UNIVERSITY OF AMSTERDAM

Ergodic Theorems for Polynomials in Nilpotent Groups
P. Zorin-Kranich

Furstenberg's groundbreaking ergodic theoretic proof of Szemerédi's theorem on arithmetic progressions in dense subsets of integers suggested at least two possible directions for generalization. One is connected with earlier work of Furstenberg and consists in investigating actions of groups other than $\mathbb{Z}$. The other looks at polynomial, rather than linear, sequences. Indeed, in the same article Furstenberg gave a short qualitative proof of Sárközy's theorem on squares in difference sets.

Furstenberg's proof of the multiple recurrence theorem involves three main steps: a structure theorem for measure-preserving systems that exhibits dichotomy between (relative) almost periodicity and (relative) weak mixing, a coloring argument that deals with the almost periodic part of the structure, and a multiple weak mixing argument dealing with the weakly mixing part. It was the structure theorem whose generalization to $\mathbb{Z}^d$-actions required most of the additional work in the multiple recurrence theorem for commuting transformations due to Furstenberg and Katznelson . The coloring argument carried through using Gallai's multidimensional version of van der Waerden's theorem. Also the multiple weak mixing argument worked similarly to the case of $\mathbb{Z}$-actions, namely by induction on the number of terms in the multiple ergodic average.

However, in the polynomial case, induction on the number of terms does not seem to work. This difficulty has been resolved by Bergelson who has found an appropriate induction scheme, called *PET induction*. Later, jointly with Leibman , he completed his program by proving a polynomial multiple recurrence theorem by Furstenberg's method, using a polynomial van der Waerden theorem as the main new ingredient.

This is when polynomials in nilpotent groups appeared on the stage. Both the coloring and the multiple weak mixing steps in Furstenberg's framework involve polynomial maps and PET induction when carried out for nilpotent groups, even if one is ultimately interested in linear sequences. A similar phenomenon occurs in Walsh's recent proof of norm convergence of nilpotent multiple ergodic averages (which we extended to arbitrary amenable groups). Thus polynomials in nilpotent groups, with Leibman's axiomatization , seem indispensable for understanding multiple recurrence for nilpotent group actions.

While the work mentioned above concerns Cesàro averages of multicorrelation sequences, there are by now at least two alternative approaches to the study of asymptotic behavior of dynamical systems: using IP-limits or using limits along idempotent ultrafilters. It is the former approach on which we concentrate. The proof of the IP multiple recurrence theorem due to Furstenberg and Katznelson parallels Furstenberg's earlier averaging arguments, although rigidity replaces almost periodicity, mild mixing weak mixing, and the Hales-Jewett theorem the van der Waerden theorem. A direct continuation of their work in the polynomial direction has been carried out by Bergelson and McCutcheon , mixing the techniques outlined above and a polynomial extension of the Hales-Jewett theorem proved earlier by Bergelson and Leibman .

One of our main results is a nilpotent extension of the IP multiple recurrence theorem . We take a somewhat novel approach to the structure theorem, obtaining dichotomy between compactness and mixing not on the level of the acting group, but on the level of the group of polynomials with values in the acting group. We have also found it necessary to prove a new coloring result, sharpening the nilpotent Hales-Jewett theorem due to Bergelson and Leibman . On the other hand, the mixing part is handled in essentially the usual way using PET induction. As remarked earlier, this method compels us to deal with polynomial mappings. Our arguments heavily rely on an efficient axiomatization of IP-polynomials along the lines of Leibman's work, but incorporating some more recent ideas.

Another reason to study polynomial, rather than linear, sequences in nilpotent groups comes from quantitative equidistribution theory on nilmanifolds (that is, compact homogeneous spaces of nilpotent Lie groups). Nilmanifolds play an important role in the theory of multiple ergodic averages , where one is interested in linear orbits of the form $(g^n x)$, where x is a point in the nilmanifold and g an element of the structure group. An obstacle for establishing results that are uniform in all such orbits is the fact that g is drawn from the possibly non-compact structure group. This can be circumvented by considering polynomial orbits, since every linear orbit on a nilmanifold can be represented as a polynomial orbit with coefficients that come from fixed compact subsets of the structure group. This is an important ingredient in the proof of the quantitative equidistribution theorem of Green and Tao .

We review this circle of ideas in order to motivate our proof of the uniform extension of the Wiener-Wintner theorem for nilsequences. This is a result about universally good weights for the pointwise ergodic theorem, that is, sequences (a_n) such that for every ergodic measure-preserving system (X, T) and every $f \in L^\infty(X)$ the averages

$$\frac{1}{N} \sum_{n \le N} a_n T^n f$$

converge as $N \to \infty$ pointwise almost everywhere. The Wiener-Wintner theorem for nilsequences states that nilsequences are universally good weights, the full measure set on which convergence holds being independent of the nilsequence. We show that convergence in this result is in fact uniform over a class of nilsequences of bounded complexity provided that f is orthogonal to a certain nilfactor of (X, T) (this is joint work with T. Eisner). The explicit description of a full measure set on which the above averages converge also allows us to deduce a version of the Wiener-Wintner theorem for non-ergodic systems (note that an appeal to the ergodic decomposition does not suffice for this purpose).

An opposite extreme to nilsequences in the class of universally good weights for the pointwise ergodic theorem are the random weights provided by Bourgain's return times theorem . This result has been generalized to certain multiple ergodic averages by Rudolph using the machinery of joinings. In a different direction, a Wiener-Wintner type extension of the return times theorem has been obtained by Assani, Lesigne, and Rudolph using the Conze-Lesigne algebra. This suggested to attack the multiple term return times theorem using Host-Kra structure theory, which we do in the last chapter. This leads us to a joint extension of all aforementioned weighted pointwise ergodic theorems, in which we also identify characteristic factors. The proof involves a version of Bourgain's orthogonality criterion valid for arbitrary tempered Følner sequences .