UNIVERSITY OF AMSTERDAM

Several Topics in Complex Variables

I.M. Smit

Summary: Several Topics in Complex Variables

This thesis is based on three articles in complex analysis, the field that studies functions of complex variables.

We will look at functions defined on some domain $U \subseteq \mathbb{C}^m$, for example. The domain U can also be regarded as a domain in $\mathbb{R}^{2m}$, but the additional structure offered by the complex numbers creates new properties to study for our function f.

The holomorphic bijective functions are of particular importance here. A function $f \colon U \to V$ between domains in $\mathbb{C}^m$ is a holomorphic bijection when it creates a pairwise correspondence between the points of U and V in a way that preserves the complex structure. If there exists holomorphic bijection between two domains U and V, they are identical in some sense; this concept is called biholomorphic.

One of the main topics in complex analysis is the contrast between functions of real variables, and those depending on complex variables. Which properties of a function f are compatible to the complex structure, and are, for example, preserved when we consider the composition $f \circ g$ with a holomorphic bijection?

Another recurring theme is the comparison of the one-dimensional setting with the case where a function f depends on multiple complex variables. One-dimensional complex analysis is an extensively developed field, but the introduction of a second variable opens new possibilities.

The three articles contained in this theses can be regarded as part of a hunt for analogies.

The first article concerns maximality properties in *plurifine potential theory*. The words 'pluri' and 'plurifine' already indicate that this topic is part of a system of analogies and generalizations.

The basic piece of this puzzle is given by the harmonic and subharmonic functions on a domain $U \subseteq \mathbb{R}^m$. When $m = 1$, these functions are equal to the affine linear functions $f(x) = ax + b$ and the convex functions respectively.

Such a linear function can be regarded as a *maximal* convex function: there is no convex function that locally exceeds a linear function.

The *plurifinely pluri(sub)harmonic* functions form a generalization of the complex analogue to the (sub)harmonic functions. Here, maximality is much more interesting: not every maximal plurifinely plurisubharmonic function is plurifinely pluriharmonic. The reverse inclusion remains valid.

In Chapter 2 we study this form of maximality, and search for analogies with known maximality properties in the other puzzle pieces.

The second article deals with a topic in complex dynamics. Let $f_0, f_1, f_2, \ldots$ be a sequence of holomorphic bijections of $\mathbb{C}^m$ to itself, with the origin as a uniformly attracting common fixed point:

$$C||z|| \leq ||f_n(z)|| \leq D||z|| \quad \text{for all } z \in \mathbb{C}^m \text{ with } ||z| \leq 1, \text{ and for all } n \in \mathbb{N},$$

where $0 < C < D < 1$.

According to the *stronger Bedford Conjecture*, the set

$$\Omega_{(f_n)} = \{z \in \mathbb{C}^m : f_n \circ \cdots \circ f_0(z) \to 0 \text{ as } n \to \infty\}$$

of points that converge to the origin under compositions of the maps f_n, should be biholomorphic to $\mathbb{C}^m$.

This is well-known in $\mathbb{C}^1$, but the question remains open in higher dimensions. In our article in Chapter 3, we refine an argument by Abbondandolo and Majer, and prove the stronger Bedford Conjecture in $\mathbb{C}^2$ under the additional assumption that $D^{11/5} < C$. This improves the previous record, which stood at $D^{29/14} < C$.

In Chapter 4 we once again study compositions of functions, but this time we look at the iterations of a fixed polynomial. (Think of $p(z) = z^2 + 2$ for a one-dimensional example). A Fatou component U of a polynomial p is a maximal connected open set of points that remain close together when we apply iterates of p: $\{p^n|_U\}$ is a normal family.

For a polynomial p on $\mathbb{C}$, Sullivan's Theorem states that every Fatou component will eventually become periodic. That is, there are natural numbers n and $k \geq 1$ such that $p^{n+k}(U) = p^n(U)$.

For a polynomial p of two variables, this is no longer true; Fatou components can wander around forever without coming back to the same spot. In Chapter 4, we study polynomials of the form $F(z, w) = (f(z, w), \lambda w)$ with $|\lambda| < 1$, and investigate whether or not this setting admits wandering Fatou components. We identify a sizable class of functions where Sullivan's Theorem holds: all Fatou components will eventually become periodic.

These three topics are certainly different, but not unrelated. In both Chapters 3 and 4, we study the behavior of a set under compositions of functions. And the subharmonic Green's function will be a key player in the proof in Chapter 4.