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Orthogonality and Quantum Geometry: Towards a Relational Reconstruction of Quantum Theory

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Abstract

This thesis is an in-depth mathematical study of the non-orthogonality relation in quantum theory.

Quantum mechanics is a crucial part of modern physics. Although it is very successful in describing microscopic phenomena, its conceptual essence is still not well understood. Its standard formalism in Hilbert spaces is one of the reasons. This formalism is good for doing calculation and making precise prediction, but to some extent it obscures the conceptual picture of quantum theory. In quantum logic researchers take physically intuitive notions as the primitives and model them in simple mathematical structures which make their quantum features transparent. Then quantum theory can be reconstructed by proving representation theorems of these simple mathematical structures via Hilbert spaces. In this way the conceptual essence of quantum theory is captured and made clear.

In this thesis I mainly take as primitive the non-orthogonality relation between the (pure) states of quantum systems. There are several reasons. First, the non-orthogonality relation is a binary relation on the states, and thus is simple from a mathematical perspective. Second, this relation describes the transitions between states triggered by measurements, and thus is intuitive from a physical perspective. Third, it proves important in quantum theory according to existing work in the literature. Therefore, a relational reconstruction of quantum theory based on this relation will be able to reveal the conceptual essence of quantum theory in a clear form. The results in this thesis show the possibility of such a reconstruction, and constitute a promising start.

In Chapter 2, I define quantum Kripke frames, the protagonists of this thesis. A quantum Kripke frame is a Kripke frame in which the binary relation possesses some simple properties of the non-orthogonality relation in quantum theory. The structure of quantum Kripke frames is studied extensively from a geometric perspective. Based on this, I prove a representation theorem of quantum Kripke frames via generalized Hilbert spaces. Moreover, the quantum Kripke frames induced by Hilbert spaces over the complex numbers are characterized.

This suggests that quantum Kripke frames can be useful in modelling quantum systems. In the meantime, several kinds of projective geometries are discovered to be Kripke frames in disguise.

Many operators on Hilbert spaces are crucial in formalizing quantum theory, so maps between quantum Kripke frames are worth studying. This is undertaken in Chapter 3. I define continuous homomorphisms between quantum Kripke frames and prove a representation theorem of them via continuous quasi-linear maps. Parallel to the unitary operators, self-adjoint operators and projectors on Hilbert spaces, I define three special kinds of continuous homomorphisms and study their properties from the perspective of the non-orthogonality relation. Moreover, I prove that under some conditions two quantum Kripke frames can be amalgamated into one, and this is the counterpart of the tensor product construction on Hilbert spaces. A preliminary to this result is a characterization of the endo-homomorphisms on a Pappian projective geometry induced by the linear maps. This solves a special case of an open problem in projective geometry.

Chapter 4 concerns the automated reasoning of quantum Kripke frames. I provide decidable, sound and strongly complete axiomatizations in modal logic for state spaces and for state spaces satisfying Superposition. These are Kripke frames more general than quantum Kripke frames. I prove that the first-order theory of quantum Kripke frames is undecidable. Moreover, I characterize the first-order definable, bi-orthogonally closed subsets in a special kind of quantum Kripke frame. This kind of quantum Kripke frame includes those induced by Hilbert spaces. This characterization hints that the first-order theory of such quantum Kripke frames may be finitely axiomatizable. The results in this chapter show that the appropriate formal language for the automated reasoning of quantum Kripke frames should be some fragment of the first-order language.

Chapter 5 is a pilot study of the transition probabilities between the states of quantum systems. They are the quantitative, more fine-grained version of the non-orthogonality relation. A probabilistic quantum Kripke frame is defined to be a quantum Kripke frame in which every pair of elements is assigned a real number between 0 and 1. The assignment of these numbers captures some properties of the transition probabilities in quantum theory. I show that in a probabilistic quantum Kripke frame every element induces a quantum probability measure in an expected way. I also discover an important property of the non-orthogonality relations in probabilistic quantum Kripke frames. Moreover, I define quantum transition probability spaces in which only the transition probabilities are primitive. I show that they correspond to probabilistic quantum Kripke frames. The results in this chapter suggest that probabilistic quantum Kripke frames, or quantum transition probability spaces, can be useful in the quantitative modelling of quantum systems and thus are worth further study.