

*Quantum Entanglement in Non-Local Games, Graph Parameters and Zero-Error Information Theory*

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## Abstract

We study quantum entanglement and some of its applications in graph theory and zero-error information theory.

In Chapter 1 we introduce entanglement and other fundamental concepts of quantum theory. Entanglement was first described in 1935 by Einstein, Podolsky and Rosen, who observed that it allowed for stronger-than-classical correlations between distant particles. They did not believe such correlations existed in nature, and interpreted this as evidence for incompleteness or incorrectness of quantum mechanics. Bell took the next step in 1964, proposing an experiment to test whether such non-classical behavior occurs in nature. He showed that classical input-output correlations satisfy a certain inequality, called a Bell inequality. Then, he showed that quantum mechanics allows for correlations that violate it. Aspect *et al.*, in the early 1980s, realized such an experiment for the first time, showing that Nature violated a Bell inequality. Therefore, Nature does not follow the rules of classical physics!

In this thesis we make use of quantum entanglement as a resource for various information-processing tasks. In Chapter 2 we address the question of *how much* quantum correlations can deviate from classical predictions. We focus on *non-local games*: experiments in which two players are separated and forbidden to communicate, and have to collaborate to accomplish a task. In this case, a Bell inequality is an upper bound on the maximum success probability of classical players. We have a Bell inequality violation when quantum players use an entangled state to achieve a larger success probability. Upper bounds on Bell inequality violation of  $G_s$  were given by Junge *et al.* in 2009. They proved that the maximum violation depends on the number of possible inputs and outputs of each player, and on the dimension of the entangled state. We give *lower bounds* in the form of two games exhibiting violations that are very close to the upper bounds of Junge *et al.* Remarkably, our results in theoretical physics are inspired by theoretical computer science. The first non-local game is based on a communication complexity problem called “hidden matching”, while the second

one derives from the work of Khot and Vishnoi on the famous Unique Games Conjecture from computational complexity theory.

Chapter 3 is dedicated to the study of *quantum graph parameters*. Interestingly, well-known quantities such as the chromatic number and the independence number of a graph can be interpreted as parameters for non-local games. For example, a “referee” can test if two players, Alice and Bob, have a  $k$ -coloring of a graph  $G$ , *i.e.*, an assignment of one out of  $k$  colors for each vertex of the graph such that adjacent vertices get different colors. He separates the two players, gives each of them a vertex of  $G$ , and asks them for the color (out of the  $k$  possible) of that vertex. If he gave Alice and Bob the same vertex, he expects to receive back the same color, if he gave them two adjacent vertices, he expects to get two different colors. The minimum number of colors  $k$  such that Alice and Bob have a classical strategy that always succeeds is the chromatic number of  $G$ . If Alice and Bob can use quantum entanglement, then the minimum  $k$  such that they always win is called the *quantum* chromatic number of the graph. For some graphs, it can be strictly smaller than the classical one. The study of quantum graph parameters officially started with two papers by Cameron *et al.* in 2010 and Cubitt *et al.* in 2009, but was implicit in earlier work. We contribute to the field in a number of ways. We study the relationship between these quantum graph parameters and other parameters such as the Lovász  $\vartheta$  number and the orthogonal rank. We find a surprising characterization of the graphs having a separation between the quantum and classical chromatic numbers. This is related to the Kochen-Specker theorem, a result in the foundations of quantum mechanics from 1967. We also find various constructions of graphs that feature separations between the quantum and classical independence numbers, based for example on graph products and graph states. Additionally, we use quantum graph parameters to give bounds on the value of general non-local games.

In Chapter 4, we move to *zero-error information theory*. We study the *zero-error capacity* of a classical noisy channel when the sender and the receiver can use quantum entanglement. We also study the *source problem*, where the receiver has partial information about the message that is going to be delivered, as well as the combined *source-channel problem*. The main tools in this field are graphs and their parameters, so zero-error information theory with entanglement is a fertile field for the concepts studied in previous chapters. For example, the capacity of a channel can be calculated using the independence number of a graph and the source-problem is related to the chromatic number of a graph. We define the *entangled* chromatic number, the *entangled* independence number and other related quantities. The exact relation with the parameters of Chapter 3 is still an open problem. We initiate the study of the source problem and source-channel problem with entanglement and we find channels and sources that exhibit a strong divergence in quantum and classical behaviors. To do that, we use results in combinatorics, linear algebra, optimization and number theory.

This thesis is based on the following articles:

- “*Near-optimal and explicit Bell inequality violations*”, by H. Buhrman, O. Regev, the author and R. de Wolf. Theory of Computing, December 2012.
- “*Kochen-Specker Sets and the Rank-1 Quantum Chromatic Number*”, by the author and S. Severini. IEEE Transactions on Information Theory, April 2012.
- “*New Separations in Zero-error Channel Capacity through Projective Kochen-Specker Sets and Quantum Coloring*”, by L. Mančinska, the author and S. Severini. IEEE Transactions on Information Theory, June 2013.
- “*Exclusivity structures and graph representatives of local complementation orbits*”, by A. Cabello, M. G. Parker, the author and S. Severini. Journal of Mathematical Physics, July 2013.
- “*Zero-error source-channel coding with entanglement*”, by J. Briët, H. Buhrman, M. Laurent, T. Piovesan and the author. Proceedings of Eurocomb’13, September 2013.