Valuation of Contingent Pension Liabilities and Implementation of Conditional Indexation

Renxiang Dai* Hans Schumacher†

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Abstract

We formulate the circularity problem that may arise in the valuation of conditionally indexed pension liabilities. Namely, the funding ratio determines the indexation level through a chosen indexation rule (often known as a "policy ladder"), but at the same time the indexation level may, with market valuation of pension liabilities, have a feedback effect on the liability value and in turn on the funding ratio. We develop a backward recursion approach to the valuation of liabilities subject to the circularity constraint. Numerical examples are used to show the impact of investment strategies and indexation rules on the liability value. The current practice of conditional indexation uses as the basis for indexation decisions a proxy of funding ratio, rather than the funding ratio based on market-based valuation, and in this way avoids the circularity problem. Our findings show that the proxy of funding ratio may be misleading in assessing the actual financial status of pension funds, and for this purpose the actual funding ratio needs to be computed and used.

*Department of Econometrics and Operations Research, Tilburg University and Netspar, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: r.dai@uvt.nl

†Department of Econometrics and Operations Research, Tilburg University and Netspar, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: j.m.schumacher@uvt.nl

INCOME IS A SERIES OF EVENTS.
——Irving Fisher, *The Theory of Interest* (1937)

1 Introduction

In the insurance and pension industries, some contracts are formulated in such a way that the payoff to beneficiaries is linked to the financial status of insurers or pension funds. In insurance, a typical example is given by participating life insurance policies (also known as with-profits policies), which provide a guaranteed return and allow beneficiaries to participate in the profit of a life insurance company through a profit-sharing scheme. Participating contracts make up a significant part of the life insurance market in many industrial countries. In the field of pensions, there exists a similar practice known as *conditional indexation*, which links the pension benefits to the financial position of a pension fund. The practice of conditional indexation has been adopted by many pension funds in the Netherlands in recent years, and its introduction is under discussion in the UK.

The way in which a conditional indexation scheme operates can be best illustrated by an example. Let the financial position of a pension fund be measured by its *funding ratio*, i. e. the ratio of pension asset value to pension liability value. The level of compensation for inflation (indexation) applied in a certain year is determined according to a rule of the following form: (i) if the funding ratio is below some threshold (e. g. 110%), there is no indexation to inflation; (ii) if the funding ratio is above some upper threshold (e. g. 140%), the pension rights will be fully indexed to indexation; (iii) if the ratio is in between, some intermediate level of indexation will apply. The rule that links the funding ratio to the indexation decision is referred to as a "policy ladder". Several large pension funds in the Netherlands have published such policy ladders.

Against the backdrop of the international move promoted by the International Accounting Standards Board (IASB) towards the market-based, fair value accountancy standard, the valuation of these contracts contingent on the financial position of insurers and pension funds have become a subject of increasing interest both to academics and practitioners. For participating life insurance contracts, the valuation has been analyzed in the contingent claim pricing framework by a number of recent papers, for example, Aase and Persson [1997], Grosen and Jørgensen [2002], Bauer, Kiesel, Kling, and Ruß [2006], Ballotta, Haberman, and Wang [2006], and Gatzert and Kling [2007]. In this paper, we consider the valuation of conditionally indexed pension liabilities in the same framework.

The conditional indexation of pension benefit may lead to some interesting and challenging issues in the market valuation of pension liabilities. First of all, the value of liabilities and the funding ratio have to be determined *simultaneously* for conditional indexation schemes. The reason is that the indexation level for the current year depends on the funding ratio via a policy ladder, and at the same time, the indexation level has a feedback effect on the liability value and in turn on the funding ratio through market valuation. In other words, a *circularity* problem arises in the valuation of conditionally indexed pension liabilities:

$$\boxed{\textbf{Funding ratio} = \tfrac{\text{Asset}}{\text{Liability}}} \;\underset{\textit{conditional indexation}}{\overset{\textit{market valuation}}{\longleftrightarrow}}\; \boxed{\textbf{Indexation level}}$$

Therefore, the determinations of the indexation level, liability value, and funding ratio are interdependent, and hence need to be addressed simultaneously through looking for a coherent solution in the sense that the indexation level and the funding ratio are consistent with each other with respect to the policy ladder and the market valuation approach.

Another complication arises from the *intertemporal* dimension of market valuation of pension liabilities. The lion's share of pension liabilities is from pension rights payable in the future. Under conditional indexation, the future pension payments can be thought of as collar-structured contingent payoffs: the payoffs are subject to floors and caps determined by minimum and maximum indexation, and contingent on the future funding ratio. In principle, one can reach a market valuation of the liabilities using contingent payoff pricing techniques developed along the lines of Black and Scholes [1973] and Merton [1973]. A complication resulting from conditional indexation is that *pension payments to be made at different time points are inter-dependent.* To see the intertemporal dependence, consider a pension fund that is to make pension payment at only two future dates (i. e. there are two European options with different expiry dates). The intertemporal dependence may arise in two ways. Firstly, the first pension payment, which can be said to be the actual payoff of the first option in the language of financial derivatives, will reduce the pension asset value. As a consequence, it has an impact on the future funding ratio, in terms of which the second payment (or the second option) is defined. Therefore the actual payoff of the first option has an impact on the *underlying* of the second, and in turn an impact on the valuation of the second. Due to the intertemporal dependence, these two options need to be considered together for their valuation, rather than one by one as the usual case of pricing a portfolio of options where the underlying dynamics is (or is assumed to be) not affected by the exercise of consisting options.

The second source of the intertemporal dependence is the practice that once an indexation level is granted in a certain year, it cannot be canceled in later years. For example, if participants are granted full indexation in 2005 and retirees receive pension benefits increased by the inflation, then the minimum amount of pension in 2006 is the amount paid in the previous year whatever the indexation level is decided for 2006 (assuming positive inflation). In the language of options, the irrevocability of conditional indexation leads to a case where the actual payoff of earlier options affects the *parameters* characterizing later options (the lower bound of pension rights in this context). In other words, the parameters which define the options are specified by some function (mechanism), but not specified as a known number as in most cases in options pricing.

From the perspective of the balance sheet of pension funds, the intertemporal dependence from the underlying can be referred to as the intertemporal dependence of the *asset* side, and the dependence from the parameters as the intertemporal dependence of the *liability* side. In comparison with the intertemporal dependence of assets, the dependence of liabilities results in stronger path-dependence in option pricing, and hence leads to a new dimension of complication in the value of pension liabilities as will be shown below.

The circularity problem arises from our assumption that the funding ratio used for indexation decision is computed using the true liability value. By this assumption, we actually impose the requirement that the funding ratio used for indexation decisions is also reliable as an indicator of the financial soundness of pension funds. For this reason, we refer to as the *consistent* implementation of conditional indexation the case where the funding ratio used in indexation decisions is based on the true value of liabilities.

A way to avoid solving circularity problems is to replace the true value of liabilities (including the value of the indexation options) by a value that is easier to compute, such as the value of the liabilities without indexation, and to assume that indexation will be based on the funding ratio as computed from that value. Current practice at pension funds implementing conditional indexation is actually of this type. Similar assumptions are made in the recent papers Nijman and Koijen [2006] and de Jong [2008] which address the valuation of the liabilities of conditional indexation schemes. Both papers focus on the inflation risk stemming from the link to inflation which is uncertain, while abstracting from the circularity problem and intertemporal dependence inherent in this type of schemes. Similar to the current practice of conditional indexation, profit-sharing decisions in participating life insurance contracts are based on the *book* value, rather than the market value, of such contracts, and hence there does not exist the circularity problem either. So in the analysis of participating contracts as carried out in the papers mentioned earlier, no circularity problems need to be solved.

The way in which a proxy of funding ratio, rather than that based on the true value of liabilities, is used for indexation decisions is labeled as the *proxy-based* implementation of conditional indexation in this paper. Though circumventing the circularity problem, this type of implementation leads to another complication. The funding ratio proxy assuming a fixed indexation level may not reflect the financial soundness of pension funds from the perspective of contingent claims pricing, since actually the indexation levels to be decided in the future are contingent and varied. To know the actual financial profile, the actual liability value and the actual funding ratio are needed. Therefore the implementation using a proxy calls for two funding ratios in the system: a proxy for indexation decisions, and the *actual* funding ratio for measuring the financial soundness. In the absence of the circularity problem, the actual values can be obtained by applying classical option pricing techniques. As will be shown below, for a given funding ratio proxy, the actual funding differs because the investment policies of pension funds differ, because the policy ladders differ, and because the demographic composition differs. It may be a point of concern from the regulatory perspective, if only the funding ratio proxy assuming a fixed indexation level is reported.

Legitimately, the very fundamental idea underlying conditional indexation is that a pension fund pays more if when it possesses more *vis-à-vis its liability*. Because a fixed-indexation proxies presume indexation to constant, but the actual indexation will be contingent and varied, *any* fixed-indexation proxy is not a right measure how much a fund possesses vis-à-vis its liability. Therefore the proxy-based implementation in principle cannot ensure that this fundamental idea is actually enforced, and it allows the scenario that a pension pays more even when it has less relative to its liability. Given the inconsistency between the intended idea and the actual implementation, one may ask what consequences the proxy-based implementation may have?

We proceed as follows. Section 2 gives a general formulation of valuation of pension liabilities with *continuous-time* indexation decisions, which enables one to learn the different nature of computation required to value conditionally indexed pension liabilities in two different types of implementation: the consistent implementation, and the proxy-based implementation. In order to draw more concrete conclusions, a more specific and stylized model with *discrete-time* indexation decisions is introduced in Section 3; this model is the basis of the investigation on liabilities valuation in Sections 4 and 5. Assuming the consistent implementation, Section 4 proposes a backward recursion approach to the market valuation of pension liabilities, and illustrates the findings by numerical examples. In the case of the proxy-based implementation, Section 5 shows how the discrepancy between the funding ratio proxy and the actual funding ratio depends on investment policies and policy ladders. Some concluding observations are in Section 6. For pension readers who are more interested in the implementation of conditional indexation and valuation issues than in the nature of computation in this study, skipping Section2 and proceeding directly to Section 3 lead to no loss of continuity.

2 The mathematical formulation for continuous-time indexation

In this section, we aim to derive a general formulation of the valuation of pension liabilities in a setting where conditional indexation decisions are made in continuous time and there is a continuum of generations. Each generation is labeled by the time s for which it has been in the system. When a generation enters the system, it is entitled to an initial pension rate (the pension amount per unit of time), $J(s)$, which is subject to updating through indexation to inflation. $J(\cdot)$ is a known function defined on the interval $[0, D]$, where D is the number of years that a generation spends in the system. We refer to as "indexation status", $x_t(s)$, the *cumulative* result at time t of the indexation history. That is, $x_t(s)$ is the ratio of the accrued pension rate at time t to the original pension rate $J(s)$. It is a positive function which satisfies $x_t(0) = 1$ (no indexation backlog at the time of entering the system). Define $X_t(s) = \log x_t(s)$, and the infinitely dimension function $X_t(\cdot)$ is referred to as the *indexation profile* at time t. it will be used as a state variable in the following state-space formulation.

The other state variable that will be used is the value of asset at time t, A_t. The evolution of asset value is given by the stochastic differential equation

$$dA_t = \mu A_t \, dt + \sigma A_t \, dW_t - dB_t \tag{1}$$

where $\{B_t\}$ is the cumulative process of benefits paid. The cumulative payment process is taken to satisfy

$$dB_t = \Big(\int_R^D \exp(X_t(s)) J(s) \, ds \Big) dt, \tag{2}$$

where it is assumed that generations receive benefits after spending R years in the system.

The evolution of the indexation profile depends on whether the true funding ratio or a proxy is used for indexation decisions, and we first consider the system using the true funding ratio, i. e. the consistent implementation.

2.1 The consistent implementation

Since indexation decisions are made on the basis of the true liability value, we characterize the policy ladder by a function $p(A_t, L_t)$ where L_t denotes the value of the liabilities at time t, which will be defined later. By definition of the indexation profile, we can write

$$X_{t+\Delta t}(s+\Delta t) = X_t(s) + \log p(A_t, L_t)\,\Delta t + o(\Delta t).$$

By a standard Taylor expansion, assuming sufficient smoothness of $X_t(s)$ as a function of s, we have

$$X_{t+\Delta t}(s+\Delta t) = X_{t+\Delta t}(s) + \frac{\partial X_t}{\partial s}(s)\,\Delta t + o(\Delta t).$$

Therefore we can write

$$X_{t+\Delta t}(s) - X_t(s) = -\frac{\partial X_t}{\partial s}(s)\,\Delta t + \log p(A_t, L_t)\,\Delta t + o(\Delta t).$$

This motivates the following stochastic differential equation for the evolution of the indexation profile:

$$dX_t(s) = -\Big(\frac{\partial X_t}{\partial s}(s) - \log p(A_t, L_t)\Big)\,dt. \tag{3}$$

The market value of the liabilities at time t is defined by

$$L_t = Z_t E_t^{\mathbb{Q}} \int_t^T Z_\tau^{-1}\, dB_\tau \tag{4}$$

where Z_t is the price of a numéraire, $\mathbb{Q}$ is the martingale measure corresponding to the chosen numéraire, and T is the time of liquidation. Since the state variables A_t and $X_t(\cdot)$ contain sufficient information to determine the conditional expectation in the above formula, we can also write

$$L_t = \mathcal{L}(t, A_t, X_t(\cdot))$$

where $\mathcal{L}$ is a function that is to be determined. It is our purpose to write an equation for the unknown function $\mathcal{L}$. As a numéraire we shall take the riskless bond, so that $Z_t = e^{rt}$. As Appendix A.1 shows, one obtains the following *nonlinear* Black-Scholes equation for the evolution of the liability value $\mathcal{L}(t, A, X)$:

$$-\frac{\partial \mathcal{L}}{\partial t} = rA\frac{\partial \mathcal{L}}{\partial A} + \tfrac{1}{2}\sigma^2 A^2 \frac{\partial^2 \mathcal{L}}{\partial A^2} + D_{\mathrm{rs}}\mathcal{L} + \log p(A, L) D_{\mathrm{us}}\mathcal{L} - r\mathcal{L}. \tag{5}$$

In the above equation,

$$D_{\mathrm{rs}}\mathcal{L}(t, A_t, X_t) := D_X \mathcal{L}(t, A_t, X_t; \frac{\partial X_t}{\partial s})$$

(subscript "rs" for "right shift") and

$$D_{\mathrm{us}}\mathcal{L}(t, A_t, X_t) := D_X\mathcal{L}(t, A_t, X_t; 1)$$

(subscript "us" for "up shift"), where $D_X\mathcal{L}(t, A_t, X_t; Z)$ denotes the directional derivatives of $\mathcal{L}$ with respect to X_t in the direction of Z.

We observe that Sircar and Papanicolaou [1998] and Schonbucher and Wilmott [2000] also derive a nonlinear Black-Scholes partial differential equation in modeling the feedback effect of hedging on the underlying asset price. In their cases, the nonlinearity arises from the feedback effect from the dynamic hedging strategy on the underlying asset, and from there back onto the price of the derivative.

2.2 The proxy-based implementation

When indexation decisions are made on the basis of a funding ratio which is computed assuming a fixed indexation level applied in the future, the pension liability is bond-like and the future funding ratio is contingent only on the future asset value. So the policy ladder is represented by $p(A_t)$. Accordingly, (3) reduces to

$$dX_t(s) = -\Big(\frac{\partial X_t}{\partial s}(s) - \log p(A_t)\Big)\, dt. \tag{6}$$

With the three equations (6), (2) and (4) and through discretization, one can obtain the liability value using Monte Carlo simulations. Alternatively, one can show that the evolution of the liability value satisfies the following *linear* Black-Scholes equation.

$$-\frac{\partial \mathcal{L}}{\partial t} = rA\frac{\partial \mathcal{L}}{\partial A} + \tfrac{1}{2}\sigma^2 A^2 \frac{\partial^2 \mathcal{L}}{\partial A^2} + D_{\mathrm{rs}}\mathcal{L} + \log p(A) D_{\mathrm{us}}\mathcal{L} - r\mathcal{L}. \tag{7}$$

Although a closed-form solution is not available generally, the linear partial differential equation is more amenable to existing numerical approaches like standard Monte Carlo methods.

The mathematical formulation for continuous-time indexation crystallizes the different nature of the computation required for the market valuation of liabilities in the two different types of implementation. For more concrete insights, we now turn to a model for discrete-time indexation which allows further investigation.

3 The model for discrete-time indexation

3.1 The economy and the pension fund

Although the computation procedure we present below applies in settings where the risk-neutral measure can be solved for or is specified, for simplicity we consider the standard Black-Scholes economy. In particular,

- The only risk factor is stock market risk, and it is traded through a stock index S_t following geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dZ_t,$$

where μ and σ are the constant drift and volatility parameters, and Z_t is the standard Brownian motion.

- The riskless asset is a cash bond with constant interest rate r, whose price, B_t, changes according to
$$dB_t = rB_t dt.$$

- The inflation rate, ρ, is constant.

We consider a pension fund which is liable for making pension payments at two time points T_1 and T_2. The payment at T_1 is subject to a decision-making at T_1 on the indexation level applied to the inflation over the period from T_0 to T_1. Likewise, the payment at T_2 is determined by a decision made at T_2 on the indexation level applied to the inflation over the period from T_1 to T_2. We are now at time is t $(< T_1)$ and to compute the liability value and funding ratio.

The pension fund is assumed to adopt a constant-proportion investment strategy. That is, the pension fund keeps the asset value invested in stocks (represented here by the stock index) a constant fraction of the total asset value, and we denote the constant proportion by α.

3.2 The policy ladder and the circularity problem

A policy ladder is a formula that produces a indexation level applied over a certain period to the pension rights paid within a short time (usually one year in practice) on the basis of the current funding ratio.[1] Because the span is short between the time when the funding ratio is computed and the decision-making on indexation level is made, and the time when the pension rights with the decided indexation level are actually paid, it is reasonable to model the policy ladder as a formula that determines the pension rights paid *immediately*, denoted by L_n, on the basis of the current funding ratio.

As mentioned in the introduction, the policy ladders in practice distinguish three regions: minimum, partial, and maximum indexation. In the paper we consider a piecewise linear form:

$$L_n = \begin{cases} L_\ell & \text{for } FR < K_\ell, \\ L_\ell + \beta(FR - K_\ell) & \text{for } K_\ell \leq FR \leq K_u, \\ L_u & \text{for } FR > K_u, \end{cases} \tag{8}$$

where FR is the current funding ratio, K_u and K_ℓ are the upper and lower thresholds of funding ratio, L_u and L_ℓ are the pension rights with maximum and minimum indexation levels, and $\beta = \frac{L_u - L_\ell}{K_u - K_\ell}$ to ensure that the function is continuous. For fixed L_u, L_ℓ and K_ℓ, the value of K_u (or β) determines how generous the policy ladder is. While in practice the policy ladder is usually formulated in terms of the indexation percentage, rather than in terms of the amount of pension rights, the formulation (8) can simplify computation, and this difference does not affect the conclusions we draw below.

Next we consider the consistent implementation, where the funding ratio used in indexation conditions is computed based on the true value of the liabilities. For the computation of the funding ratio, we can decompose the pension

[1] In an average-salary scheme, the indexation level also apply to active workers, but we abstract from the added complication from the practice.

liability value into two components: the value of pension right paid immediately L_n, and the value of pension rights to be paid in the future, denoted by L_f. We can write

$$FR = \frac{A}{L_n + L_f}, \tag{9}$$

where A is the current asset value. The distinction between L_n and L_f relies on the time when the liability value and funding ratio are computed. For instance, L_n is zero and all the liability value is captured by L_f at time t, whereas L_n is the actual payment and L_f is zero at time T_2.

Now turn to the parameters and variables in equations (8) and (9). K_ℓ and K_u are parameters whose values are determined by the adopted policy ladder. We also assume that L_u and L_ℓ are parameters with known values for the moment. By this assumption, we abstract at this point from the intertemporal dependence of the liability side,[2] which will be taken into account later on. The current asset value A is easy to measure because it is simply the sum of investment value in the bond and the stock index.[3] Given the realization of the stock index, A is known. In the Black-Scholes economy, which is a dynamic setting with randomness, for a given investment strategy, A captures all the randomness and dynamics of the economy and hence can be used as the state variable. FR, L_n and L_f are variables whose values we are to solve for. When L_f is *known*, we can solve for the left two unknowns, FR and L_n, by the two equations (8) and (9):

$$L_n = \begin{cases} L_\ell & \text{for } A < (L_\ell + L_f)K_\ell, \\ \frac{1}{2}\Big[L_\ell - L_f - \beta K_\ell + \\ \qquad \sqrt{4\beta A + (L_f + L_\ell - \beta K_\ell)^2}\Big] & \text{for } (L_\ell + L_f)K_\ell \leq A \leq (L_u + L_f)K_u, \\ L_u & \text{for } A > (L_u + L_f)K_u. \end{cases}$$

Or in more compact form,

$$L_n = \min\left\{\max\left[\frac{1}{2}\left(L_\ell - L_f - \beta K_\ell + \sqrt{4\beta A + (L_f + L_\ell - \beta K_\ell)^2}\right), L_\ell\right], L_u\right\}. \tag{10}$$

The above equation solves the circularity problem by finding the coherent solution in the sense that the pension rights paid immediately and the current funding are consistent.

At time T_2, L_f is indeed easy to know since it is just zero. At time T_1, however, the determination of L_f is not that easy, and it is actually one of the keys to the liability valuation of the conditional indexation scheme, to which we now turn.

[2]If the intertemporal dependence via parameters were considered, then L_u and L_ℓ for the payment to be made at T_2 would be dependent on the payment made at T_1, and cannot be said to be parameters with known values.

[3]That is to say, the value of contingent contributions in the future is excluded from the asset value.

4 Valuation in the consistent implementation

We shall describe the procedure to compute the liability value at t, and then through some numerical example to show how the liability value depends on the asset, and the impact of investment strategies and policy ladders on the liability value and funding ratio. First, let us introduce the following notations we need:

- A_τ: the asset value at time τ *before* pension payment
- A'_τ: the asset value at time τ *after* pension payment
- L_τ: the liability value of the pension fund at time τ *before* pension payment if there is any
- $\mathcal{L}_\tau(\cdot)$: the function associating the state variable(s) to L_τ
- L_ℓ^τ: the pension paid at τ with *minimum* indexation
- L_u^τ: the pension paid at τ with *maximum* indexation
- L_n^τ: the pension made immediately from the vantage point at time τ
- $\mathcal{L}_n^\tau(\cdot)$: the function relating the state variable(s) to L_n^τ
- L_f^τ: the pension to be made in the future from the vantage point at time τ
- $\mathcal{L}_f^\tau(\cdot)$: the function relating the state variable(s) to L_f^τ

4.1 The computation procedure

The liability value at time t is computed using *backward* method. For those familiar with the backward methods of pricing Bermudan options, we note that the computation procedure is similar to that for pricing Bermudan options. The main difference is that for Bermudan options, we solve an optimization problem at each early exercise date, whereas for the liability valuation, we solve the circularity problem at each pension payment date. As mentioned in the introduction, We distinguish two types of intertemporal dependence: dependence of assets, and dependence of liabilities. For ease of exposition, we start with the case where there is only intertemporal dependence of assets, and then address the case with both types of dependence.

4.1.1 Intertemporal dependence of assets only

We **start from time** T_2, when the pension fund makes the second payment, $L_n^{T_2}$, and comes to an end. Given $L_f^{T_2} = 0$, we can obtain $L_n^{T_2}$ by applying (10)to solve the circularity problem:

$$L_n^{T_2} = \min\left\{\max\left[\frac{1}{2}\left(L_\ell^{T_2} - \beta_{T_2}K_\ell + \sqrt{4\beta_{T_2}A_{T_2} + (L_\ell^{T_2} - \beta_{T_2}K_\ell)^2}\right), L_\ell^{T_2}\right], L_u^{T_2}\right\}. \tag{11}$$

where $\beta_{T_2} = \frac{L_u^{T_2} - L_\ell^{T_2}}{K_u - K_\ell}$. As can be seen from the above equation, the second payment is dependent on the asset value at T_2, A_{T_2}, and hence we can write

$$L_n^{T_2} = \mathcal{L}_n^{T_2}(A_{T_2}). \tag{12}$$

Now **move back to** T_1, when the first payment is made. From the vantage point at T_1, the second payment can be thought of as an option written on the pension asset value. Since the fund follows constant-proportion strategies in the Black-Scholes economy, A_{T_2} follows a lognormal distribution conditioning on A'_{T_1} at time T_1, i. e.

$$A_{T_2} = A'_{T_1} \exp\left([\alpha\mu + (1-\alpha)r - \frac{1}{2}\alpha^2\sigma^2](T_2 - T_1) + \alpha\sigma\sqrt{T_2 - T_1} z \right),$$

where z follows the standard normal distribution. Please note that A'_{T_1} is the pension asset value at T_1 *after* the first payment, rather than that *before* the payment, A_{T_1}, and that

$$A'_{T_1} = A_{T_1} - L_n^{T_1}.$$

The option value, i. e. $L_f^{T_1}$, can be obtained using option pricing techniques, for example by solving the Black-Scholes partial differential equation (PDE) of $\pi(\tau, X)$

$$r\pi(\tau, X) = \frac{1}{2}\alpha^2\sigma^2 X^2 \frac{\partial^2\pi}{\partial X^2}(\tau, X) + rX\frac{\partial\pi}{\partial X}(\tau, X) + \frac{\partial\pi}{\partial\tau}(\tau, X) \tag{13a}$$

with the boundary condition

$$\pi(T_2, A_{T_2}) = L_n^{T_2}. \tag{13b}$$

And $L_f^{T_1}$ is simply $\pi(T_1, A'_{T_1})$. The PDE (13) does not have an analytical solution. Nevertheless, either through numerical methods like finite different methods, or using an analytical approximation method as described in Appendix A.2, one can obtain $L_f^{T_1}$ as a function of A'_{T_1}, which can also be formulated in terms of A_{T_1}:

$$L_f^{T_1} = \mathcal{L}_f^{T_1}(A'_{T_1}) = \mathcal{L}_f^{T_1}\left(A_{T_1} - L_n^{T_1}\right). \tag{14}$$

With the above functional expression for $L_f^{T_1}$, one can solve for the first payment $L_n^{T_1}$ by applying (10) once again:

$$\begin{aligned} L_n^{T_1} \quad = \quad & \min\left\{ \max\left[\frac{1}{2}\left(L_\ell^{T_1} - \mathcal{L}_f^{T_1}\left(A_{T_1} - L_n^{T_1}\right) - \beta_{T_1} K_\ell + \right.\right.\right. \\ & \left.\left.\left. \sqrt{4\beta_{T_1} A_{T_1} + \left(\mathcal{L}_f^{T_1}(A_{T_1} - L_n^{T_1}) + L_\ell^{T_1} - \beta_{T_1} K_\ell \right)^2} \right), L_\ell^{T_1} \right], L_u^{T_1} \right\} \end{aligned}$$

where $\beta_{T_1} = \frac{L_u^{T_1} - L_\ell^{T_1}}{K_u - K_\ell}$. The above equation characterizes the circularity problem at time T_1 stemming from the conditional indexation. Through numerical methods, one can solve the equation to obtain $L_n^{T_1}$ as a function of A_{T_1}:

$$L_n^{T_1} := \mathcal{L}_n^{T_1}(A_{T_1}). \tag{15}$$

The total liability value at time T_1 is

$$L_{T_1} = L_n^{T_1} + L_f^{T_1} = \mathcal{L}_n^{T_1}(A_{T_1}) + \mathcal{L}_f^{T_1}\left(A_{T_1} - \mathcal{L}_n^{T_1}(A_{T_1})\right).$$

So one can also write L_{T_1} as a function of A_{T_1}:

$$L_{T_1} := \mathcal{L}_{T_1}(A_{T_1}) \tag{16}$$

Eventually, we can come **back to the current time** t. For the current liability value L_t, one can, through numerical methods, solve the PDE (13a) with the boundary condition $\pi(T_1, A_{T_1}) = \mathcal{L}_{T_1}(A_{T_1})$. Finally we arrive at the current liability value and funding ratio:

$$L_t = \pi(t, A_t), \quad FR_t = \frac{A_t}{\pi(t, A_t)}. \tag{17}$$

We summarize the computation procedure. It starts from the last payment date T_2, and obtains the amount of the last payment as a function of the asset value as Equation (12) by solving the circularity problem at that time point. Move one period back to the first payment date T_1. Using options pricing techniques, the value at T_1 of the last payment can be formulated in the form of (14), with which the amount of the first payment can be written as a function of the asset value as Equation (15) by solving the circularity problem at T_1. Therefore we can have the total liability value as a function of the asset value as Equation (16). Moving back the valuation date t, and applying options pricing techniques, we eventually obtain the liability value and the funding ratio as in Equation (17).

The computation procedure can, with minor adaption, be used to address the case with both types of intertemporal dependence.

4.1.2 Both types of intertemporal dependence

The practice that indexation can not be canceled later on once it has been granted makes the *minimum* pension benefit paid by pension funds in certain year dependent on the indexation decisions made in previous years, or equivalently on the pension benefit actually paid in previous years. It leads to what we refer to as the intertemporal dependence of liabilities. In the stylized two-period model, we formulate this type of intertemporal dependence through letting the lower bound of the second payment depend on the *actual* first payment. In particular, the lower bound of the second payment is specified to be equal to the actual payment at T_1, so we have

$$L_\ell^{T_2} = L_n^{T_1}, \quad L_u^{T_2} = L_n^{T_1} e^{\rho(T_2 - T1)}. \tag{18}$$

The computation procedure for this case also works backwards and start from the last payment date T_2. Applying (10) to solve the circularity problem at T_2 leads to Equation (11) as in the case of the asset dependence alone. The difference is that, given (18), the second payment is dependent on the asset value at T_2, A_{T_2}, as well as on the actual first payment, $L_n^{T_1}$. So we write

$$L_n^{T_2} = \mathcal{L}_n^{T_2}(A_{T_2}, L_n^{T_1}). \tag{19}$$

Now move back to T_1, when the first payment is made. For a given value of $L_n^{T_1} \in [L_\ell^{T_1}, L_u^{T_1}]$, one can, using the options pricing techniques, obtain $L_f^{T_1}$ as a function of A'_{T_1}, which can also be formulated in terms of A_{T_1}. That is

$$L_f^{T_1} = \mathcal{L}_f^{T_1}(A'_{T_1}, L_n^{T_1}) = \mathcal{L}_f^{T_1}\left(A_{T_1} - L_n^{T_1}, L_n^{T_1}\right). \tag{20}$$

With the above functional expression for $L_f^{T_1}$, one can solve for the first payment

$L_n^{T_1}$ through solving the circularity problem at T_1 by means of (10) once again:

$$\begin{aligned} L_n^{T_1} \;&=\; \min\left\{\max\left[\frac{1}{2}\left(L_\ell^{T_1} - \mathcal{L}_f^{T_1}\left(A_{T_1} - L_n^{T_1}, L_n^{T_1}\right) - \beta_{T_1}K_\ell + \right.\right.\right. \\ &\left.\left.\left.\sqrt{4\beta_{T_1}A + \left(\mathcal{L}_f^{T_1}(A_{T_1} - L_n^{T_1}, L_n^{T_1}) + L_\ell - \beta_{T_1}K_\ell\right)^2}\right), L_\ell^{T_1}\right], L_u^{T_1}\right\} \end{aligned}$$

where $\beta_{T_1} = \frac{L_u^{T_1} - L_\ell^{T_1}}{K_u - K_\ell}$. As in the previous case, through numerical methods, one can solve the equation to obtain $L_n^{T_1}$ as a function of A_{T_1}:

$$L_n^{T_1} := \mathcal{L}_n^{T_1}(A_{T_1}). \tag{21}$$

Therefore the total liability value at time T_1 is computed as

$$L_{T_1} = L_n^{T_1} + L_f^{T_1} = \mathcal{L}_n^{T_1}(A_{T_1}) + \mathcal{L}_f^{T_1}\left(A_{T_1} - \mathcal{L}_n^{T_1}(A_{T_1})\right).$$

So one can also write L_{T_1} as a function of A_{T_1}:

$$L_{T_1} := \mathcal{L}_{T_1}(A_{T_1}) \tag{22}$$

At last we can come back to the current time t. For the current liability value L_t, one can, through numerical methods, solve the PDE (13a) with the boundary condition $\pi(T_1, A_{T_1}) = \mathcal{L}_{T_1}(A_{T_1})$. Finally we arrive at the current liability value and funding ratio:

$$L_t = \pi(t, A_t), \quad FR_t = \frac{A_t}{\pi(t, A_t)}. \tag{23}$$

The computation procedure can be extended to other market settings with a given pricing probability measure, for instance, in the presence of interest rate risks and inflation risks. The method would be essentially the same, but the computational load would increase with the number of risk factors in the model. Furthermore, as long as the intertemporal dependence of liabilities is modeled as the boundaries of pension payment at a time point being determined *only* by the actual payment of the previous time point, the procedure can be extended to the case where the pension fund has finitely many pension payments in a straightforward way: we only have to repeat the computation at time T_1 for each of the intermediate payment dates. These extensions, however, are beyond the scope of the paper. Next we shall, through some numerical examples, look into what insights result from applying the computation procedure.

4.2 Numerical examples

For the purpose of numerical illustration, the following parameter values are assumed:

$$\begin{aligned} &r = 3\%, \quad \mu = 7\%, \quad \sigma = 20\%, \quad \rho = 4\%, \quad \alpha \in \{0.25, 0.50, 0.75\} \\ &T_0 = 0, \quad t = 9, \quad T_1 = 10, \quad T_2 = 20 \\ &K_\ell = 110\%, \qquad K_u \in \{115\%, 140\%, 160\%\} \\ &L_\ell^{T_1} = 100, \qquad L_u^{T_1} = L_\ell^{T_1} e^{\rho(T_1 - T_0)} = 149 \end{aligned}$$

As mentioned before, the lower and upper bounds of the second payment are given by (18) to reflect the intertemporal dependence of liabilities.

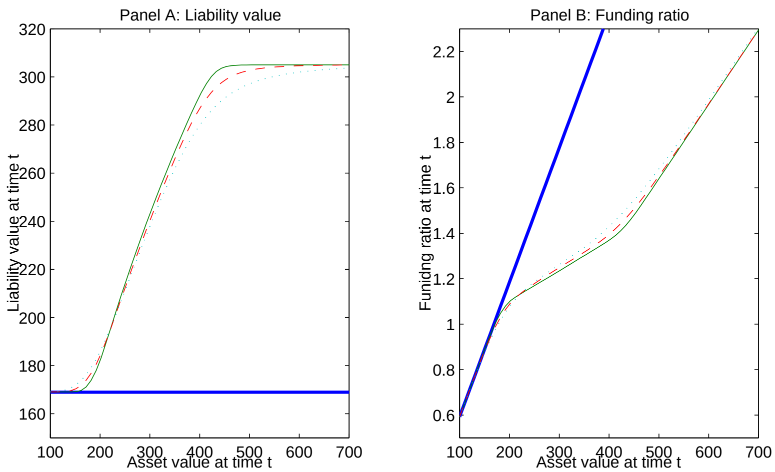

Figure 1: **Liability value and funding ratio at time t for different investment policies** This figure shows both the liability value (the left panel) and the funding ratio (the right panel) at time t as a function of the then asset value for three asset mixes: the stock weight is assumed to be 25% (the solid line), 50% (the dashed line), and 75% (the dotted line). The bold lines are for the financial measure on the basis of the FTK requirement, which will be discussed in Section 5.

4.2.1 The impact of investment strategies

To see the impact of the aggressiveness of investment strategies, we fix the policy ladder ($K_\ell = 110\%$ and $K_u = 140\%$), and consider three stock weights $\alpha \in \{25\%, 50\%, 75\%\}$.

Before looking into the liability value and funding ratio of the conditional indexation schemes, let us recall some defining properties of defined-benefits (DB) and defined-contribution (DC) schemes as reflected by liability value and funding ratio. For a pure DB scheme, because of the bond-like property of pension benefits, the liability value is independent of changes of pension asset value, and the funding ratio is therefore a linear function of the pension asset value. On the other hand, for a typical DC scheme, the liability value is equal to (i. e. a linear function of) the pension asset value whereas the funding ratio is constant at 100%.

As can be seen from the behavior of its liability value and funding ratio illustrated in Figure 1, conditional indexation schemes are neither typical of DB nor of DC, but strike some balance between the two stereotypes. If the asset value is either very low or very high, the liability values for all the three different investment policy show little or even no dependence of the asset value, bearing the hallmark of DB schemes. The same point can be seen from the funding ratio of the conditional indexation scheme as it approaches a linear function of the asset value for both ends of the asset value domain as shown in the figure. In contrast, for the intermediate domain of asset values, say between 200 and 400, the conditional indexation scheme bears a close resemblance to DC schemes, in that the liability value is increasing with the asset value, and the funding ratio is insensitive to the variation of asset value.

Therefore the practice of conditional indexation has the effect of stabilizing the funding ratio through introducing a DC element to the originally DB system.

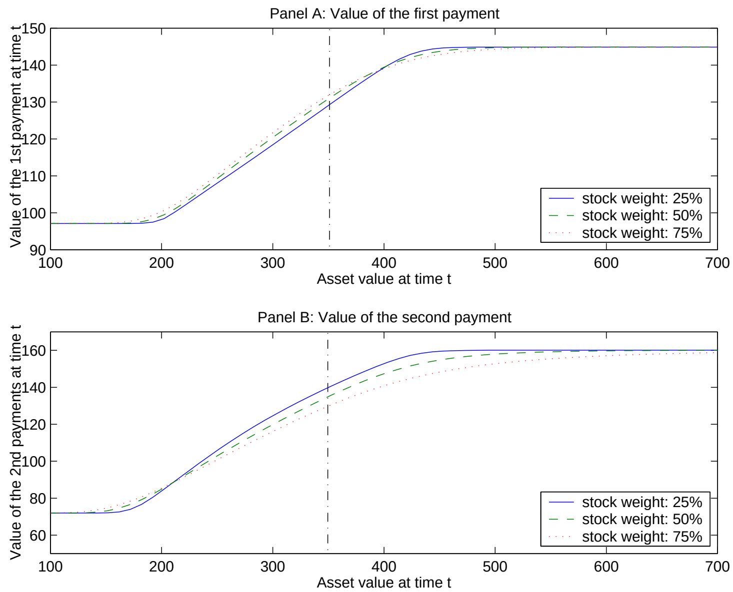

Figure 2: **Value of the two pension payments to be made at T_1 and T_2**
This figure shows the liability value of the first payment(the upper panel) and the second payment (the lower panel) at time t as a function of the asset value for three asset mixes: the stock weight is assumed to be 25% (the solid line), 50% (the dashed line), and 75% (the dotted line). The vertical dash-dotted line is corresponding to asset value equal to 350.

The magnitude of the stabilizing effect, however, depends on the investment policy. The stability of funding ratio in the intermediate region of asset values becomes more pronounced with a lower stock weight, for in that case the curve for the funding ratio is flatter as shown in the figure. An intuitive explanation is as follows. In comparison with a scheme with a high stock weight, a low stock weight scheme has less volatility of asset value. For a conservative asset mix, a given amount of asset value increase implies an increased future asset value with more certainty than for an aggressive asset mix. And hence a larger pension increase in the form of an increased indexation level can be granted to participants, which implies a larger increase in pension liabilities. That is, the liability value of is more sensitive to the variation of asset value for a conservative asset mix. Translated into the funding ratio, it implies that the funding ratio has greater stability when the investment strategy is less risky.

Whether increased riskiness of asset mix will raise or lower the liability value depends on the current asset value. In particular, when the asset value is low, say equal to 200, a higher stock weight increases the liability value. In contrast, when the asset value is high, say equal to 400, a higher stock weight decreases the liability value. The dependence on the current asset value stems from the collar structure of pension rights under conditional indexation. A higher stock weight may enhance the probability of granting greater-than-the-minimum indexation, and *increase* the liability value. On the other hand, it is also possible that an increased stock weight dampens the possibility of the maximum indexation, and hence *decreases* the liability value. The balance of the two countervailing effects depends on the current asset value. The increasing effect dominates for low

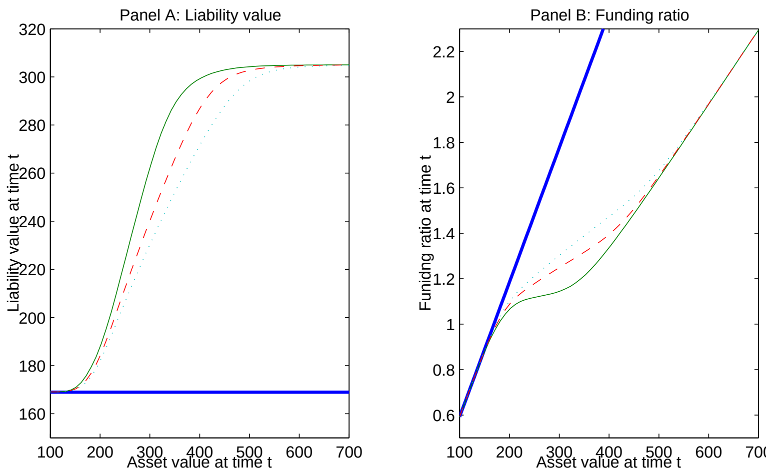

Figure 3: **Liability value and funding ratio at time t for different policy ladders** This figure shows both the liability value (the left panel) and the funding ratio (the right panel) at time t as a function of the then asset value for three policy ladders: K_u is assumed to be 1.15 (the solid line), 1.40 (the dashed line), and 1.60 (the dotted line). The bold lines are on the basis of valuation following the FTK requirement, which will be discussed in Section 5.

asset values due to the guaranteed pension floor, whereas the decreasing effect prevails for high asset values because of the full indexation ceiling of pension rights. At some intermediate values of asset these two effects compensate each other, making the liability value insusceptible to changes of asset mix. The impact of investment policy can also been expressed in terms of the funding ratio: an raised stock weight decreases the funding ratio at low asset values, and increase the funding ratio at high asset values.

The pension liability consists of the two payments to be made in the future in this example, and the liability value can be decomposed into two components: the value of the early payment, and that of the late payment. To analyze the impact of investment policy on the two components can lead one to see how the change of investment policy results in *intertemporal* redistribution. Figure 2 shows the values of the two payments as a function of the asset value for the three investment policies. When the current asset is either very high or very low, say 200 and 450, the change of asset mix drives the two components in the same direction. In contrast, for some intermediate asset values, the change of asset mix decreases the value of one payment, but increases the value of another. For instance, consider the asset value equal to 350, where a rise in stock weight will increase the value of the first payment, but decrease the value of the second considerably. That is to say, the improved growth potential from higher stock weight is captured by the first payment preemptively, leaving the second payment only a loss. If the first payment is interpreted as the pension rights of an old generation, and the second as that of a young generation, then it implies that the investment policy has implications for *inter-generational* redistribution.

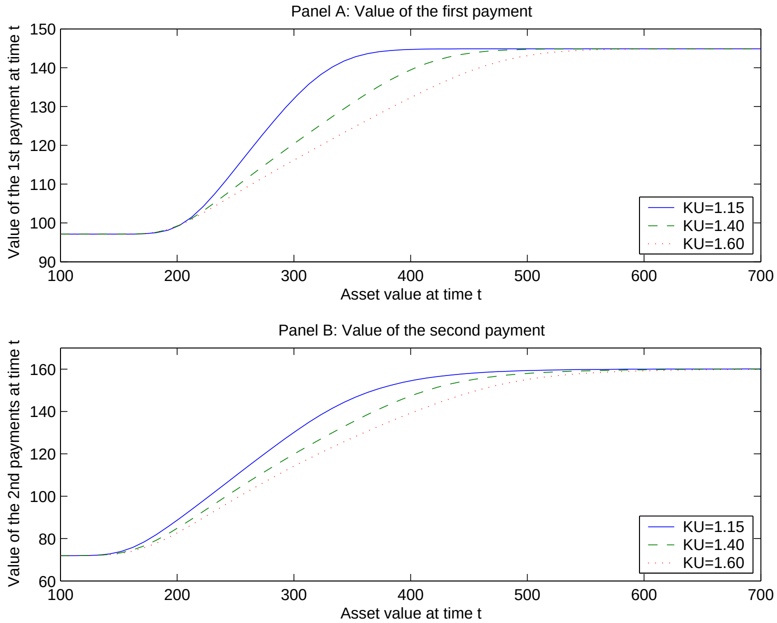

Figure 4: **Liability value and funding ratio at time t for different policy ladders** This figure shows the liability value of the first payment(the upper panel) and the second payment (the lower panel) at time t as a function of the asset value for three asset mixes: K_u is assumed to be 1.15 (the solid line), 1.40 (the dashed line), and 1.60 (the dotted line).

4.2.2 The impact of policy ladders

To see the impact of the policy ladders, we fix the stock weight ($\alpha = 0.5$) and vary the threshold of funding ratio for full indexation: $K_u \in \{115\%, 140\%, 160\%\}$. To the extent that a lower value of K_u implies that it is easier for participants to receive full indexation, the three values for K_u are used to stylize how generous the policy ladder is.

As would be anticipated, a more generous policy ladder, *ceteris paribus*, results in a greater liability value and hence a lower funding ratio whatever the current asset value is (Figure 3). Another interesting point is that the funding ratio is more stable under a more generous policy ladder as the curve of funding ratio is flatter in such a case. An intuitive explanation is that a more generous policy ladder makes the granted indexation level and hence the liability value more responsive to the variation of asset value, which implies more stability of the funding ratio.

As Figure 4 shows, the values of both payments are increasing with the generosity of policy ladder, irrespective of the current asset value. However, generally the first payment seems to benefit more from a generous policy ladder than the second does, reflecting the preemptive advantage of the first payment over the second.

4.3 More realistic modeling of the intertemporal dependence of liabilities

Typically indexation decisions are made on an annual basis, and the pension benefit paid in a year is the *cumulative* result of the indexation decisions over last many years. For instance, a 75-year-old retiree's pension benefit depends on each of the annual indexation levels that she has been granted over the last 50 years (assuming entry into the system at the age of 25). In the previous modeling of the intertemporal dependence of liabilities, it is assumed that the pension right at a time point depends only on the pension benefit (or equivalently the indexation decision) in the previous period, and therefore one state variable is sufficient to capture the information on indexation decisions. The gradual accrual of indexation and the irrevocability of indexation decisions as seen in practice, however, mean that the assumption is restrictive. A realistic model of the intertemporal dependence of liabilities needs a large number of state variables (up to 80) to carry the relevant information on indexation decisions over the last many years.

With such realistic modeling of the intertemporal dependence of liabilities, the valuation of conditionally indexed pension liabilities in the consistent implementation leads to a computational task of high dimension. The model can be solved in principle through backward recursion as in our stylized model, solving circularity problems at each step, but a major computational problem is the dimension of the state space which may be beyond the computational capacity usually available. The dimensionality problem cannot be solved by standard Monte Carlo methods since the liability value is need to determine what indexation decisions are made; adaptations akin to Monte Carlo methods for American option pricing may be possible but are not addressed in this paper.

5 Valuation in the proxy-based implementation

In the consistent implementation discussed above, the funding ratio used for indexation decisions is financially valid, and hence it is a reliable indicator in assessing the financial status of the pension fund. (The funding ratio obtained in this way will be referred to as the "consistent funding ratio" in the following.) In the valuation of pension liabilities, the circularity problem needs to solved to keep the consistency of the resultant funding ratio between its role as the basis of indexation and its role as an indicator of financial solvency. The current practice of conditional indexation is different though. Indexation decisions are made on the basis of a proxy of funding ratio for which the liability value is computed assuming a fixed indexation, typically zero indexation.[4] The proxy-based implementation avoids the circularity problem inherent in the consistent implementation.

A caveat for the current practice is that the zero-indexation proxy used for indexation decisions may be misleading in assessing the financial status of pension funds. From the perspective of market valuation, it is straightforward

[4] According to the current Dutch pension regulation (FTK), the liability value is computed on the basis of the guaranteed pension rights only, excluding the contingent rights from conditional indexation. Those who read Dutch may see the document available at http://docs.szw.nl/pdf/35/2004/35_2004_349957.pdf.

that the zero-indexation proxy overstates the financial solvency of pension funds, and hence is an overestimate of the "actual" funding ratio, as long as it is possible to grant indexation to participants. Nevertheless, one may defend the validity of the zero-indexation proxy as an indicator of financial solvency by the argument that the proxy, if being read with some "discount", can still make sense. As will be shown below, however, the proxy overestimates the latent actual funding ratio in a complicated manner, making such discounting rather difficult. Moreover, the funding ratio computed on the basis of *any* fixed level of indexation is not reliable indicator of financial soundness, because the indexation levels decided in the future are contingent and varied. Therefore for the purpose of assessing the financial status of pension funds using this implementation, the actual funding ratio needs to be computed and used. In the following, we shall address the market valuation of the liability and the computation of the actual funding ratio in the proxy-based implementation.

A different funding ratio used for indexation decisions, *ceteris paribus*, leads to different indexation decisions and hence to a different pension scheme. The zero-indexation proxy is of course different from the consistent funding ratio, so the scheme resulting from zero-indexation proxy, referred to as the the "ZIP scheme" hereafter, is different from that generated by the consistent implementation. The question we face is what the actual funding ratio of the ZIP scheme is. It is worth noting that the consistent funding ratio is *not* the answer since it is the actual funding ratio for a *different* pension scheme.

To obtain the actual funding ratio of the ZIP scheme, we need to know the zero-indexation proxy first, for the reason that it is on the basis of the zero-indexation proxy that indexation decisions and pension rights are determined. We still work in the two-payment model. With the assumption of zero-indexation, the (proxy) liability value at t is simply the two minimum payment at T_1 and T_2 discounted by the risk-free rate: $L_t^{zip} = L_\ell^{T_1} e^{-r(T_1-t)} + L_\ell^{T_1} e^{-r(T_2-t)}$. Therefore the zero-indexation proxy is

$$FR_t^{zip} = \frac{A_t}{L_t^{zip}}. \tag{24}$$

Their values as a function of the asset value are shown in Figure 1 and 3. They are of DB nature, and independent of investment policy and policy ladder. We assume that the two indexation decisions at T_1 and T_2 are based on the two zero-indexation proxies immediately before payments respectively. The zero-indexation proxies at T_1 and T_2 can be computed in a similar way.

Given the way the indexation decisions are made and the way the proxies for the indexation decisions are computed, one can obtain the actual funding ratio of ZIP schemes by applying classical options pricing techniques. Key to the computation funding ratio is the market valuation of pension liabilities. In this context, the valuation of liabilities is to price two contingent pension payments at T_1 and T_2. Applying Monte Carlo simulation methods (see the Appendix A.3 for an algorithm), one can obtain the market value of the liability and the actual funding ratio of the ZIP scheme.

Using the same parameter values as before, The upper panels in Figures 5 and 6 show the relation between the actual funding and the zero-indexation proxy of the ZIP scheme. Because the zero-indexation proxy, as can be seen from (24), is a linear function of the asset value, these two figures can also be

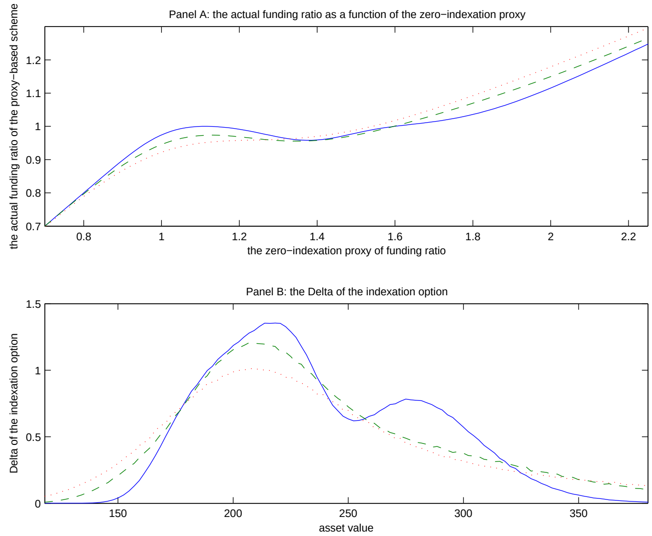

Figure 5: **The proxy-based implementation for different stock weights**
In this figure, Panel A shows the actual funding ratio at time t as a function of the zero-indexation proxy, and Panel B shows the first-order derivative of the actual liability value with respect to the asset value (the Delta of the indexation option). The stock weight is assumed to be 25% (the solid line), 50% (the dashed line), and 75% (the dotted line). The horizontal axes of both panels are set to correspond to each other.

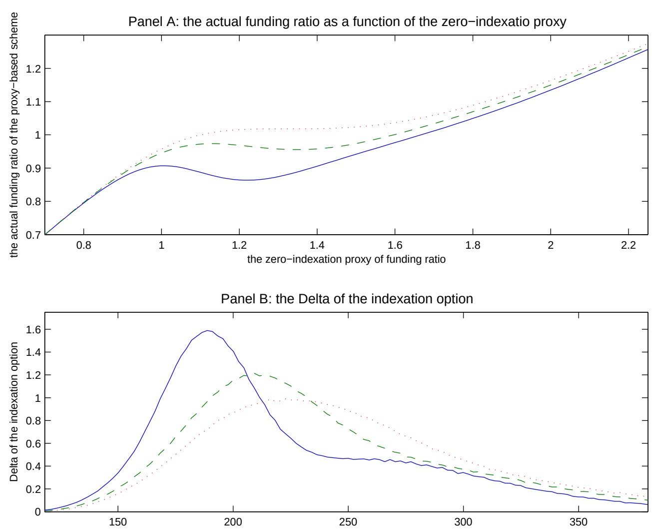

Figure 6: **The proxy-based implementation for different policy ladders**
In this figure, Panel A shows the actual funding ratio at time t as a function of the zero-indexation proxy, and Panel B shows the first-order derivative of the actual liability value with respect to the asset value (the Delta of the indexation option). The upper threshold of funding ratio K_u is assumed to be 1.15 (the solid line), 1.40 (the dashed line), and 1.60 (the dotted line). The horizontal axes of both panels are set to correspond to each other.

interpreted as representing the actual funding ratio as a function of the asset value. A striking point, as can be seen from the downward-sloping section of some curves in the figures, is that an increase in the zero-indexation proxy reduces the actual funding ratio for some investment strategies and policy ladders. It is because that an increase in the asset value, *ceteris paribus*, would raise the first payment, which in turn raise the lower bound of the second payment, and the boosting effect on both payments can raise the liability value to such an extent that is greater than the increase in the asset value. It makes no sense, based solely on the numerical exercises of a stylized model as used in this paper, to rush to the conclusion that a stock market crash and a dampened asset value should be welcomed with open arms by pensions funds that wish to improve their financial status. Nevertheless, it highlights the fact that it is possible for an increased in the asset value to actually hurt the actual funding ratio of a pension fund.

The existence of such a counterintuitive scenario is accounted for by two elements inherent in the proxy-based implementation. One is the dynamic inconsistency of this implementation. The very fundamental idea of conditional indexation is that a pension fund pays more only when it possesses more vis-à-vis its liability. The proxy-based implementation is not able to ensure that this idea is realized, because a proxy of funding ratio simply uses a "financially incorrect" measure of the liability, and hence can not tell one whether a pension fund possesses more relative its liability. There may exist the case where a pension fund pays more when it actually possesses less relative liability. The other element contributing to this scenario is the irrevocability of indexation, namely indexation cannot be canceled once granted. It allows an indexation decision in a year to have "repercussions" on the pension benefit over many years that follows. For instance, if a participant receives 2% indexation at the year of retirement, her pension benefit in every year till her decease will be raised by

zero-indexation proxy		**1.00**	**1.10**	**1.20**	**1.40**	**1.60**	**1.80**
Panel A investment policy: $\alpha = \cdots$	25%	*0.97*	*1.00*	*0.99*	*0.96*	*1.00*	*1.04*
		`0.99`	`1.06`	`1.10`	`1.15`	`1.20`	`1.24`
	50%	*0.95*	*0.97*	*0.97*	*0.96*	*1.00*	*1.07*
		`0.97`	`1.04`	`1.09`	`1.16`	`1.21`	`1.25`
	75%	*0.92*	*0.95*	*0.96*	*0.97*	*1.02*	*1.09*
		`0.96`	`1.03`	`1.08`	`1.16`	`1.22`	`1.27`
Panel B policy ladder: $K_u = \cdots$	115%	*0.91*	*0.89*	*0.86*	*0.91*	*0.98*	*1.05*
		`0.97`	`1.03`	`1.07`	`1.11`	`1.13`	`1.15`
	140%	*0.95*	*0.97*	*0.97*	*0.96*	*1.00*	*1.07*
		`0.97`	`1.04`	`1.09`	`1.16`	`1.21`	`1.25`
	160%	*0.96*	*1.00*	*1.02*	*1.02*	*1.04*	*1.09*
		`0.98`	`1.05`	`1.11`	`1.19`	`1.25`	`1.31`

Table 1: **Comparison of funding ratios** The table shows the actual funding ratio of the ZIP scheme and the consistent funding ratio corresponding to various values of the zero-indexation proxy. Panel A is for three stock weights with a fixed policy ladder ($K_\ell = 110\%$, and $K_u = 140\%$) whereas Panel B is for three policy ladders with a fixed stock weight ($\alpha = 50\%$). The actual funding ratios of the ZIP scheme are rendered in *italics*, and the consistent funding ratios in `typewriter` font.

2% compared to the case of no indexation. Paying 2% more in one year is no big deal, but an increase of 2% in all 30 annual payments may have noticeable impact on the liability value.

Therefore it is possible for a pension fund to grant participants too much indexation in the sense that an increase in asset value leads to a lower actual funding ratio. The possibility of over-indexation can also be seen from the Delta of the indexation option, i. e. the first-order derivative of the liability value as a function of the asset value. When the Delta is greater than one, say 1.5, it means that an increase of 1 dollar in the asset value leads to an increase of 1.5 dollars in the liability value stemming from more indexation. As shown in the lower panel in Figures 5 and 6, the Delta is indeed larger than one for some combinations of the current asset value, the stock weight, and the policy ladder, corresponding to what we see from the behavior of the actual funding ratio in the corresponding upper panels. It is intuitively clear that a more generous indexation rule is more likely to have over-indexation (Figure 6). In a less intuitive way, Figure 5 implies that a pension fund with a more aggressive investment strategy is less likely to have over-indexation.

Also evident in the upper panels in Figures 5 and 6 is that the actual funding ratio of the ZIP scheme is insensitive to the change of the asset value, a phenomenon also seen in the consistent implementation. Akin to the case of the consistent implementation, the actual funding ratio as a function of the asset value depends on the investment strategy and the policy ladder chosen by the pension fund. Compare Figures 5 with Panel B of Figure 1, and compare Figure 6 with Panel B of Figure 3. One can find that the pattern of the dependence on the investment strategy and on the policy ladder is similar to the case in the consistent implementation. The intuitive explanation we offered in the case of the consistent implementation seems also plausible here.

Given the relevance of these various factors, it is possible that the actual funding ratio of Fund A is greater than that of Fund B, but the zero-indexation proxies of the two funds indicate the other way around. It implies that the comparison of financial solvency of different funds based solely on their zero-indexation proxies can be misleading. Furthermore, even for a single fund, an improved zero-indexation proxy may not indicate improved financial solvency if there are major changes in relevant aspects. The complicated relation between the zero-indexation proxy and the actual funding ratios implies there does not exist a simple way to "discount" the zero-indexation proxy to learn about the actual financial status of pension funds. Ideally, reliable discounting should take into account current asset value, investment policy, policy ladder, and demographic composition[5] of pension funds.

Table 1 compares the three funding ratios discussed in this paper: the consistent funding ratio, the zero-indexation proxy, and the actual funding ratio of the ZIP scheme. For a given asset value, the zero-indexation proxy is greater than the consistent funding ratio, which in turn is greater than the actual funding ratio of the ZIP scheme. So it is predictable that a change from the implementation based on zero-indexation proxy to the consistent implementation, *ceteris paribus*, will improve the financial soundness of pension funds. The reason for the improvement is that after the change of valuation method, indexation deci-

[5]It is conceivable that the demographic composition is relevant through its impact on the payoff structure of pension rights, though this issue is not discussed here owing to space consideration.

sions are made based on less favorable (to participants) funding ratios, resulting in less indexation, and thus less pension liabilities.

6 Conclusion

In this paper, we addressed the market valuation of conditionally indexed pension liabilities in two types of implementation. One is the current practice, where a proxy of funding ratio, typically the zero-indexation proxy, is used for indexation decisions. The other is to use the market-based and financially correct funding ratio for indexation decisions. Given its conformity with the ongoing move to market-based accounting standards, the latter implementation, labeled as the consistent implementation here, may be able to find its way into practice; the valuation of conditionally granted pension liability is on the agenda of the pension regulator in the Netherlands,[6] where many pension funds have adopted conditional indexation.

The mathematical formation of the liability valuation for continuous-time indexation indicates the different nature of the computation required for the market valuation of pension liabilities in these two types of implementation. In the proxy-based implementation, the liability value satisfies a *linear* Black-Scholes equation, implying that standard options pricing and computational methods are applicable. In the consistent implementation, by contrast, we obtain a *nonlinear* Black-Scholes equation for the evolution of the liability value, where the nonlinearity arise from the circularity problem involved in this type of implementation. The solution of the nonlinear equation defies standard option pricing approaches, and calls for a new methodology.

In the model for discrete-time indexation, we developed a backward recursion approach to the liability valuation in the consistent implementation. Based on the same model, we also considered the liability valuation in the proxy-based implementation. Numerical examples show that in both types of implementation, pension funds with conditional indexation find a middle road between DB and DC in terms of the way that the financial status depends on the asset value of pension funds. Conditional indexation provides a shield of the financial solvency against the fluctuation of the asset value through introducing a DC element into the originally DB system. The numerical examples also highlight the impact on the financial status of the investment strategy and the indexation rule that a pension fund adopts.

In the proxy-based implementation, the actual funding ratio, as opposed to the proxy, needs to be computed and used in assessing the financial soundness of pension funds. Some numerical exercises show that in this respect, the zero-indexation proxy may be misleading: an increase in the zero-indexation proxy (or equivalently in the asset value) may hurt the actual funding ratio of a pension fund. The existence of such peculiar scenarios is related to the dynamic inconsistency inherent in this type of implementation. Moreover, the comparison of the financial status among pension funds based solely on the proxy is rendered invalid by the fact that the relation between the proxy and the actual funding ratio depends on the investment policy, the indexation rule and the demographics of pension funds.

[6]See “Principlesfor a financial assessment framework: more transparency and clearer information” by the Dutch central bank, DNB.

In the consistent implementation, the irrevocability of indexation leads to a computational problem of high dimension if a more realistic model is adopted. The dimensionality problem is not addressed here, and subject to further research.

A Appendix

A.1 Derivation of the nonlinear Black-Scholes equation

For the derivation of the Black-Scholes equation, we need the following calculus rule.

A calculus rule for derivatives of functionals

Consider the set $\mathcal{C}$ of continuous real-valued functions on an interval $[0, D]$. Let ϕ be a functional defined on this space, i.e. a mapping from $\mathcal{C}$ to $\mathbb{R}$. The *Fréchet derivative* of the functional ϕ at $X \in \mathcal{C}$ is a linear mapping, denoted by $(\partial\phi/\partial X)(X)$, which is such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\phi(X+Y) = \phi(X) + \frac{\partial\phi}{\partial X}(X)Y + Z, \qquad \|Z\| < \varepsilon\|Y\|$$

for all $Y \in \mathcal{C}$ such that $\|Y\| < \delta$. The *directional derivative* of ϕ at X in the direction Y, for given $Y \in \mathcal{C}$, is the derivative at $\lambda = 0$ of the scalar function $\psi(\lambda)$ defined by

$$\psi(\lambda) = \phi(X + \lambda Y).$$

The directional derivative of ϕ at X in the direction of Y is denoted by $D\phi(X; Y)$.

Now suppose that $f(\lambda)$ is a $\mathcal{C}$-valued function defined on an interval of the form $[0, c)$ with $c > 0$, such that $f(0) = X$ (a "curve" starting at X). Suppose moreover that there is a function Y defined on $[0, D]$ such that

$$f(\lambda) = X + \lambda Y + o(\lambda), \quad \lambda \downarrow 0.$$

We then write $(df/d\lambda)(0) = Y$. As an example, let $X \in \mathcal{C}$ be given and consider the curve $f(\lambda)$ defined by

$$(f(\lambda))(s) = \begin{cases} X(s-\lambda) & \text{if } s \in [\lambda, D] \\ X(0) & \text{if } s \in [0, \lambda]. \end{cases}$$

Then we have (under suitable smoothness assumptions):

$$\frac{df}{d\lambda}(0) = -\frac{dX}{ds}\,.$$

The following calculus rule may be formulated which relates the derivative along a given curve to the directional derivative:

$$\frac{\partial\phi}{\partial X}(X)\frac{df}{d\lambda}(0) = D\phi(X; \frac{df}{d\lambda}(0)).$$

A nonlinear Black-Scholes equation

On the one hand, the process

$$E_t^{\mathbb{Q}} \int_0^T Z_\tau^{-1} \, dB_\tau = \int_0^t Z_\tau^{-1} \, dB_\tau + Z_t^{-1} L_t$$

is a $\mathbb{Q}$-martingale. Therefore, the drift term under $\mathbb{Q}$ of the process $Z_t^{-1} L_t$ must be equal to minus the drift term of the process $\int_0^t Z_\tau^{-1} \, dB_\tau$. We have

$$d \int_0^t Z_\tau^{-1} \, dB_\tau = Z_t^{-1} \, dB_t = Z_t^{-1} \Big(\int_R^D \exp(-X_t(s)) \, ds \Big) b \, dt. \tag{25}$$

On the other hand, on the basis of the expression $L_t = \mathcal{L}(t, A_t, X_t(\cdot))$ we can write (taking account of the fact that the differential equation for the indexation profile does not contain a Brownian term, and suppressing the argument (t, A_t, X_t) of the partial derivatives of $\mathcal{L}$)

$$dL_t = \frac{\partial \mathcal{L}}{\partial t} \, dt + \frac{\partial \mathcal{L}}{\partial A} \, dA_t + \frac{\partial \mathcal{L}}{\partial X} \, dX_t + \tfrac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial A^2} \, d[A, A]_t.$$

Of course we have $d[A, A]_t = \sigma^2 A_t^2 \, dt$. Moreover, in view of the above calculus rule and (3), we write

$$\frac{\partial \mathcal{L}}{\partial X}(t, A_t, X_t) \, dX_t = \big(D_X \mathcal{L}(t, A_t, X_t; \frac{\partial X_t}{\partial s}) - D_X \mathcal{L}(t, A_t, X_t; 1) \log p(A_t, L_t) \big) \, dt$$

where the subscript X of the directional derivative serves to indicate that this derivative is taken with respect to the function L taken as a function of its third argument. To simplify the notation, write

$$D_X \mathcal{L}(t, A_t, X_t; \frac{\partial X_t}{\partial s}) = D_{\mathrm{rs}} \mathcal{L}(t, A_t, X_t)$$

(subscript "rs" for "right shift") and

$$D_X \mathcal{L}(t, A_t, X_t; 1) = D_{\mathrm{us}} \mathcal{L}(t, A_t, X_t)$$

(subscript "us" for "up shift"). Given our choice of the riskless bond as a numéraire, the asset price dynamics (1) may be rewritten as

$$dA_t = r A_t \, dt + \sigma A_t \, d\widetilde{W}_t - dB_t$$

where $\widetilde{W}_t$ is a Brownian motion under $\mathbb{Q}$. In summary, we can write

$$dL_t = \tilde{\mu}_L(t, A_t, X_t) \, dt + \sigma_L(t, A_t, X_t) \, d\widetilde{W}_t$$

where (suppressing arguments)

$$\tilde{\mu}_L = \frac{\partial \mathcal{L}}{\partial t} + r A_t \frac{\partial \mathcal{L}}{\partial A} + \tfrac{1}{2} \sigma^2 A_t^2 \frac{\partial^2 \mathcal{L}}{\partial A^2} - \int_R^D J(s) \exp(X_t(s)) \, ds + D_{\mathrm{rs}} \mathcal{L} + \log p(A_t, L_t) D_{\mathrm{us}} \mathcal{L}$$

and

$$\sigma_L(t, A_t, X_t) = \sigma A_t.$$

Given that $Z_t = e^{rt}$, we have

$$d(Z_t^{-1} L_t) = e^{-rt}(\tilde{\mu}_L - r\mathcal{L})\,dt + e^{-rt}\sigma_L\,d\widetilde{W}_t. \tag{26}$$

Comparing now (25) to (26), we find

$$\tilde{\mu}_L - r\mathcal{L} = -b\int_R^D \exp(X_t(s))\,ds.$$

We obtain the following nonlinear Black-Scholes equation for the evolution of the liability value $\mathcal{L}(t, A, X)$:

$$-\frac{\partial \mathcal{L}}{\partial t} = rA\frac{\partial \mathcal{L}}{\partial A} + \tfrac{1}{2}\sigma^2 A^2 \frac{\partial^2 \mathcal{L}}{\partial A^2} + D_{\mathrm{rs}}\mathcal{L} + \log p(A, L) D_{\mathrm{us}}\mathcal{L} - r\mathcal{L}.$$

A.2 An analytical approximation method

An alternative approach is to find some reasonable approximation of (11) with which $L_f^{T_1}$ has an analytical solution. For the numerical values considered in the paper (see Section 4.2), $L_f^{T_1}$ is close to a piecewise linear function of A_{T_2} (see Figure 7), so we may consider the following piecewise linear approximation:

$$L_n^{T_2} = \min\left\{\max\left[L_\ell^{T_2} + \phi(A_{T_2} - L_\ell^{T_2} K_\ell), L_\ell^{T_2}\right], L_u^{T_2}\right\}, \tag{27}$$

where

$$\phi = \frac{L_u^{T_2} - L_\ell^{T_2}}{L_u^{T_2} K_u - L_\ell^{T_2} K_\ell}.$$

Equation (27) can be rewritten as

$$L_n^{T_2} = L_\ell^{T_2} + \phi \max\left(A_{T_2} - L_\ell^{T_2} K_\ell, 0\right) - \phi \max\left(A_{T_2} - L_u^{T_2} K_u, 0\right),$$

The above equation shows that the second payment can be thought of as the payoff of a portfolio which consists of: (i) riskless payoff $L_\ell^{T_2}$, (ii) payoff of *long* ϕ units of the European call option written on A_{T_2} with strike at $L_\ell^{T_2} K_\ell$, and (iii) payoff of *short* ϕ units of the European call option written on A_{T_2} with strike at $L_u^{T_2} K_u$. Because the underlying is lognormal, we can use the well-known Black-Scholes formula to obtain $L_f^{T_1} = \mathcal{L}_f^{T_1}(A'_{T_1})$. In particular,
for $A'_{T_1} \le 0$,

$$\mathcal{L}_f^{T_1}(A'_{T_1}) = L_\ell^{T_2} e^{-r(T_2 - T_1)}, \tag{28a}$$

for $A'_{T_1} > 0$,

$$\begin{aligned} \mathcal{L}_f^{T_1}(A'_{T_1}) \quad = \quad & L_\ell^{T_2} e^{-r(T_2 - T_1)} + \phi BS(A'_{T_1}, L_\ell^{T_2} K_\ell, r, \alpha\sigma, T_1, T_2) \\ & - \phi BS(A'_{T_1}, L_u^{T_2} K_u, r, \alpha\sigma, T_1, T_2), \end{aligned} \tag{28b}$$

where $BS(P, E, a, b, \tau, T)$ is the Black-Scholes formula for call options.[7] Specifically,

$$BS(P, E, a, b, \tau, T) = P\Phi(d_1) - e^{-a(T-\tau)} X\Phi(d_2),$$

[7] It is assumed that in case $A_{T_1} < L_\ell^{T_1}$, the contingent contribution at time T_1 is equal to $L_\ell^{T_1} - A_{T_1}$; for the rest of the time, the fund has zero asset value.

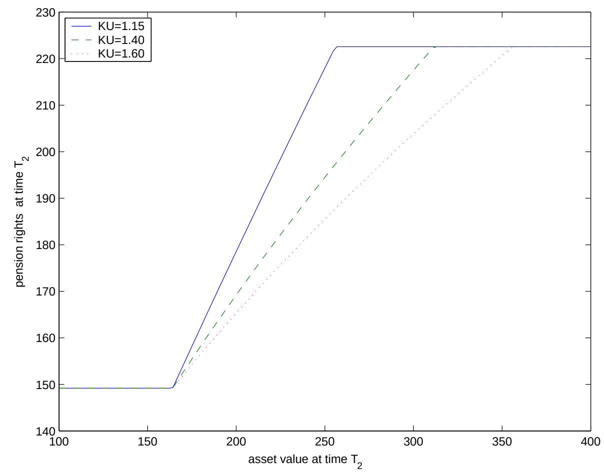

Figure 7: L_n^{T2} **as a function of** A_{T_2} This figure shows the amount of the payment at time T_2 as a function of the then asset value for different policy ladders. We fix the lower threshold of funding ratio, $K_\ell = 110\%$, and let the upper threshold, K_u, equal to 115% (the solid line), 140% (the dashed line), and 160% (the dotted line). The lower and upper bounds of the second payment are assumed to be 149 and 223 respectively. For other parameter values, see Section 4.2.

where $\Phi(\cdot)$ is the standard cumulative distribution function, and

$$\begin{aligned} d_1 &= \frac{\ln(P/E) + (a + \frac{1}{2}b^2)(T-\tau)}{b\sqrt{T-\tau}}, \\ d_2 &= \frac{\ln(P/E) + (a - \frac{1}{2}b^2)(T-\tau)}{b\sqrt{T-\tau}}. \end{aligned}$$

Whatever approach is used, one can obtain $L_f^{T_1}$ as a function of A'_{T_1}, which can also be formulated in terms of A_{T_1}. That is

A.3 An algorithm for computing the actual funding ratio in the proxy-based implementation

This appendix shows how to compute the true funding ratio of ZIP schemes using a Monte Carlo simulation algorithm. Using the risk-neutral pricing method, the true funding ratio can be computed as

$$\widetilde{FR}_t^{zip} = \frac{E_t^{\mathbb{Q}}\left[e^{-r(T_1-t)}L_{T_1} + e^{-r(T_2-t)}L_{T_2}\right]}{A_t}, \tag{29}$$

where $E_t^{\mathbb{Q}}[\cdot]$ denotes the expectation under the risk-neutral measure $\mathbb{Q}$ conditoning on information available at time t, L_{T_1} and L_{T_2} are the payments at T_1 and T_2, and hence the numerator is the market value of liabilities in the FTK scheme.

The value of $\widetilde{FR}_t^{zip}$ can be obtained through simulations. Let the number of simulations be denoted by n, and use **boldface** letters to denote $n \times 1$ vectors. Given the current asset value A_t and a constant-proportion strategy with stock weight α, the asset value at T_1 can simulated by

$$\mathbf{A}_{T_1} = A_t \exp\left([r - \frac{1}{2}\alpha^2\sigma^2](T_1 - t) + \alpha\sigma\sqrt{T_1 - t}\mathbf{Z}\right),$$

where $\mathbf{Z}$ is standard normal, and the FTK funding ratio at T_1 by

$$\mathbf{FR}^{zip}_{T_1} = \frac{\mathbf{A}_{T_1}}{L^{T_1}_\ell + L^{T_1}_\ell e^{-r(T_2-T_1)}}.$$

According to the FTK funding ratio, the first payment is determined as

$$\mathbf{L}_{T_1} = \begin{cases} L^{T_1}_\ell & \text{for } \mathbf{FR}^{zip}_{T_1} < K_\ell, \\ L^{T_1}_\ell + \frac{L^{T_1}_u - L^{T_1}_\ell}{K_u - K_\ell}(\mathbf{FR}^{zip}_{T_1} - K_\ell) & \text{for } K_\ell \le \mathbf{FR}^{zip}_{T_1} \le K_u, \\ L^{T_1}_u & \text{for } \mathbf{FR}^{zip}_{T_1} > K_u. \end{cases}$$

Subtracting the first payment, the asset value of the pension fund is

$$\mathbf{A}'_{T_1} = \max\left(\mathbf{A}_{T_1} - \mathbf{L}_{T_1}, \mathbf{0}\right),$$

where the max operator is due to the assumption specified by Footnote 7. Repeating the procedure for the first payment, one can simulate for T_2 the asset value, the FTK funding ratio and the second payment as the following:

$$\begin{aligned} \mathbf{A}_{T_2} &= \mathbf{A}'_{T_1} \exp\left([r - \frac{1}{2}\alpha^2\sigma^2](T_2 - T_1) + \alpha\sigma\sqrt{T_2 - T_1}\mathbf{Z}\right), \\ \mathbf{L}^{T_2}_\ell &= \mathbf{L}_{T_1}, \quad \mathbf{L}^{T_2}_u = \mathbf{L}_{T_1} e^{\rho(T_2-T_1)} \\ \mathbf{FR}^{zip}_{T_2} &= \frac{\mathbf{A}_{T_2}}{\mathbf{L}^{\ell}_{T_2}}, \\ \mathbf{L}_{T_2} &= \begin{cases} \mathbf{L}^{T_2}_\ell & \text{for } \mathbf{FR}^{zip}_{T_2} < K_\ell, \\ \mathbf{L}^{T_2}_\ell + \frac{\mathbf{L}^{T_2}_u - \mathbf{L}^{T_2}_\ell}{K_u - K_\ell}(\mathbf{FR}^{zip}_{T_2} - K_\ell) & \text{for } K_\ell \le \mathbf{FR}^{zip}_{T_2} \le K_u, \\ \mathbf{L}^{T_2}_u & \text{for } \mathbf{FR}^{zip}_{T_2} > K_u. \end{cases} \end{aligned}$$

where all the operations involving two or more vectors are understood to be entry-wise.

With $\mathbf{L}_{T_1}$ and $\mathbf{L}_{T_2}$, the true funding ratio of FTK schemes can be evaluated as

$$\widetilde{FR}^{zip}_t \simeq \frac{e^{-r(T_1-t)}\mathbf{L}^\top_{T_1}\mathbf{1}/n + e^{-r(T_2-t)}\mathbf{L}^\top_{T_2}\mathbf{1}/n}{A_t}.$$

References

K. Aase and S. Persson. Valuation of the minimum guaranteed return embedded in life insurance products. *Journal of Risk and Insurance*, 64:599–617, 1997.

L. Ballotta, S. Haberman, and N. Wang. Guarantees in with-profit and unitized with-profit life insurance contract: fair valuation problem in presence of the default option. *Journal of Risk and Insurance*, 73:97–121, 2006.

D. Bauer, R. Kiesel, A Kling, and J Ruß. Risk-neutral valuation of participating life insurance contract. *Insurance: Mathematics and Economics*, 39:171–183, 2006.

F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659, 1973.

F. de Jong. Pension fund investment and the valuation of liabilities under conditional indexation. *Insurance: Mathematics and Economics*, 42:1–13, 2008.

N. Gatzert and A. Kling. Analysis of participating life insurance contract: a unification approach. *Journal of Risk and Insurance*, 74:547–570, 2007.

A. Grosen and P. Jørgensen. Life insurance liabilities at market value. *Journal of Risk and Insurance*, 69:63–91, 2002.

R. C. Merton. Theory of rational option pricing. *Bell Journal of Economics*, 4: 141–183, 1973.

T. E. Nijman and R. S. J. Koijen. Valuation and risk management of inflation-sensitive pension rights. In N. E. Kortleve, T. E. Nijman, and E. P. M. Ponds, editors, *Fair Value and Pension Fund Management*. Elsevier, Amsterdam, 2006.

P. J. Schonbucher and P. Wilmott. The feedback effect of hedging in illiquid markets. *SIAM Journal of Applied Mathematics*, 61:232–272, 2000.

K. R. Sircar and G. Papanicolaou. General black-scholes models accounting for increased market volatility from hedging strategies. *Applied Mathematical Finance*, 5:45–82, 1998.