



Graph Parameters and Invariants of the Orthogonal Group
G. Regts

Summary

This thesis is concerned with links between certain graph parameters and the invariant theory of the orthogonal group and some of its subgroups. These links are given through so-called *partition functions of edge-coloring models*. These partition functions can be seen as graph parameters as well as polynomials that are invariant under a natural action of the orthogonal group.

Partition functions of edge-coloring models were introduced as graph parameters by de la Harpe and Jones [28] in 1993. For $k \in \mathbb{N}$, a *k-color edge-coloring model* (actually called *vertex model* in [28]) is a statistical physics model. Given a graph G , we can think of the edges of G as particles, the vertices as interactions between particles and the colors as states. Given a coloring of the edges of G with k colors (i.e. an assignment of states to the particles), at each vertex we see a multiset of colors to which the edge-coloring model assigns a number. The *weight* of the coloring is the product over the vertices of G of these numbers; in statistical mechanics it is called the *Boltzmann weight*. The *partition function* of the model is the sum, over all possible colorings of the edges of G with k colors, of the weights associated to these colorings.

Many interesting graph parameters are partition functions of edge-coloring models. For example, the number of perfect matchings, the number of proper k -edge-colorings for fixed $k \in \mathbb{N}$, but also the number of homomorphisms into a fixed graph.

In this thesis we characterize when a graph parameter f is the partition function of a complex-valued k -color edge-coloring model, for a fixed $k \in \mathbb{N}$, in terms of an infinite number of equations of the form $\sum_{i=1}^n \lambda_i f(G_i) = 0$, for certain $\lambda_i \in \{\pm 1\}^n$, graphs G_1, \dots, G_n and $n \in \mathbb{N}$. These equations can be thought of as describing an ideal in a polynomial ring R with infinitely many variables. The proof of the characterization is based on a combinatorial interpretation of these polynomials in R that are invariant under the orthogonal group, which in turn is proved using the First and Second Fundamental Theorem of invariant theory for the orthogonal group, and on Hilbert's Nullstellensatz.

Summary

An important tool are certain labeled graphs, called *fragments*. One can construct, for any edge-coloring model h , a natural map from the space of formal linear combinations of fragments to the tensor algebra. If h is real valued, then the image of this map turns out to be the algebra of those tensors that are invariant under the subgroup of the orthogonal group consisting of the elements leaving h invariant. This is proved using a theorem of Schrijver [58]. If h is complex valued the situation is more complicated, but a similar statement can be proved. The connection between fragments and invariant tensors allows us to answer a question posed by Szegedy [66].

Besides introducing the edge-coloring model, de la Harpe and Jones also introduced the *vertex-coloring model* (which is called spin model in statistical mechanics). Given a graph G , we can also think of the vertices of G as particles, the edges as interactions between particles and again the colors as states. Given a coloring of the vertices of G with n colors (i.e. an assignment of states to the particles), at every edge one sees a pair of colors; the vertex-coloring model assigns a number to each such pair. The *weight* of the coloring is the product over the edges of G of the numbers associated to these pairs. This is called the *Boltzmann weight* in statistical mechanics. The *partition function* of a vertex-coloring model is the sum over all possible colorings of the vertices of the graph with n colors of the associated weights. Partition functions of vertex coloring models generalize counting graph homomorphisms.

Szegedy [66] showed that any partition function of a vertex-coloring model can also be obtained as the partition function of a complex edge-coloring model. Using advanced methods from geometric invariant theory we are able to characterize in this thesis for which vertex-coloring models the edge-coloring model can be taken to be real valued.

In [45], Lovász and Szegedy introduce vertex-coloring models with infinitely many colors and show how they can be seen as limits of certain sequences of simple graphs when the set of simple graphs is equipped with a topology based on homomorphism densities. Motivated by this work, we introduce in this thesis edge-coloring models with infinitely many colors and show how they can be seen as limit objects of certain sequences of edge-coloring models with finitely many colors if the latter set is equipped with a particular topology.