio choice $\omega_t^*(a)$, $c_t^*(a)$ and the endogenous optimal wealth $w_t^*(a)$ (for $a = \{a_j\}_{j=1}^J$) can be found by following procedure. The procedure starts by defining a new variable, $a_t = w_t - c_t$, as the after-consumption-wealth. Construct a grid of $a_t = \{a_j\}_{j=1}^J$. Now we solve for the optimal consumption and portfolio policies for each given $a_j$. The first order conditions w.r.t. $\omega_t$ and $c_t$ are

$$\begin{aligned} 0 &= \beta E_t \left[ v'(w_{t+1}) \left( \widetilde{R}_{t+1}^e - R^f \right) \left( \widetilde{R}_{t+1}^G \right)^{-\gamma} \right] && (27) \\ u'(c_t) &= \beta E_t \left[ v'(w_{t+1}) \left( R^f + \omega_t \left( \widetilde{R}_{t+1}^e - R^f \right) \right) \left( \widetilde{R}_{t+1}^G \right)^{-\gamma} \right] && (28) \end{aligned}$$

Envelope theorem implies that $u'(c_t) = v'(w_t)$, since

$$v'(w_t) = \beta E_t \left[ v'(w_{t+1}) \left( R^f + \omega_t \left( \widetilde{R}_{t+1}^e - R^f \right) \right) \left( \widetilde{R}_{t+1}^G \right)^{-\gamma} \right] \quad (29)$$

Replace $v'(w_{t+1})$ by $u'(c_{t+1})$ in the two first order conditions, we have

$$\begin{aligned} 0 &= \beta E_t \left[ u'\left( c_{t+1}^*[w_{t+1}] \right) \left( \widetilde{R}_{t+1}^e - R^f \right) \left( \widetilde{R}_{t+1}^G \right)^{-\gamma} \right] && (30) \\ c_t^*(a_t) &= I_u \left( \beta E_t \left[ u'\left( c_{t+1}^*[w_{t+1}] \right) \left( R^f + \omega_t^*(a_t) \left( \widetilde{R}_{t+1}^e - R^f \right) \right) \left( \widetilde{R}_{t+1}^G \right)^{-\gamma} \right] \right) && (31) \end{aligned}$$

where $c_{t+1}^*[w_{t+1}]$ as the optimal consumption policy at time $t+1$, and $I_u(\centerdot)$ denotes the inverse function of $u'(c_t)$. Using any numerical solver, Eq (30) will give the optimal portfolio weight $\omega_t^*(a_t)$ for any given amount of investment $a_t$. Because of the borrowing and short-selling constraints, we then impose the restriction $0 \leq \omega_t^* \leq 1$. Then, eq (31) will give the corresponding consumption $c_t^*(a_t)$ for any given amount of investment $a_t$. Finally, the optimal wealth process is endogenously determined by $w_t^* = c_t^*(a_t) + a_t$. The advantage of this method is that the numerical search is only needed once in solving $\omega_t^*(a_t)$, while $c_t^*(a_t)$ can be directly obtained from (31).

Hence we obtain the corresponding policy function $c_t^*(w_t)$ for $t \geq R$. The value obtained at time $R$ can be decompose into two terms $v(w_R) = u(w_R)K(T-R)$ where $K(t) = \frac{1}{F}(1-\exp(-Ft))$, and $F = \frac{\delta - r(1-\gamma)}{\gamma} - \frac{(1-\gamma)(\mu-r)^2}{\gamma^2\sigma^2}$.[8]

Then, we solve the working periods ($1 \leq t < R$). But before moving backward into the working period, we need to map the vector of the single state variable $\{w_R^*(j)\}_{j=1}^J$ into two state variables with $w_{i,j}^\tau = w_R(j)\frac{i}{I}$, $w_{i,j}^{DC} = w_R(j)\frac{I-i}{I}(1-\tau^o)^{-1}$, with $i = 0, 1, ..., I$. This step is because both taxable and DC savings are state variables for the working period optimization problem. In a similar way, we map the vector $\{c_R^*(j)\}_{j=1}^J$ into a matrix with $c_{ij}^* = c_R^*(j)$ for $\forall i = 0, 1, ..., I$. Hence we obtain the corresponding policy function $c_{R,i,j}^*\left(w_{R,i,j}^\tau, w_{R,i,j}^{DC}\right)$ at time $t = R$.

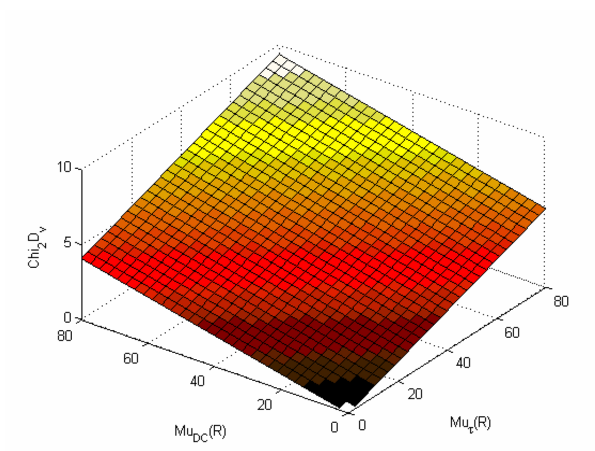


Figure: optimal consumption policy
$c_R^*\left(w_R^\tau, w_R^{DC}\right)$ at time $t = R$.

[8] After retirement, since there is no labor income nor social security available for the individual, the model is the classical Merton (1969) model. Without any portfolio constraint, the value function time time R has the following expression $v(w_R) = \frac{w^{1-\gamma}}{1-\gamma}K(T-R)$, as shown in Merton (1969) and Munk (2007). With portfolio constraint (e.g. no-borrowing constraint), Grossman and Vila (1992, proposition 3.2.) show that the value function has the expression $v(w_R) = \frac{w^{1-\gamma}}{1-\gamma}\overline{K}(T-R)$, with $\overline{K}(t) = \exp\left(r + \mu k - A\sigma^2 k^2/2\right)(1-A)t$.

## A.2 At time $t = R - 1$

At time $t = R - 1$, the individual has two accounts, TA and TDA. Therefore the normalized value function has two state variables, $w^{\tau}_{R-1}, w^{DC}_{R-1}$. The individual has to decide how much to consume, $c_{R-1}$, out of the TA wealth, and where to locate his savings among the two accounts (by choosing $m^{DC}_{R-1}$); and finally the individual has to decide the right portfolio's $(\omega^{\tau}, \omega^{DC})$ for both TA and TDA.

Formally, the normalized value function and the normalized budget constraint are

$$\varpi\left(w^{\tau}_{R-1}, w^{DC}_{R-1}\right) = \max_{c_{R-1}, \omega^{\tau}, \omega^{DC}, m^{DC}_{R-1}} \frac{c^{1-\gamma}_{R-1}}{1-\gamma} + \beta E_{R-1}\left[\left(R^G_R R^N_R\right)^{1-\gamma} v\left(w_R\right)\right] \tag{32}$$

s.t. the budget constraint

$$\begin{aligned}
w_R &= w^{\tau}_R + \left(1 - \tau^o\right) w^{DC}_R && (33)\\
w^{\tau}_R &= \left(w^{\tau}_{R-1} - c_{R-1} - (1-\tau^y) m^{DC}_{R-1}\right)\left[R^f + \omega^{\tau}_{R-1}\left(\widetilde{R}^e_R - R^f\right)\right]\left(R^G_R R^N_R\right)^{-1} + (1-\tau^o) ss - h_R && (34)\\
w^{DC}_R &= \left(w^{DC}_{R-1} + m^{DC}_{R-1}\right)\left[R^f + \omega^{DC}_{R-1}\left(\widetilde{R}^e_R - R^f\right)\right]\left(R^G_R R^N_R\right)^{-1} && (35)
\end{aligned}$$

and the non-negative constraint (i.e. no borrowing) in both accounts

$$w^{\tau}_{R-1} - c_{R-1} - (1-\tau^y) m^{DC}_{R-1} \geq 0, \text{ and } m^{DC}_{R-1} \geq 0 \tag{36}$$

Notice that the value function is changed from with one state variable, $v\left(w_R\right)$, to the value function with two state variables $\varpi\left(w^{\tau}_{R-1}, w^{DC}_{R-1}\right)$.

Follow Carroll's idea, we define two new wealth variables, namely the amount available for investment in the taxable account $a^{\tau}_{R-1} = w^{\tau}_{R-1} - c_{R-1} - (1-\tau^y) m^{DC}_{R-1}$, and the amount available for investment in the DC account $a^{DC}_{R-1} = w^{DC}_{R-1} + m^{DC}_{R-1}$. We then choose a non-negative 1-D grid to discretize $a^{\tau}_{R-1} = \{a^{\tau}_j\}^J_{j=1} \geq 0$, and do the same for $a^{DC}_{R-1} = \{a^{DC}_h\}^H_{h=1} \geq 0$.

### A.2.1 Optimize portfolio's

The first step is to compute the optimal portfolio strategies for both accounts. The first order conditions w.r.t. $\omega^{\tau}_{R-1}$ and $\omega^{DC}_{R-1}$ are

$$
\begin{aligned}
0 &= \beta E_{R-1}\left[u'\left(w_R\left(a_j^{\tau},a_h^{DC}\right)\right)\left(\widetilde{R}_R^{e}-R^{f}\right)\left(R_R^{G}R_R^{N}\right)^{-\gamma}\right] && (37)\\
0 &= \beta E_{R-1}\left[u'\left(w_R\left(a_j^{\tau},a_h^{DC}\right)\right)\left(\widetilde{R}_R^{e}-R^{f}\right)\left(R_R^{G}R_R^{N}\right)^{-\gamma}\right](1-\tau^{o}) && (38)
\end{aligned}
$$

Notice that the portfolio's are function of $a^{\tau}$ and $a^{DC}$. We need to determine $\omega_{R-1}^{\tau}$ and $\omega_{R-1}^{DC}$ for each combination of $\left(a_j^{\tau},a_h^{DC}\right)$. One special case is $\omega_{R-1}^{\tau}=\omega_{R-1}^{DC}=\omega_{R-1}$. Assuming $\omega_{R-1}^{\tau}=\omega_{R-1}^{DC}=\omega_{R-1}$ for any combination of $\{a_j^{\tau},a_h^{DC}\}$. First, for a given vale of investable wealth in TA and TDA, we simulate the next period total wealth for certain portfolio choice $\omega$ :

$$
\begin{aligned}
\widetilde{w}_R^{\tau}\left(a_j^{\tau}\right) &= a_j^{\tau}\left[R^{f}+\omega\left(\widetilde{R}_R^{e}-R^{f}\right)\right]\left(R_R^{G}R_R^{N}\right)^{-1}+(1-\tau^{o})ss-h_R && (39)\\
\widetilde{w}_R^{DC}\left(a_h^{DC}\right) &= a_h^{DC}\left[R^{f}+\omega\left(\widetilde{R}_R^{e}-R^{f}\right)\right]\left(R_R^{G}R_R^{N}\right)^{-1} && (40)
\end{aligned}
$$

The optimal portfolio $\omega_{R-1}^{*}$ is the one that solves the following equation based on the FOC w.r.t $\omega$

$$
\begin{aligned}
0 &= \beta E_{R-1}\left[v'\left(\widetilde{w}_R\right)\left(\widetilde{R}_R^{e}-R^{f}\right)\left(R_R^{G}R_R^{N}\right)^{-\gamma}\right] && (41)\\
&= \beta E_{R-1}\left[u'\left(c_R\left(\widetilde{w}_R^{\tau},\widetilde{w}_R^{DC}\right)\right)\left(\widetilde{R}_R^{e}-R^{f}\right)\left(R_R^{G}R_R^{N}\right)^{-\gamma}\right] && (42)
\end{aligned}
$$

where $v'\left(\widetilde{w}_R\right)=u'\left(c_R\left(\widetilde{w}_R^{\tau},\widetilde{w}_R^{DC}\right)\right)$, and $c_R\left(\widetilde{w}_R^{\tau},\widetilde{w}_R^{DC}\right)$ is obtained by interpolating the previously constructed policy function $c_{R,i,j}^{*}\left(w_{R,i,j}^{\tau},w_{R,i,j}^{DC}\right)$.

### A.2.2 Optimize consumption

The second step is to calculate the optimal consumption for each combination of $\{a_j^{\tau},a_h^{DC}\}$. We know the first order conditions w.r.t. $c_{R-1}$ is

$$
\begin{aligned}
u'\left(c_{R-1}\right) &= \beta E_{R-1}\left[v'\left(\widetilde{w}_R\right)\left(R^{f}+\omega_{R-1}^{*}\left(\widetilde{R}_R^{e}-R^{f}\right)\right)\left(R_R^{G}R_R^{N}\right)^{-\gamma}\right] && (43)\\
&= \beta E_{R-1}\left[u'\left(c_R\left(\widetilde{w}_R^{\tau},\widetilde{w}_R^{DC}\right)\right)\left(R^{f}+\omega_{R-1}^{*}\left(\widetilde{R}_R^{e}-R^{f}\right)\right)\left(R_R^{G}R_R^{N}\right)^{-\gamma}\right] && (44)
\end{aligned}
$$

Therefore, we compute $c_{R-1}^{*}$ directly for each underlying $\left(a_j^{\tau},a_h^{DC}\right)$ as

$$
c_{R-1}^{*}\left(a_j^{\tau},a_h^{DC}\right)=\left\{\beta E_{R-1}\left[u'\left(c_R\left(\widetilde{w}_R^{\tau},\widetilde{w}_R^{DC}\right)\right)\left(R^{f}+\omega_{R-1}^{*}\left(\widetilde{R}_R^{e}-R^{f}\right)\right)\left(R_R^{G}R_R^{N}\right)^{-\gamma}\right]\right\}^{-1/\gamma} \quad (45)
$$

Define a new variable called before-consumption wealth $b_{R-1} = a^{\tau} + c^{*}_{R-1} = w^{\tau}_{R-1} - (1 - \tau^{y})m^{DC}_{R-1}$. Now we can generate an endogenous 2-D grid for $b_{R-1}$ using

$$b^{*}_{R-1}\left(a^{\tau}_{j}, a^{DC}_{h}\right) = a^{\tau}_{j} + c^{*}_{R-1}\left(a^{\tau}_{j}, a^{DC}_{h}\right). \tag{46}$$

In addition, we can evaluate the expected utility of the next period, for the given value of $\{a^{\tau}_{j}, a^{DC}_{h}\}$, as

$$\begin{aligned} EV_{R-1}\left(a^{\tau}_{j}, a^{DC}_{h}\right) &= \beta E_{R-1}\left[\left(R^{G}_{R}R^{N}_{R}\right)^{1-\gamma} v\left(w_{R}\right)\right] && (47) \\ &= \beta E_{R-1}\left[\left(R^{G}_{R}R^{N}_{R}\right)^{1-\gamma} u\left(w_{R}\right)\right] K\left(T-R\right) && (48) \end{aligned}$$

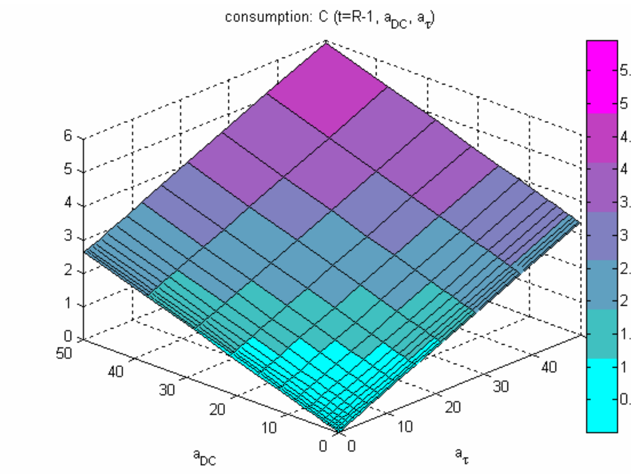


Figure: Consumption as function of investable wealth $c^{*}_{R-1}\left(a^{\tau}, a^{DC}\right)$ at time $R-1$.

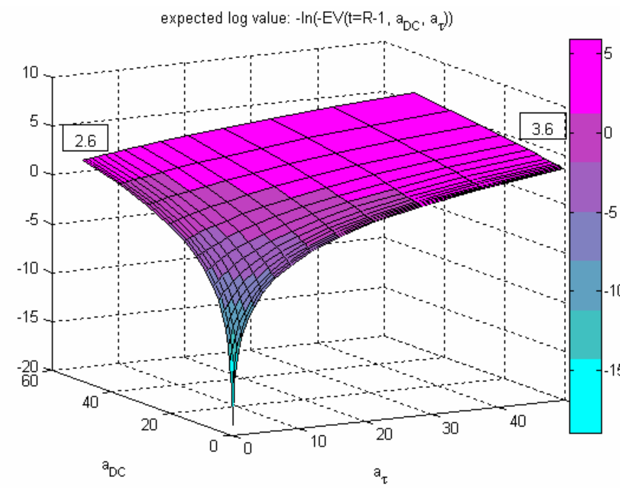


Figure: The expected value at time $R-1$, $EV_{R-1}\left(a^{\tau}, a^{DC}\right)$.

### A.2.3 Optimize contribution rate

The third step is to compute the optimal contribution rate $i)$ $m_{R-1} \geq 0$ or $ii)$ $1 \geq m_{R-1} \geq 0$. Case i) says that borrowing or premature withdraw is not allowed, but there is no upper limit on contributions to DC plans. Case ii) adds an upper limit on contribution rate, i.e. the contribution is no larger than 100% of gross salary income. Let's consider Case ii) first.[9]

[9] Case i) means $m^{\max} = \min\left(1, \widehat{w}^{\tau}/(1-\tau^{y})\right)$; Case ii) means $m^{\max} = \widehat{w}^{\tau}/(1-\tau^{y})$ in the calculation procedure below.

FOC condition w.r.t. $m_{R-1}$ can not solve the optimal contribution rate.[10] We need to use the value function itself to search the optimal contribution rate numerically. First construct two exogenous grids for possible values of wealth $\widehat{w}^{\tau} = \{W_n\}_{n=1}^{N}$ and $\widehat{w}^{DC} = \{W_n\}_{n=1}^{N}$. Then, we construct a grid of possible contribution rates. We start with the case without employer match. The grid of possible contribution rates is denoted by $\widehat{m}_{R-1} = \{m_i\}_{i=1}^{M} \subset [0, m^{\max}]$, with $m^{\max} = \min\left(1, \widehat{w}^{\tau}/(1-\tau^y)\right)$. These implies a set of before-consumption wealth $\widehat{b}$ and investable wealth $\widehat{a}^{DC}$ for any given combination of $\{\widehat{w}^{\tau}, \widehat{w}^{DC}, \{m_i\}_{i=1}^{M}\}$, as

$$\widehat{b}_i = \widehat{w}^{\tau} - (1-\tau^y)\widehat{m}_i > 0 \tag{49}$$

$$\widehat{a}_i^{DC} = \widehat{w}^{DC} + \widehat{m}_i \tag{50}$$

With employer match, e.g. $m_t^{DC} = \min\left(m_t + \min\left(m_t, 6\%\right), 20\%\right)$, the grid of possible contribution rates remains the same, but the implied $\widehat{b}$ and $\widehat{a}^{DC}$ become

$$\widehat{b}_i = \widehat{w}^{\tau} - (1-\tau^y)\widehat{m}_i > 0 \tag{51}$$

$$\widehat{a}_i^{DC} = \widehat{w}^{DC} + \widehat{m}_i^{DC} \tag{52}$$

If $\widehat{b}_i$ and $\widehat{a}_i^{DC}$ are known, then with the help of the interpolation relation $b_{R-1}^{*}\left(a^{\tau}, a^{DC}\right)$ we can back out the corresponding private wealth $\widehat{a}_i^{\tau}$. It then leads to the optimal consumption $\widehat{c}_i^{*} = \widehat{b}_i - \widehat{a}_i^{\tau}$, for any given set of $\{\widehat{w}^{\tau}, \widehat{w}^{DC}, m_i\}$. We denote the implied consumption as $c_i^{*}\left(\widehat{w}^{\tau}, \widehat{w}^{DC}, m_i\right)$. Furthermore, we can compute the expected value of the next period $\widehat{EV}_i\left(\widehat{a}_i^{\tau}, \widehat{a}_i^{DC}\right)$ based on relation $EV_{R-1}\left(a^{\tau}, a^{DC}\right)$. Finally, we can evaluate the trade-off between the consumption and the contribution using the recursive objective function as

$$\varpi_{R-1}(\widehat{w}_{R-1}^{\tau}, \widehat{w}_{R-1}^{DC}) \equiv \max_{\{m_i\}_{i=1}^{M}} \frac{\left(\widehat{c}_i^{*}\right)^{1-\gamma}}{1-\gamma} + EV_{R-1}\left(\widehat{a}_i^{\tau}, \widehat{a}_i^{DC}\right) \tag{53}$$

The optimal contribution rate is the one that maximize the above expression for given values of $(\widehat{w}_{R-1}^{\tau}, \widehat{w}_{R-1}^{DC})$. Let's denote it as $m^{*}(\widehat{w}^{\tau}, \widehat{w}^{DC})$. As a by-product, we also get the corresponding consumption policy $c^{*}(w^{\tau}, w^{DC}) = c^{*}\left(\widehat{w}^{\tau}, \widehat{w}^{DC}, m^{*}(\widehat{w}^{\tau}, \widehat{w}^{DC})\right)$.

[10] FOC w.r.t. $m_{R-1}^{DC}$ is

$$(1-\tau^y)\beta E_{R-1}\left[\left(R_R^G R_R^N\right)^{-\gamma} v'\left(w_R\right)\widetilde{R}_R^{P,\tau}\right] = (1-\tau^o)\,\beta E_{R-1}\left[\left(R_R^G R_R^N\right)^{-\gamma} v'\left(w_R\right)\widetilde{R}_R^{P,DC}\right]$$

which clearly doesn't hold in general. Since $\tau^y > \tau^o$, this FOC implies marginal cost of contribution < martinal benefit of contribution, therefore $m_{R-1}^{DC}$ should take some maximum value (if exists). However the danger of doing so is that it implies $w_{R-1}^{\tau}$ might be unlimited. Due to this reason, we need to resort to value function to find the optimal $m_{R-1}^{DC}$ for any given level of $\left\{w^{\tau}, w^{DC}\right\}$.

### A.2.4 Euler equation

Finally, the Envelope theorem tells us

$$
\begin{aligned}
\varpi'_\tau\left(w^\tau_{R-1}, w^{DC}_{R-1}\right) &= \beta E_{R-1}\left[\left(R^G_R R^N_R\right)^{-\gamma} v'\left(w_R\right)\left[R^f + \omega_{R-1}\left(\widetilde{R}^e_R - R^f\right)\right]\right] && (54)\\
&= u'\left(c^*_{R-1}\left(w^\tau_{R-1}, w^{DC}_{R-1}\right)\right) && (55)\\
\varpi'_{DC}\left(w^\tau_{R-1}, w^{DC}_{R-1}\right) &= \left(1-\tau^o\right)\beta E_{R-1}\left[\left(R^G_R R^N_R\right)^{-\gamma} v'\left(w_R\right)\left[R^f + \omega_{R-1}\left(\widetilde{R}^e_R - R^f\right)\right]\right] && (56)\\
&= \left(1-\tau^o\right) u'\left(c^*_{R-1}\left(w^\tau_{R-1}, w^{DC}_{R-1}\right)\right) && (57)
\end{aligned}
$$

where $\varpi'_\tau$ and $\varpi'_{DC}$ denote the partial differentials.

### A.2.5 Optimal policies for $t = R-1$

For the last year of working, $t = R-1$, the optimal contribution rate is largely 100% of the labor income, except for when TA wealth ($w^\tau$) is very limited, but the TDA wealth ($w^{DC}$) is abundant. The optimal consumption increases with TA and TDA wealth in general. The special feature for the optimal consumption is that there are kinks due to the liquidity constraint, i.e. individual can not consume more than what they have in the taxable account. Similarly there are also kinks for the value function. Except for this, the value function increases with both wealth accounts.

### A.2.6 Repeat the procedure

Now we are able to proceed to $t = R-2, R-3, ....1$, by repeating the similar procedure as for $t = R-1$. To generate the average pattern of life cycle portfolio holding, as in Figure 1, we simulate the model from time 1 to T for 10,000 scenario's, and take the average over all simulated scenario's.

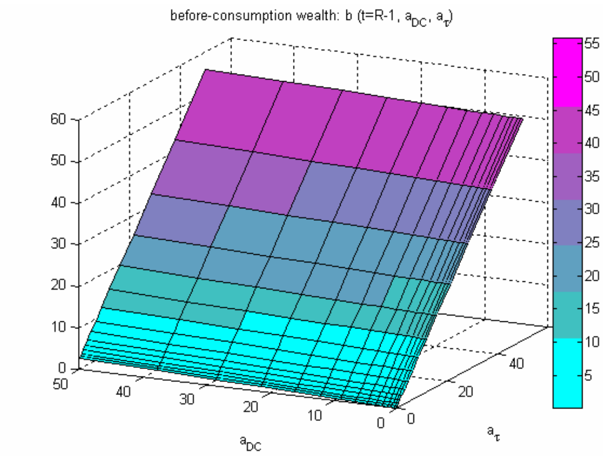


Figure: The implied before-consumption wealth $b$ as function of investable wealth $b_{R-1}\left(a^{\tau}, a^{DC}\right)$ at time $R-1$.

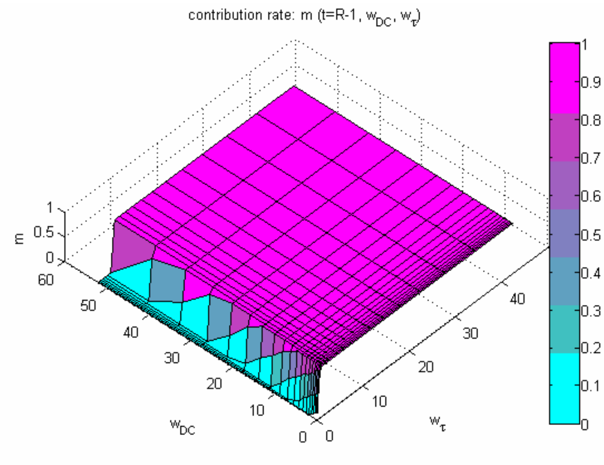


Figure: The optimal contribution rate $m^*_{R-1}\left(w^{\tau}, w^{DC}\right)$ at time $R-1$.

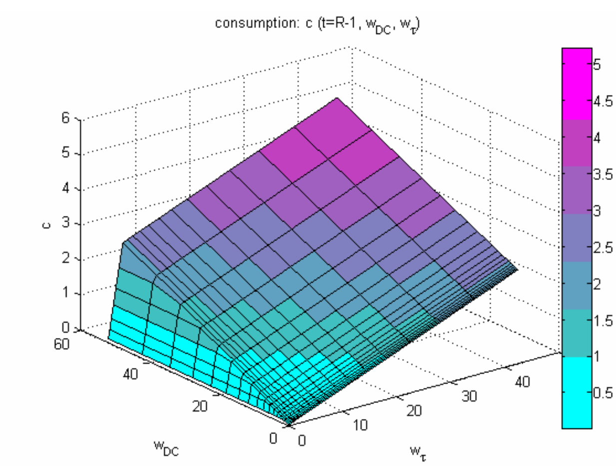


Figure: The optimal consumption policy $c^*_{R-1}\left(w^{\tau}, w^{DC}\right)$.

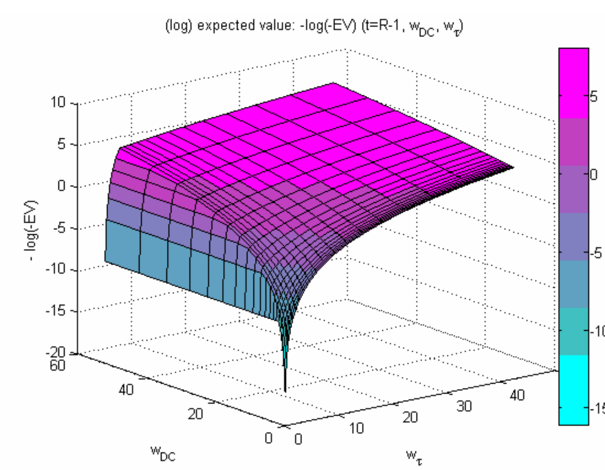


Figure: (log) value function -ln$(-\varpi(w^{\tau}, w^{DC}))$ at time R-1.

# B Solution method of Section 4

## B.1 Default #1

In Default #1, the contribution rate $m \geq 0$ and portfolio allocations $1 \geq \omega \geq 0$ are chosen at the beginning of the career ($t = 0$), and are fixed throughout life time. Notice that we the portfolio allocation for both accounts are assumed to be the same constant mix through out life time, $\omega^{\tau} = \omega^{DC} = \omega$. Therefore, the value functions, for given the chosen level of $m$ and $\omega$, are denoted as $\varpi\left(w_t^{\tau}, w_t^{DC} \mid \omega, m\right)$ for working period, and $v\left(w_t \mid \omega, m\right)$ for retirement period. We solve the model numerically using dynamic programming.

For the final period, the optimal consumption policy is to consume everything $c_T^* = w_T$. The corresponding value function is given by $v\left(w_T \mid \omega, m\right) = \frac{\left(c_T^*\right)^{1-\gamma}}{1-\gamma}$. Then we proceed to time $t = T-1$. During the retirement period ($R \leq t \leq T-1$), the normalized value function (in recursive form) and wealth process are

$$v\left(w_t \mid \omega, m\right) = \frac{c_t^{1-\gamma}}{1-\gamma} + \beta E_t\left[\left(R_{t+1}^G\right)^{1-\gamma} v\left(w_{t+1} \mid \omega, m\right)\right] \tag{58}$$

subject to the budget dynamics

$$w_{t+1} = \left(w_t - c_t\right) \widetilde{R}_{t+1}^{P,\omega} \left(R_{t+1}^G\right)^{-1} + \left(1-\tau\right) ss_{t+1} - h_{t+1} - med_{t+1} \tag{59}$$

where we use a shorthand notation $\widetilde{R}_{t+1}^{P,\omega} = R^f + \omega\left(\widetilde{R}_{t+1}^e - R^f\right)$ for the portfolio returns.

For any given value of $m$ and $\omega$, we only need to optimize the consumption choice $c_t$.

The FOC w.r.t. $c_t$ is

$$u'\left(c_t\right) = \beta E_t\left[v'\left(w_{t+1} \mid \omega, m\right) \widetilde{R}_{t+1}^{P,\omega} \left(\widetilde{R}_{t+1}^G\right)^{-\gamma}\right] \tag{60}$$

The Envelope theorem gives

$$v'\left(w_t \mid \omega, m\right) = \beta E_t\left[v'\left(w_{t+1} \mid \omega, m\right) \widetilde{R}_{t+1}^{P,\omega} \left(R_{t+1}^G\right)^{-\gamma}\right] = u'\left(c_t\right) \tag{61}$$

So, pushing one period ahead, we have $v'\left(w_{t+1} \mid \omega, m\right) = u'\left(c_{t+1}\right).$

Define a new variable, $a_t = w_t - c_t$, as the after-consumption-wealth. Construct a grid of $a_t = \{a_j\}_{j=1}^J$. Now we solve for the optimal consumption and portfolio policies for each given $a_j$.

$$\begin{aligned} c_t &= I_u\left(\beta E_t\left[v'\left(w_{t+1} \mid \omega, m\right) \widetilde{R}_{t+1}^{P,\omega} \left(\widetilde{R}_{t+1}^G\right)^{-\gamma}\right]\right) && (62) \\ &= I_u\left(\beta E_t\left[u'\left(c_{t+1}^*[w_{t+1}]\right) \widetilde{R}_{t+1}^{P,\omega} \left(\widetilde{R}_{t+1}^G\right)^{-\gamma}\right]\right) && (63) \end{aligned}$$

Following the EGM, the optimal wealth process is endogenously determined by $w_t^* = c_t^*\left(a_t\right) + a_t$, for a given value of $\{m, \omega\}$.

At time $t = R$, we split the single wealth variable $w_R^*$ into two wealth variables $w_R^\tau$ and $w_R^{DC}$, and obtain the corresponding policy function $c_R^*\left(w_R^\tau, w_R^{DC}\right)$, as done in Appendix B.

During the working period ($0 \leq t \leq R-1$), the normalized value function (in recursive form) and wealth process are

$$\varpi\left(w_t^\tau, w_t^{DC} \mid \omega, m\right) = \frac{c_t^{1-\gamma}}{1-\gamma} + \beta E_t\left[\left(R_{t+1}^G\right)^{1-\gamma} \varpi\left(w_{t+1}^\tau, w_{t+1}^{DC} \mid \omega, m\right)\right]$$

s.t. the wealth dynamics and no borrowing constraint as

$$\begin{aligned}
w_{t+1}^\tau &= \left(w_t^\tau - c_t - (1-\tau^y)m\right)\widetilde{R}_{t+1}^{P,\omega}\left(R_{t+1}^G R_{t+1}^N\right)^{-1} + (1-\tau^y) - h_{t+1} - med_{t+1} && (64)\\
w_{t+1}^{DC} &= \left(w_t^{DC} + m^{DC}\right)\widetilde{R}_{t+1}^{P,\omega}\left(R_{t+1}^G R_{t+1}^N\right)^{-1} && (65)\\
0 &\leq w_t^\tau - c_t - (1-\tau^y)m && (66)
\end{aligned}$$

Construct two investable wealth grids $a_t^\tau \equiv (w_t^\tau - c_t - (1-\tau^y)m) = \{a_j^\tau\}_{j=1}^J \geq 0$, and $a_t^{DC} \equiv \left(w_t^{DC} + m^{DC}\right) = \{a_h^{DC}\}_{h=1}^H \geq 0$. Then calculate the optimal consumption for each combination of $\{a_j^\tau, a_h^{DC}\}$. We know the first order conditions w.r.t. $c_t$ is

$$\begin{aligned}
u'(c_t) &= \beta E_t\left[v'\left(\widetilde{w}_{t+1}\right)\widetilde{R}_{t+1}^{P,\omega}\left(R_{t+1}^G R_{t+1}^N\right)^{-\gamma}\right] && (67)\\
&= \beta E_{R-1}\left[u'\left(c_{t+1}\left(\widetilde{w}_{t+1}^\tau, \widetilde{w}_{t+1}^{DC}\right)\right)\widetilde{R}_{t+1}^{P,\omega}\left(R_{t+1}^G R_{t+1}^N\right)^{-\gamma}\right] && (68)
\end{aligned}$$

Therefore, we compute $c_t^*$ directly for each underlying $\left(a_j^\tau, a_h^{DC}\right)$ as

$$c_t^*\left(a_j^\tau, a_h^{DC}\right) = \left\{\beta E_t\left[u'\left(c_{t+1}\left(\widetilde{w}_{t+1}^\tau, \widetilde{w}_{t+1}^{DC}\right)\right)\widetilde{R}_{t+1}^{P,\omega}\left(R_{t+1}^G R_{t+1}^N\right)^{-\gamma}\right]\right\}^{-1/\gamma} \tag{69}$$

Following the EGM, the optimal wealth process is endogenously determined by $w_t^{*,\tau} = c_t^*\left(a_t^\tau, a^{DC}\right) + a_t^\tau + (1-\tau^y)m$ and $w_t^{*,DC} = a^{DC} - m^{DC}$, for a given value of $\{m, \omega\}$.

Repeat the above procedure for all values of $\{m, \omega\}$. Finally, the optimal $\{\omega^*, m^*\}$ are the ones that maximize the value function

$$\{\omega^*, m^*\} = \arg\max \varpi\left(w_0^\tau, w_0^{DC} \mid \omega, m\right)$$