Correlation Functions of In- and Out-of-Equilibrium Integrable Models

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Summary

Since the birth of the scientific method it became widespread in the physical sciences the idea that natural phenomena can always be understood by studying the physics of each of their elementary constituents. While this attitude led to immense discoveries on the sub-atomic particles that brought to a unified theory of electromagnetism, weak and strong interactions inside the nuclei (the so called standard model), on the other hand it left many big questions with no proper answer. For example, one of the most fascinating concepts of modern-day physics is emergence, the process whereby complex macroscopic properties arise from the interaction of many simple-behaved constituents. For example, when interactions among the single constituents of a system becomes large, the properties of the whole are in general completely different from the properties of the single element. New undiscovered physics therefore emerges from the presence of such non-trivial interactions which cannot usually be treated perturbatevely, namely they cannot be addressed assuming that the interactions are small. When the constituents are confined to a one-dimensional geometry for example interactions become more and more effective. The traffic on a single line road can switch from fluid to jammed by slightly increasing the density of cars above a certain critical value. In the same way the movement of many interacting quantum particles in one dimension is collective and they behave as a fluid. When prepared in an out-of-equilibrium situation they reorganize themselves in a unusual, hard-to-predict manner that possibly leads to new exotic states of matter.

Quantum systems with a large number of strongly interacting constituents are usually very hard to address. To solve a quantum mechanical problem one indeed has to diagonalize the Hamiltonian of the system which is a matrix of the same size as the Hilbert space, whose dimension grows exponentially with the number N of elementary particles in the system. The study of quantum systems at equilibrium is based on a number of tools that either rely on the weakness of the interaction, like perturbation theory, or on the assumption that the fluctuations of physical observables around their equilibrium values are assumed to be small, as it is done in mean-field approaches. These approaches are based on assumptions that can rarely be checked with an exact solution for any number of particles N. However if we restrict ourselves to one spacial dimension it turns out that it is possible to find exact solutions for models which are truly interacting and physically relevant. These models are called *integrable* and the two examples addressed in this thesis are the XXZ Heisenberg spin chain [1] and the Lieb-

[1]With Hamiltonian $H = J\sum_{j=1}^{L}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z\right)$ with σ_j^α the Pauli matrices associated to each spin in the lattice.

Liniger model for point-interacting bosons [2]. Like non-interacting theories are reducible to single particle physics, in these models the many-body scatterings among the particles are reducible to sequences of two-body processes [3]. This reduction is enforced by a certain number of internal symmetries which also play the role of constrains for the non-equilibrium time evolution of the density matrix associated to the system. As a remarkable consequence the time evolution from a generic initial state keeps much more memory of the initial conditions than in a generic interacting system, where the density matrix simply relaxes to the usual Gibbs distribution.
If the initial state is a pure state which is not an eigenstate of the Hamiltonian (global quantum quench) then the asymptotic steady state, which effectively describes the equilibrium expectation values of all the local observables, is not trivially characterized by a single temperature, as it is the case in the Gibbs ensemble, but by many more additional data on the initial state. A typical problem is to determine the thermodynamically relevant information on the initial state that one needs to know in order to reproduce the equilibrium expectation values of all the local observables (represented by operators that act non-trivially only on a finite sub-region of the system), which are indeed the ones that we can directly observe in a laboratory [4] (see figure 1).

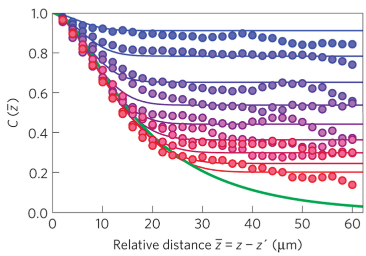

Figure 1: Experimental data for the time evolution (from top to bottom) of phase-phase correlations $\langle \Psi^{+}(z)\Psi(z')\rangle$ (with $\Psi^{+}(z), \Psi(z')$ the bosonic creation/annhilation operator) as a function of the relative distance $\bar{z} = z - z'$ of a one-dimensional Bose gas after having being coherently split into two parts. The green line corresponds to the expectation value of the operator $\Psi^{+}(z)\Psi(z')$ on the steady state, which can be described by the Generalized Gibbs Ensemble. The experiment is realized with contact interacting cold atoms in an effectively one-dimensional trap such that the quantum gas can be treated as a Lieb-Liniger gas of bosons.

The aim of this thesis is to develop a method to study the non-equilibrium dynamics of integrable systems in their ther-

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[2] With Hamiltonian $H = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j)$ acting on bosonic wave functions $\Psi(x_1, \ldots, x_N)$

[3] Namely there is no creation of new momenta in the scattering. If N particles scatter with initial momenta $(k_1, \ldots, k_N)$ then the final state is described by a new set of momenta which is simply a permutation $\mathcal{P}$ of the initial one $(k'_1, \ldots, k'_N) = \mathcal{P}(k_1, \ldots, k_N)$. On the other hand each particle gets a non-trivial phase shift which forces the momenta $(k_1, \ldots, k_N)$ to satisfy a set of non-linear coupled equations, the so called Bethe equations.

[4] We define a local observable as represented by an operator $\hat{O}(x)$ acting on a sub-region $[x, x+\delta x]$ of the entire volume L, such that $\lim_{\text{th}} \delta x / L = 0$, and with a negligible intensity, compared to the total energy of the system $\lim_{\text{th}} \frac{\langle \hat{O} \rangle}{\langle H \rangle} = 0$. We denote with $\lim_{\text{th}}$ the thermodynamic limit with a number $N \to \infty$ of particles in a volume $L \to \infty$ with fixed density $D = N/L$.

modynamic limit. This is the regime where we can expect to obtain some explicit results and where we can describe the system in terms of its thermodinamically emerging degrees of freedom. This is usually a much easier description than the one in terms of the positions or momenta of all the constituents. Moreover it is only in the thermodynamic limit where we can formulate general statements on the thermalization of an isolated system, where we observe true quantum relaxation of local observables when we restrict to a finite part of the whole system. In this case indeed the complementary part acts as an effective infinite bath which induces relaxation processes on the local observables.

In the following part we list the main results of this thesis.

Overlaps in integrable models

In order to have an exact description of the non-equilibrium time evolution given by a quantum quench one needs to know the overlaps between the basis of eigenstates of the model, which is given by the Bethe states, and the initial state, which in general is very different from any of the eigenstates [5]. A physically interesting class of initial states is constituted by the eigenstates of the same model with a different coupling constant. While the algebraic approach to the exact solution of integrable models, the algebraic Bethe ansatz, provides handy expressions for the overlaps between Bethe states with the same coupling, an analogous result valid for states with two arbitrarily different couplings has not been found yet. In this thesis we provide the exact overlaps for two specific cases where the initial state is the ground state of the free theory. For the Lieb-Liniger model this is achieved by setting the coupling constant of the interactions between the particles in the gas to zero $c \to 0$, leading to a many-body free bosonic theory. Equivalently we obtain the overlaps of the Bethe state of the XXZ spin chain with the ground state of the model in the limit of infinite anisotropy $\Delta \to \infty$, when the the XXZ model reduces to the classical longitudinal Ising spin chain. These two expressions for the overlaps are remarkably elegant in terms of the quasi-momenta which parametrize the Bethe states and they are very similar to the formula giving the norm of the Bethe states. This suggests that we may be able in the future to find explicit expressions for more generic overlaps.

[5] To compute the time evolution after a quench from the initial state $|\Psi_0\rangle$ of a generic operator $\hat{O}$ one typically inserts two resolutions of the identity $\mathbf{1} = \sum_\lambda |\lambda\rangle\langle\lambda|$ using the basis of the eigenstates of the model such that $\langle\Psi_0|\hat{O}(t)|\Psi_0\rangle = \sum_\mu \sum_\lambda \langle\mu|\hat{O}|\lambda\rangle e^{it(E_\mu - E_\lambda)}\langle\Psi_0|\mu\rangle\langle\lambda|\Psi_0\rangle$. It is then necessary to know the overlaps between any eigenstate and the initial state $\langle\Psi_0|\mu\rangle \;\forall\mu$. We say that the initial state $|\Psi_0\rangle$ is far from any of the eigenstates of the system if the overlaps decay exponentially with the number of constituents N, $\langle\Psi_0|\mu\rangle \sim e^{-Nc[\mu]} \;\forall\mu$ with $\Re c \geq 0$.

Post-quench steady state in the Lieb-Liniger gas and in the XXZ spin chain

This thesis provides the first non-Gibbs steady states after a non-equilibrium time evolution of truly interacting models. The non-equilibrium evolution is implemented by letting an initial state unitarily evolve under an integrable Hamiltonian as the Lieb-Liniger gas for point-interacting bosons and the XXZ spin chain, a model for interacting spins on a lattice. Due to the presence of non-trivial conserved quantities in the models, the expectation values of local operators, as the density-density correlation or the nearest-neighbors spin-spin correlation, do not relax to expectation values which are reproducible by a Gibbs-like ensemble.

Even a generalization of it, the Generalized Gibbs Ensemble [6] which aims to include all the other local constraints in the system, fails to give correct predictions. In the case of the Lieb-Liniger model the conserved quantities are indeed plagued with creepy divergences that do not allow to properly evaluate them on the initial state. In the XXZ spin chain the set of local conserved quantities turns to not be complete and more unknown additional constraints need to be taken into account. We then address the problem with an alternative approach, the so called quench action approach, which does not rely on any assumption on the internal symmetries of the system but only on the emerging form of the overlaps between the eigenstates and the initial state in the thermodynamic limit. This gives a set of effective constraints that are used to determine the steady state, which turns to be the exact one. The question of

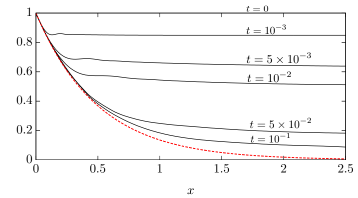

Figure 2: Time evolution (black lines from top to bottom) of phase correlations $\langle BEC(t)|\Psi^+(x)\Psi(0)|BEC(t)\rangle$ as a function of the relative distance x of a one-dimensional Bose gas after the quench from the ground state of the free theory (Bose-Einstein condensate $|BEC\rangle$) to the infinite repulsive regime (hard-core bosons). The red dotted line shows the steady state expectation values which is not given by an effective thermal Gibbs ensemble $\lim_{t\to\infty}\langle BEC(t)|\Psi^+(x)\Psi(0)|BEC(t)\rangle \neq \mathrm{Tr}\left(e^{-\beta H}\Psi^+(x)\Psi(0)\right)$ but by the saddle point state $|\rho_{sp}\rangle$ given by quench action approach $\lim_{t\to\infty}\langle BEC(t)|\Psi^+(x)\Psi(0)|BEC(t)\rangle = \langle\rho_{sp}|\Psi^+(x)\Psi(0)|\rho_{sp}\rangle$. This approach also reproduces the whole time evolution towards the equilibrium with the minimal amount of information on the initial state.

[6]The Gibbs ensemble is expected to reproduce the equilibrium values of the physical operators in most interacting systems $\lim_{t\to\infty}\langle\Psi_0|\hat{O}(t)|\Psi_0\rangle = \mathrm{Tr}\left(e^{-\beta H}\hat{O}\right)/\mathrm{Tr}\left(e^{-\beta H}\right)$ where β^{-1} is the effective temperature of the initial state. If there are more local conserved quantities than just the single Hamiltonian then this is expected to generalize to the Generalized Gibbs Ensemble $e^{-\sum_n \beta_n Q_n}$ where $\{Q_n\}$ is the maximal set of local operators that commute with the Hamiltonian H.

how to reproduce such exact steady
states with the alternative approach of the Generalized Gibbs Ensemble is not answered yet.

Time evolution

While some facts on the steady state after a non-equilibrium time evolution are in general established, on the contrary much less is known on the time evolution towards equilibrium. This is in general a much more complicated problem. A quite common statement for example is that in order to reproduce its whole time evolution a larger amount of information on the initial state is needed, compared to the one necessary to reconstruct the steady state at infinite times. However we show in this thesis that this is not the case in a quenched integrable model (see figure 2). The thermodynamic limit indeed selects only a restricted amount of data on the initial state, which are responsible for the whole time evolution up to infinite times, when the system has relaxed to its steady state. This is a direct consequence of the quench action approach, which also provides a general tool to access the effective time scales of the non-equilibrium dynamics which are encoded in the post-quench effective velocity of propagation of the excitations. This is also a functional of the initial state and does not depend on all its details but only on its macroscopic parameters (like its energy density for example). Different initial states therefore will lead to effective field theories with different light velocities.

Besides its theoretical relevance the whole time evolution is also much more related to experimental realizations. Cold atoms experiments are indeed usually limited in the range of time, making usually impossible to observe the true post-quench steady state in a real setting (See figure 1).