



On the Articulation of Systematic-Dialectical Methodology and Mathematics

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***ON THE ARTICULATION OF SYSTEMATIC-DIALECTICAL
METHODOLOGY AND MATHEMATICS***

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Summary and General Conclusions

The aim for this dissertation was twofold. First, it has established how a systematic dialectical perspective elucidates the nature of mathematics by clarifying the nature of the concepts that the mathematical sciences depend upon. Secondly, this dissertation has shown how a systematic dialectical perspective on capitalism may inform assumptions for mathematical models and how the results of these models can further the systematic dialectical exhibition that gave rise to the assumptions. Thus, it was shown how the two approaches mutually reinforce each other and are far from being mutually exclusive. The former was achieved by means of an investigation of Hegel's dialectical account of the mathematical (Chapter 2). The latter aim was achieved by a critical evaluation of the assumptions Marx makes when drawing up his schemes (i.e. models) of reproduction from a systematic dialectical perspective. Since these models are presented alongside Marx's systematic dialectical account of Capitalism, an exploration of the way Marx has integrated them into his overall account (Chapter 3) and the possibilities for improvement in this respect (Chapter 4), have been instrumental in indicating how a tighter fit between the systematic dialectical foundations of model assumptions, the assumptions themselves and the model may be achieved.

Before Hegel's dialectical determination of the mathematical could be presented, however, a more detailed understanding of Hegel's and Marx's historical and systematic dialectics was required (Chapter 1). Thus, the dissertation opened with a methodological chapter describing Hegel's dialectics and Marx's critique thereof as well as its implications for Marx's own dialectics. In a nutshell, the systematic dialectical method reconstructs the knowledge about a given object totality, whose intelligibility is fully dependent on one category: its universal principle. One can of course only reconstruct knowledge if one has first acquired enough of it (in the phase of appropriation).

The universal principle is a category without which no sense can be made of the totality under scrutiny. A central reference in this dissertation was Hegel's *Encyclopädie der philosophischen Wissenschaften*, in which Hegel outlines the systematic interconnections between all fields of knowledge in their totality. In his view three *object* totalities can be distinguished within this totality of all knowledge: the Logic, the realm of 'the idea in and for itself'; Nature, the realm of 'the idea in its otherness'; and society, the realm of 'the idea that returns into itself out of its otherness'. Hegel identifies Being, Space and Free Will as their respective universal principles.

In order to reconstruct the knowledge about an object totality from such a universal principle, Hegel asks himself three questions: α) how does this universal

principle appear in total categorial isolation (that is if one tries to imagine it without taking recourse to any examples)?; β) how does it express itself in the world (that is, how does it appear if one tries to behold all of its instances/examples at once)?; and finally γ) how can the tension between an α) abstract thought and its β) instances be resolved? By asking the questions α) and β) again about the category found under γ) the process can start anew, until finally some γ) is found that is at one with its expression and thus would no longer yield oppositional answers to α) and β).

After his systematic dialectical accounts were complete, Hegel applied these principles to the philosophy of history, that he describes as a battle of the α) 'abstract Generality' of the state against the β) principle of specific Subjectivity that would eventually give rise to the new nexus of the γ) Ideal State. In Hegel's view this process could potentially be completed in the post French revolution society he lived in.

Marx's criticism of Hegel is two-fold. First, in his *Zur Kritik der Hegelschen Rechtsphilosophie*, he criticizes Hegel's account of society for being too harmonious. Secondly, and in my view relatedly, in his *Critique of Hegel's Philosophy and Dialectics in General* he criticizes Hegel's dialectical obsession with the resolution of conflicts, for in Marx's view this precludes Hegel from recognizing misrepresentations of Nature in thought as well as actually existent conflicts. Marx's historical materialism that largely got shape in *Die Deutsche Ideologie* (which Marx wrote in collaboration with Engels) indicates Marx's partial solution to both problems. In response to the first critique, historical materialism describes material inequalities as an ongoing cause of change and revolution, and thus identifies every society thus far, including capitalism, as a battlefield in which Free Will is anything but actualized. In response to the second critique, this conflict-ridden take on Capitalism also allows for a description of actually existent conflicts. Furthermore, Marx and Engels' account of history takes distance from the received view, thus enabling the recognition of misrepresentations in historical accounts thus far.

So, if one holds that Marx intended his social theories in the *Grundrisse* and *Capital* to be systematic dialectical (as all authors reviewed in Section 1.4 do – and I concur) it is likely to differ from Hegel's in three respects:

1. The knowledge reconstructed should be appropriated critically, so as to allow for the chance that categories developed in the empirical sciences misrepresent the matter at hand.
2. Its starting point should be materially grounded in the relations of production emanating from material inequalities.
3. Its starting point should allow for unresolved conflicts at every stage of the dialectical exhibition.

Whether Marx's attempt at formulating a systematic dialectical alternative for Hegel's social theory was successful and the ramifications this has for model building can best be discussed after Hegel's take on the mathematical is described, as was done in Chapter 2.

Hegel discusses the Quantitative and its moments as part of 1) the Doctrine of Being, which is the first of the three subdivisions in his *Logic* (the other two being 2) the Doctrine of Essence and 3) the Doctrine of the Concept). This doctrine is itself subdivided into the sections A) Quality, B) Quantity and C) Measure. The three object totalities Hegel discusses in his *Encyclopädie*, as well as the subdivisions therein and the sections of those, all relate to each other in the same way as α , β and γ do. Hence, the place Hegel reserves for his determination of the Quantitative implies that 1) Hegel conceives of the Quantitative as a fairly abstract field that is nevertheless indispensable for the understanding of everything else and 2) the Quantitative is a reflection on the hopeless multiplicity one is confronted with when trying to get to grips with all concrete instances and examples of the Qualitative at once. In overview, Hegel reasons as follows: On the basis of Quality alone we are unable to make qualitative distinctions, so we enter the realm of Quantity, which is governed by external reflections on sets of elements. The elements are arbitrarily chosen Units One and the sets are Amounts of them expressed through Number in an Intensive Magnitude that has its ultimate meaning in the bad potential infinity that develops in its Extensive Magnitude, but can only be negatively defined as being beyond the finite. So just as Quality is not sufficient to understand the absolute, so is Quantity. So we need both. That is, we need a qualitative Quantum: γ) Measure. Only through Measure there can be any hopes for practical applications of the Quantitative and its categories.

Since the Qualitative must thus dialectically precede the Quantitative, the mathematical requires the Qualitative for its existence in thought (and no one can be aware of anything that cannot be thought). The fact that mathematics helps us to comprehend reality is a result of this. It does not fit reality because 'the book of nature is written in the mathematical language' (as if nature is somehow ontologically quantitative), but because language has evolved a tight fit with reality and by implication so does mathematics.

As to clarifying the nature of the concepts that the mathematical sciences depend upon, Hegel's dialectical treatment of the mathematical, when considered in more detail, clarifies the nature of mathematical concepts like the One, the successor function and sets and elements from their systematic dialectical relations to other categories in language, rather than from their mathematical relationships only. To begin with, our understanding of Hegel's dialectical treatment of the Quantitative and the fact that its thinkability springs from the intellect's failure to comprehend the Qualitative all at once and on its own terms,

implies further that the One and the successor function have a qualitative base and need not be presupposed, as they usually are in mathematics.

Secondly, bringing Hegelian terminology to bear on set theory helps the mathematically minded to understand what Hegel was probably on about as well as helping Hegelians to understand set theory, particularly regarding the proper understanding and use of ordinal and cardinal Numbers. The ordinal number is the number you are arbitrarily assigning to each element as you are counting (that is ordering) the elements in a set. For Hegel this counting operation involves a continuous move from the elements already contained in the set we Numerated and thus considered and those for which we have not yet done so. The former as Intensive Magnitude determines what the set is and as such positively defines it, while the latter determine what it is not and thus negatively define it. Numeration then, is expanding an Intensive Magnitude into an Extensive Magnitude. So although these magnitudes change, the operation by which this is done (i.e. Numeration by using the successor function) does not. This leads Hegel to consider mathematical infinity, ∞ , as the bad potential infinity that is never reached and the operation of Numeration as the true philosophical Infinity. When the size of a set's Intensive Magnitude is determined by completing the Numeration of all its elements, it does no longer matter which element was counted first and which second, for no matter where you started, the Number reached will be the same for any particular given set. From a mathematical perspective we have then reached a cardinal Number, which from a Hegelian perspective is best understood as the Intensive Magnitude or size of a finite set. As such, the size of the set is itself a Unit: it expresses the Number of elements it contains while denying them autonomy.

Hegel's point about the relationship between Intensive and Extensive Magnitudes not only elucidates the use of finite ordinal and cardinal Numbers, but also that of infinite ones. If a set is expanded with all subsets contained within it, this is equivalent to raising 2 to the power of the number of elements in that set. By analogy, since the Intensive Magnitude of the denumerable infinite set of all natural numbers \mathbb{N} is defined as \aleph_0 , it contains 2^{\aleph_0} subsets, so the size or Intensive Magnitude of the power set of \mathbb{N} , $P(\mathbb{N})$, that is of the continuum \mathbb{R} is 2^{\aleph_0} and that of $P(\mathbb{R})$ is $2^{2^{\aleph_0}}$, etc. It can be proven that each set thus obtained is of a higher order of infinity than the previous set. This implies that infinite cardinal Numbers may themselves be ranked in a well ordering. So just as transfinite iterations of a successor function lead a finite Intensive Magnitude into the bad potential infinity Hegel associates with its Extensive Magnitude, transfinite iterations of the power operation lead an infinite Intensive Magnitude into the 'worst' potential infinity associated with the size of the class of all sets V . In

short: even for an infinite Intensive Magnitude there exists an Extensive Magnitude through which it gains meaning.

‘[S]ince the power set of \mathbb{N} [...], contains an enormous Amount of infinite sets as elements that therefore must be seen as complete, ‘finished’, limited objects’, this way of thinking implies ‘the existence of an enormous amount and enormously big *actually infinite sets*’ (Horsten 2004: 27, my translation). This fact, together with the fact that even infinite Numbers can be ordered to fit Hegel’s conceptual apparatus, at least partially dispenses with the ‘badness’ of Hegel’s bad potential infinity in that infinity is no longer just defined as an unreachable Extensive Magnitude beyond every finite Intensive Magnitude, but within the well ordering of the infinite cardinal Number associated with that set, can itself also be viewed as an Intensive Magnitude. As a result we can now distinguish two principles of philosophical infinity: 1) the principle of Numeration that leads a finite Intensive Magnitude into its potentially infinite Extensive Magnitude and 2) the principle of the power operation that ultimately leads an infinite Intensive Magnitude into the Extensive Magnitude associated with the set of all sets, V . Because the founding father of set theory, Georg Cantor, was born after Hegel’s death, Hegel cannot possibly have been aware of these points.

Chapter 3 next elaborated on Marx’s dialectics. It follows Smith (1990) in his identification of Marx’s universal principle as being the need for *exchange* that arises from the indirect sociality of capitalist production. This indirect sociality itself stems from the institutional separation of the site of production from that of consumption that is so characteristic of capitalism (cf. Reuten & Williams 1989: 56-57). Not only is this starting point materially grounded in Marx’s historical materialist account of human history up until capitalism, it also allows for unresolved conflicts, for producers may fail to sell and consumers may be unable to buy. In either case real people have real problems caused by the other group. Therefore, in Marx’s account, capitalist society is not governed by Free Will, but by the imperative for exchange and thus by ‘commodification’. If products are produced to be exchanged (which determines them as *commodities*), they are produced for their *value* rather than their use. *Value in exchange* does not Measure Qualitative usefulness but rather is imposed on the product in capitalism. As such it does not predate the capitalist mode of production and would cease to exist when that mode of production would. As such, the further determination of *value in exchange* as *the money form of value* is not a Measure in a Hegelian sense at all, for rather than pinning a Quantity on a pre-existing Quality it is ontologically quantitative through and through. *Value in exchange* is an abstraction actualized through *money*. It is an abstraction-in-practice that rules our daily lives and arbitrates between life and death.

The exchange imperative not only allows for real conflicts, it also serves as an inescapable determination of human behavior and thus functions in ways that are similar to forces of nature. So, the mentioned imperative implies at least a similar potential for the application of mathematical models as is present in the natural sciences. Forces of nature and their determinants, however, must be Measured before they can be modeled. Models in the natural sciences therefore work with quantitative representations of pre-existing Qualities (such as e.g. length in physics), they are not ontologically Quantitative, whereas the entities that go into modeling capitalism are. The upshot of this is, that pure mathematical quantities can now be seen as a driving force of human behavior and modeled using purely quantitative techniques and categories. As a result, what Hegel holds for determinations on the level of abstract thinking only can be directly applied to the study of capitalism. It is this observation that led Arthur to contend that the grand structure of the three volumes of *Capital* is homologous to the structure of Hegel's *Logic*.

With indirect sociality the qualities of the product produced are immaterial to the producer. As a result, *money as capital* must be the *end of exchange* and each sum of *money* invested in means of production (making it into *constant capital*) must be worked up by labor (employed as *variable capital*) in order to be exchanged for a higher sum of *money* to be reinvested and *accumulated*. From a capitalist perspective, this accumulation takes the form of constant increments in money, but from the perspective of reinvestment it seems constant increases in the scale of operation are the driving force. Finally, from the perspective of society the qualitative change in the product (from raw materials to a finished commodity) seem to be the main point of production. The time a production cycle takes, can vary and so can the number of cycles for which particular means of production last. If they last for just one cycle, Marx dubs them *circulating capital* and if they last for more than one he calls them *fixed capital*. At this stage, Marx introduces his schemes of reproduction.

Though Marx's schemes of reproduction are commonly considered the first two-sector macro-economic model ever conceived of, they do not live up to their full potential as mathematical models of systematic dialectically conceived of interrelationships within capitalism. They model the interactions between *capital's* two main departments: one producing means of production, the other consumption goods. As such, the model pertains to a rather abstract and general level in Marx's dialectical account. For the model to reflect that, its formulation should refrain from filling in any specific parameters or numbers. Instead, it should be kept as general as possible and therefore formulated purely algebraically (as Chapter 4 did).

Secondly, Marx does not present his model assumptions as though they result from his systematic dialectics, even though many of them might very well be presented like that. Since Volume II of *Capital*, in which Marx presents his schemes, was only posthumously published on the basis of the drafts and notebooks Marx bequeathed to Engels, we can only guess whether Marx would have done so had he lived long enough to write a more final draft. Either way, the schemes being presented as they are, there is room for improvement in both respects.

Specifically, a systematic dialectical account may inform four different types of assumptions:

1. foundational assumptions, that outline the important categories to model and indicate how they are related;
2. heuristic assumptions that can be used to create ever more concrete model generations, for instance by setting out a static model and allowing dynamics in later (this is what is going on when Marx first models *simple reproduction* and moves on to *expanded reproduction* later);
3. absency assumptions, that stipulate that certain influences, though not empirically absent are absent at the level of abstraction the model pertains to;
4. anticipatory assumptions that outline conditions of existence and anticipate them.

Foundational assumptions may follow directly from a systematic dialectical exhibition. After all, such an exhibition shows the categories that are important and suggests how they are interdependent, thus providing a first indication of a possible model specification. Heuristic assumptions should be sought after next. A dialectical defense of such assumptions can be that the (static) possibility of the very existence of a posited relationship should be investigated, before the (dynamic) development of such a relationship can even be contemplated. If the dialectical account indicates that a dynamic relationship is imperative, but the model cannot even ground the existence of a static one, the dialectical account so far apparently still lacks completeness and thus further mediating conditions must be dialectically determined before moving on to the dynamic model of the relationship's development. If, as was the case with Marx's schemes of reproduction, the static model does not preclude dynamics, one may of course move on to drop the heuristic assumption(s) without first returning to the systematic dialectics.

Together these two types of assumptions describe rather positively what the model is about and by implication this gives a lot of information on what is excluded from it. Yet sometimes it is helpful to reflect on the level of abstraction reached in the dialectical exhibition and explicate the influences that one has not

yet exhibited and can thus safely abstract from at the level of abstraction depicted in the model (as is the case with foreign trade at the level of abstraction Marx's schemes of reproduction pertain to). The explication of as yet unexhibited influences happens in absence assumptions. These, however, should be no more than a check on the model specification emanating from the foundational and heuristic assumptions. After all, if a certain influence was dialectically determined to be important at the level of abstraction of the model, it should have been a factor in the foundational assumptions or specifically neglected for heuristic reasons.

Finally, the assumptions that anticipate conditions of existence are in place to enable the unhindered expression of a force or tendency whose articulation has been dialectically determined to be necessary for the functioning of the object totality under scrutiny (for Marx's schemes of expanded reproduction the case in point is that of sufficient monetary accommodation and availability of labor power enabling unhindered accumulation of capital). This type of assumption differs from the absence type. The latter explicates the complete absence of something at the level of abstraction modeled (such as foreign trade). In the case of the former, by contrast, some general tendency or entity *has* been determined to exist in the abstract (such as the need for *capital to accumulate*), but the exact mechanism by which it might concretely come about has *not* yet been. In such cases it is dialectically OK to anticipate that possible impediments to the development that was determined as dialectically necessary, will somehow be taken away or rendered harmless at more concrete levels (e.g. monetary accommodation must somehow be offered if the abstract requirement of accumulation is to hold in the concrete). So instead of stipulating that something is *absent* at this level of abstraction, it stipulates that something we cannot yet specify must somehow be *present*. If the force or tendency in question, can be successfully expressed in a model under this type of assumption, but not when it is removed, this indicates the need to introduce institutions in the remainder of the systematic dialectical exhibition that allow the assumption to hold in practice (such as e.g. credit). (Of course, if removal of the assumption is unproblematic, it should not have been necessary to formulate it in the first place.) Thus, this fourth type of assumption is particularly useful for indicating the road ahead.

When this conceptual apparatus is applied to the assumptions Marx makes in drawing up his schemes of simple reproduction, we can identify the following as foundational (the letters indicate the order in which they were presented in *Capital II*):

- b.** No revolution in values takes place in the component parts of the productive capital

- d.** Society's total yearly product breaks down into I) means of production and II) means of consumption
- f.** The rate of valorization s/v (i.e. surplus-value over variable capital) is constant and given for each department (and set at 100% for both departments)

When we focus on the term 'component parts', rather than values, we can accept **b** as simply saying that productive capital can always be decomposed into *constant capital*, *variable capital* and surplus-value, a decomposition that follows directly from Marx's dialectical exhibition. Similarly, the institutional divide between the site of production and that of consumption means **d** is fully warranted dialectically. Though the terms Marx formulates it in are too strict (focusing on the 100% I placed between brackets), it is dialectically defensible to stipulate that s/v is more or less constant over time since only labor can produce value. Except then for the specific number Marx assumes, assumption **f** is defensible as a foundational assumption as well.

He also makes two heuristic assumptions:

- c.** The value of a department's yearly produce is constant and all surplus-value and wages are consumed (so there is no accumulation)
- g.** The ratio of variable to constant capital is equal, constant and given across departments

Marx himself treats **c** as a heuristic assumption. It is in place to focus on what the system is when viewed as static, before it is dynamized. When moving on to expanded reproduction he retains only one element in it: all wages are fully consumed. **g** effectively assumes technical change away. Marx retains this assumption for expanded reproduction, but since technical change is inherent in (the intensive aspect of) accumulation, this is not defensible.

Furthermore he explicates the absence of some things in 3 further assumptions:

- a.** Products are exchanged at their values
- e.** Depreciation costs equal replacement expenditure
- h.** There is no foreign trade.

Assumptions **a** and **h** follow from the fact that only departmentalization of capital has been dialectically determined, so distinctions within it that might induce a divergence between prices and values (such as competition or foreign trade), cannot be made yet. For the same reason, only macro-economic aggregates can be considered at this stage, so any mismatches between depreciation costs and replacement expenditure are likely to level out and **e** can be accepted as well.

Since assumption **c** implies that all values are constant, so will the ratios between them be. As a result, all more specific assumptions on values and ratios (i.e. **b**, **f** and **g**) are redundant for simple reproduction. The only assumptions Marx really needed to call upon in order to formulate his proportionality condition were

d (departmentalization as the foundation of his models), **c** (simple reproduction as the first case to be considered heuristically) **e** (anything depreciated gets replaced and thus represents a cost of production) and **h** (the absence of foreign trade). This condition, $(v + s)_I = c_{II}$, says that the value of consumptive expenditure in the department producing means of production is equivalent to the means of production used up in the department producing means of consumption. Thus, the condition indicates the interdependency between the two departments.

For Expanded reproduction Marx adds two more absence assumptions:

- i.** There has already been production on an expanded scale
- j.** The sum total of replacement expenditure on *fixed capital* equals the sum total of depreciation allowances for *fixed capital* in each department

Previous expansion (**i**) must be assumed when discussing full-fledged capitalism, for without it, it cannot be considered full-fledged. Assumption **j** effectively says that capital has no funds other than the value of their produce available from which to replace and accumulate capital and indeed the possibility of outside finance has not yet been dialectically determined.

Finally he anticipates that possible obstacles to the dialectically necessary imperative of *accumulation* will be taken away at later more concrete stages of the dialectical exhibition:

- k.** '[T]he amount of money present in a country is sufficient for both hoarding and accumulation.'
- l.** There is always enough labor power on hand

Here the argument is that the requirement of accumulation can only materialize concretely if somehow these conditions are met in later more concrete stages of the exhibition, even though we have not yet determined how this might be brought about.

With all these assumptions in place the proportionality condition for expanded reproduction becomes:

$$(v + s)_I - \Delta c_I = c_{II} + \Delta c_{II}$$

So, with expanded reproduction, the funds spend in the other department by the department producing means of production are lowered by the amount they accumulate, while the capital needs of the department producing means of consumption are actually higher, as it too has to accumulate.

Using the insights developed in Chapter 3, Chapter 4 proceeded to reconstruct Marx's reproduction schemes along dialectical lines. In particular it aimed to:

- 1) retain only the dialectically defensible (elements of) assumptions,
- 2) let dialectical reasoning rather than mathematical ease and rigor determine the order in which assumption and equations are presented,

- 3) present the whole model in its abstract generality and therefore algebraically throughout,
- 4) explicitly define all time-dependent variables as such.
- 5) integrate technical change and expansion, or the intensive and extensive aspects of accumulation respectively, in a comprehensive model of expanded reproduction.

Since Marx never brings technical change back in, a model generation had to be added to Marx's schemes to achieve the fifth aim. So Chapter 4 first discussed *simple reproduction*, next extensive growth and finally *expanded reproduction*. The first two are algebraically formulated and (where applicable) dynamized versions of Marx's own models of simple and expanded reproduction respectively. My model of expanded reproduction goes beyond Marx's. I nevertheless used Marx's term, for his *concept* encompasses technical change and, in order to achieve as tight a fit as possible between Marx's dialectics and his modeling, the model had to reflect that.

As indicated, simple reproduction requires just four assumptions, which - in keeping with my second aim - were presented in the following order:

1. Foundational: 'Society's total yearly product breaks down into *two great departments*': a department producing means of production (i.e. current and additional constant capital) (department p) and one producing means of consumption (i.e. commodities intended for consumption out of wages paid out to variable capital, and out of capitalists' surplus-value) (department c) (cf. assumption **d** in Section 3.3).
2. Heuristic: *Simple reproduction* means that 1) the total value of the yearly produce of both departments (x_p and x_c) is constant, 2) all surplus-value is consumed and 3) there are no savings out of wages (cf. **c**).
3. Absency: There is no foreign trade (cf. **h**).
4. Absency: Aggregate depreciation costs incurred yearly equal aggregate yearly replacement expenditure (cf. **e**).

With these in place, all of Marx's results could be formulated. Since the model for simple reproduction is static, there was no insight to be gained from modeling variables as time-dependent yet. Of course this changes when modeling extensive growth.

To model extensive growth in a dialectically defensible way, the following can be assumed:

- 2a. Absency: There are no savings out of wages (cf. **c**).
5. Absency: There are no price changes (cf. **a** and **b**)
6. Foundational: The ratio of s/v (notation: ε) is constant and given in each department (cf. **f**)

7. Heuristic: the organic composition of capital ($\kappa = c/(c + v)$) is constant and given for each department (cf. **g**).
8. Absency: All expansion of capital is financed out of surplus-value (cf. **j**).
9. Anticipation: There is always enough money available to finance hoarding for replacement purposes and accumulation at the desired rate (cf. **k**).
10. Anticipation: An accumulating department will always find sufficient labor power on hand to increase the variable capital it employs by as much as its growth rate requires (cf. **l**).
11. Anticipation: Changes in department p's rate of accumulation are always actualized. Department c's actualizable rates are therefore constrained by the amount left by department p. Anticipating eventual recovery, it is assumed department c accumulates in such a way that all of department p's produce is actually sold.

Of these, assumptions 5 through 7 are consequences from the rewriting of assumption 2 into 2a. That is, when all the dialectically undefendable aspects of that assumption are dropped, it is no longer a heuristic, but rather an absence assumption. That is, at the level of abstraction we are now considering, wages are unlikely to exceed subsistence wages by any significant amount and so we can safely assume that savings out of wages do not occur. But now that capital is explicitly allowed to expand, we do need to rethink the way in which this might occur, leading to assumptions 5 through 7. Note that in its current formulation assumption 5 integrates the original assumptions **a** and **b** by focusing on their value aspects, changes in which and price divergences of which, they declare *absent* at this level of abstraction. So the *foundational* elements ('component parts') of assumption **b** are no longer emphasized here and its character too changes. The order in which assumptions 8 through 10 were presented and the reason they can be adopted has been sufficiently explained.

Finally, assumption 11 (which Marx never explicates at all) is only called for if we want to model the effects that a unilateral decision by department p to alter its rate of accumulation has on the other department and the economy as a whole. With the original goal being to see what happens to the conditions for valorization when introducing accumulation, we did not need the assumption immediately. The assumption can best be viewed as an anticipatory assumption. With only two departments dialectically determined, a department that wants to keep a larger part of its produce for itself (or conversely, sell a larger part of it) can do so. But since doing so regarding consumption goods does not affect accumulation, department p can take the lead. Department c is to respond according to assumption 11 so as to limit the duration of emerging capitalist crises, and so assumption 11 is anticipatory to the extent that crises are usually overcome eventually.

As a result, the maximum rate of accumulation out of surplus-value possible for department c, θ_c , is a function of θ_p :

$$\theta_{ct} = \frac{(\varepsilon_p + 1 - \kappa_p \varepsilon_p \theta_{pt})(1/\kappa_p - 1)}{(1 - \kappa_c) \varepsilon_c} \times \frac{c_{pt}}{c_{ct}} - \frac{1}{(1 - \kappa_c) \varepsilon_c} \quad (4.16)$$

Alternatively, given that the relationship $\theta_i = \frac{g_i}{(1 - \kappa_i) \varepsilon_i}$ holds for both departments (albeit with different parameters) the actualizable growth rate for department c (g_c) can be expressed in terms of the growth rate set by department p (g_p):

$$g_{ct} = \frac{c_{pt}}{c_{ct}} \left[\frac{(1 - \kappa_p)(1 + \varepsilon_p)}{\kappa_p} - g_{pt} \right] - 1 \quad (4.17)$$

This equation shows that when department p sets its g_p to g_p' during the period under scrutiny, the growth rate for both departments starts to diverge, but it has also been shown that as long as assumption 11 holds, this effect lasts only one year. Thus The following would then hold:

$g_{c0} < g_0 < g_{p0} = g_1 = (x_{p1} - \delta c_1) / c_1 = \Delta c_1 / c_1 = g_{c1} = g_{p1}$. But, says Marx, this is utterly unlikely since there is no dialectically determined mechanism in place yet that will entice department c somehow to consistently buy up the left-overs. But even though smooth adaptation may be unlikely, some adaptation mechanism must exist for capitalism to survive its recurring crises.

In order to go beyond Marx and model technical change as an integral part, heuristic assumption 7 and anticipatory assumption 11 from the model of extensive growth had to be modified to become:

7a. Foundational: The organic composition of capital ($\kappa = c/(c + v)$) rises with each round of accumulation (operationalized here as a calendar year).

11a. Anticipation: Changes in department p's rate of accumulation are always actualized in conformity to the latest technology. Department c's actualizable rates are therefore constrained by the amount left by department p. Anticipating eventual recovery, it is assumed department c accumulates in such a way that all of department p's produce is actually sold.

Thus formulated, assumption 7 is finally on a par with Marx's dialectics and is therefore no longer a heuristic assumption, but has become foundational. After all, *capital's* preference for accumulating low risk *constant capital* over high risk *variable capital* that leads to a rising occ has long been shown dialectically to ensue from the workings of *accumulation*. So 7a flows immediately from the

dialectics leading up to the model. Assumption 11 is essentially retained, albeit that department p's accumulation now restricts department c in both setting the rate of accumulation and securing the latest technology for itself.

In analogy to 4.16. we may then write:

$$\frac{\Delta c_{ct}}{c_{ct}} = \left[1 + (1 - \theta_{pt}) \varepsilon_p - \varphi_p + \frac{\Delta c_{pt}}{c_{pt}} \right] \frac{1}{\zeta_{pt}} \frac{c_{pt}}{c_{ct}} - 1 \quad (4.19)$$

This result enables one to prove that balanced growth is impossible with ongoing technical change.

These models show how the models for the concepts of *simple reproduction*, *intensive* and *extensive growth* and *expanded reproduction* can be developed alongside and in line with the dialectics that gave rise to the concepts themselves. In so doing, the models grow ever more intricate and complex, which reflects how our understanding of the way in which the two great departments are interrelated grows more intricate and complex as well. Finally, the algebraic generalizations presented, ensure that the results found (such as the equations describing the way that the factors determining g_c interact during transition) are perfectly general and will work with all numbers. Additionally, the algebraic formalizations allow one to see how all vantage points taken until we arrived at the model for expanded reproduction can be conceived of as special cases of this most intricate and complicated model.

So when *extensive growth* is integrated with *intensive growth* a solution can still be constructed (at least theoretically) in which both departments can grow at the same rate. Moreover, the solution still allows for changes to be made to this rate. But it is also clear that the relevant equations have grown into quite inelegant monsters and that finding a solution somehow requires department c to take into account its own changing occ (κ_{c1}) its previous occ (κ_{c0}) its rate of surplus-value (ε_c) and those of the other great department (κ_{p1} , κ_{p0} and ε_p respectively). Meanwhile, it is unclear how this information could reach the other department and if it did somehow, what mechanism would entice it to act upon it so as to make sure all produce is valorized.

In overview, it can be concluded that the dialectical move from *capital's* static renewal to its dynamic expansion can be depicted not only verbally through a systematic dialectical exhibition, but can also be represented mathematically through a series of model generations, the last of which eventually dispenses with all heuristic assumptions. All in all, careful contemplation of the dialectics of a system has been shown to give extra support to assumptions figuring in formal and/or mathematical representations (i.e. models) of (aspects of) it, which in turn

makes such mathematical representations more defensible. I expect both model builders and dialectically inclined system analysts to benefit from this result, for in the very least it has been shown here that the two ways of reasoning are actually quite compatible. I have high hopes for a fruitful debate between the two groups based on the linkages between their fields that this book has shown do exist.