$$
\begin{aligned}
&= \int_{B\cap S_1^+\cap S^+} g(s)\, m^+(ds) + \int_{B\cap S_1^+\cap S^-} g(s)\, m^-(ds) \\
&- \int_{B\cap S_1^-\cap S^+} g(s)\, m^+(ds) - \int_{B\cap S_1^-\cap S^-} g(s)\, m^-(ds) \\
&= \int_{B\cap S_1^+\cap S^+} f(s)\, m^+(ds) - \int_{B\cap S_1^+\cap S^-} f(s)\, m^-(ds) \\
&+ \int_{B\cap S_1^-\cap S^+} f(s)\, m^+(ds) - \int_{B\cap S_1^-\cap S^-} f(s)\, m^-(ds) \\
&= \int_{B\cap S^+} f(s)\, m^+(ds) - \int_{B\cap S^-} f(s)\, m^-(ds) = \hat{\mu}(B)\,,
\end{aligned}
$$

as required. □

4.4 Occupation measures and local times

Let $\big(X(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}^d\big)$ be a measurable stochastic process. For a Borel set $D\in\mathbb{R}^{d+1}=\mathbb{R}^d\times\mathbb{R}$ we define

$$\mu_{\mathbf{X}}(D) = \lambda_d\Big(\Big\{\mathbf{t}\in\mathbb{R}^d:\ \big(\mathbf{t}, X(\mathbf{t})\big)\in D\Big\}\Big)\,. \tag{4.4}$$

Clearly, $\mu_{\mathbf{X}}$ is a σ-finite measure on $\mathbb{R}^{d+1}$; it is the *occupation measure* of the stochastic process $\big(X(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}^d\big)$. For Borel sets $A\in\mathbb{R}^d$ and $B\in\mathbb{R}$, the value of $\mu_{\mathbf{X}}(A\times B)$ describes, informally, the amount of time in the set A the process spends in the set B.

Fix a "time set" $A\in\mathbb{R}^d$ of a finite positive Lebesgue measure and consider the measure on $\mathbb{R}$ defined by

$$\mu_{\mathbf{X},A}(B) = \mu_{\mathbf{X}}(A\times B),\ B\in\mathbb{R},\ \text{Borel}.$$

By the definition of the occupation measure we have the following identity valid for every measurable nonnegative function f on $\mathbb{R}$:

$$\int_A f\big(X(\mathbf{t})\big)\,\lambda_d(d\mathbf{t}) = \int_{\mathbb{R}} f(x)\,\mu_{\mathbf{X},A}(dx)\,. \tag{4.5}$$

If on an event of probability 1 $\mu_{\mathbf{X},A}(B)$ is absolutely continuous with respect to the Lebesgue measure λ, we say that the process has a *local time* over the set A. The local time is a version of the Radon-Nykodim derivative

$$l_{\mathbf{X},A}(x) = \frac{d\mu_{\mathbf{X},A}}{d\lambda}(x),\ x\in\mathbb{R}\,. \tag{4.6}$$

A local time is otherwise known as an *occupation density*. The basic property of the local time follows from (4.5): for every measurable nonnegative function f on $\mathbb{R}$,

$$\int_A f\big(X(\mathbf{t})\big)\,\lambda_d(d\mathbf{t}) = \int_{\mathbb{R}} f(x) l_{\mathbf{X},A}(x)\,dx\,. \tag{4.7}$$

If a process has a local time over a set A, the local time can also be computed by

$$l_{\mathbf{X},A}(x) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_A \mathbf{1}_{[x-\varepsilon,x+\varepsilon]}\big(X(\mathbf{t})\big)\,\lambda_d(d\mathbf{t})\,, \tag{4.8}$$

and the limit exists for almost every $x\in\mathbb{R}$. This useful representation of the local time has also an attractive intuitive meaning. It implies, in particular, that one can choose a version of a local time such that $l_{\mathbf{X},A}(x) = l_{\mathbf{X},A}(\omega;x)$ is a measurable function $\Omega\times\mathbb{R}\to\mathbb{R}$. In particular, for every $x\in\mathbb{R}$, $l_{\mathbf{X},A}(x)$ is a well defined random variable.

An immediate conclusion from (4.8) is the following monotonicity property of the local times: if a process $\big(X(\mathbf{t}),\,\mathbf{t}\in\mathbb{R}^d\big)$ has local times over sets A and B, then

$$A\subset B \text{ implies that } l_{\mathbf{X},A}(x)\le l_{\mathbf{X},B}(x) \text{ a.s.}. \tag{4.9}$$

Let $\big(X(t),\,t\in\mathbb{R}\big)$ be a measurable stochastic process with a one-dimensional time. If the process has a local time over each interval $[0,t]$ in some range $t\in[0,T]$, then it is common to use the two-variable notation

$$l_{\mathbf{X}}(x,t) = l_{\mathbf{X},[0,t]}(x),\ 0\le t\le T,\ x\in\mathbb{R}\,.$$

Using (4.8) shows that there is a version of $\big(l_{\mathbf{X}}(x,t)\big)$ that is jointly measurable in all 3 variables, ω, x, t.

As expected, existence and finite dimensional distributions of a local time are determined by the finite dimensional distributions of the underlying process.

Proposition 4.4.1 *(i) The finite dimensional distributions of the local time are determined by the finite dimensional distributions of the process. That is, let* $\big(X(\mathbf{t}),\,\mathbf{t}\in\mathbb{R}^d\big)$ *and* $\big(Y(\mathbf{t}),\,\mathbf{t}\in\mathbb{R}^d\big)$ *be measurable stochastic processes with the same finite dimensional distributions. If* $\big(X(\mathbf{t}),\,\mathbf{t}\in\mathbb{R}^d\big)$ *has a local time over a set* A*, then so does the process* $\big(Y(\mathbf{t}),\,\mathbf{t}\in\mathbb{R}^d\big)$*. Moreover, there a Borel set* $S\subset\mathbb{R}$ *of null Lebesgue measure such that the finite dimensional distributions of* $\big(l_{\mathbf{X},A}(x),\,x\in S^c\big)$ *coincide with the finite dimensional distributions of* $\big(l_{\mathbf{Y},A}(x),\,x\in S^c\big)$*.*

(ii) If $\big(X(t),\,t\in\mathbb{R}\big)$ *and* $\big(Y(t),\,t\in\mathbb{R}\big)$ *are measurable stochastic processes with the same finite dimensional distributions, and if* $\big(X(t),\,t\in\mathbb{R}\big)$ *has a local time over each interval* $[0,t]$ *in some range* $t\in[0,T]$*, then so does the process* $\big(Y(t),\,t\in\mathbb{R}\big)$*. Moreover, for every* $t_1,\ldots,t_k$ *in* $[0,T]$ *there a Borel set* $S\subset\mathbb{R}$ *of null Lebesgue measure such that the finite dimensional distributions of* $\big(l_{\mathbf{X},A}(x,t_j),\,x\in S^c,\,j=1,\ldots,k\big)$ *coincide with the finite dimensional distributions of* $\big(l_{\mathbf{Y},A}(x,t_j),\,x\in S^c,\,j=1,\ldots,k\big)$*.*

We first prove a useful lemma.

Lemma 4.4.2 *Let* $\big(X(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ *and* $\big(Y(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ *be measurable stochastic processes with the same finite dimensional distributions. Let* $A \subset \mathbb{R}^d$ *be a measurable set of a finite positive Lebesgue measure, and* $f : A \to \mathbb{R}$ *a bounded measurable function. Then*

$$\int_A f\big(X(\mathbf{t})\big)\, \lambda_d(d\mathbf{t}) \overset{d}{=} \int_A f\big(Y(\mathbf{t})\big)\, \lambda_d(d\mathbf{t})\,.$$

Proof: Suppose that the process $\big(X(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ is defined on some probability space $\big(\Omega_1, \mathcal{F}_1, P_1\big)$, while the process $\big(Y(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ is defined on some other probability space $\big(\Omega_2, \mathcal{F}_2, P_2\big)$. Let $\mathbf{T}_1, \mathbf{T}_2, \ldots$ be a sequence of i.i.d. random vectors in $\mathbb{R}^d$, whose common law is the normalized Lebesgue measure on A, and suppose that the sequence is defined on yet another probability space $\big(\Omega_3, \mathcal{F}_3, P_3\big)$. Note that for every $\omega_1 \in \Omega_1$,

$$\frac{1}{n}\sum_{j=1}^{n} f\big(X(\mathbf{T}_j)\big) \to \int_A f\big(X(\mathbf{t})\big)\, \lambda_d(d\mathbf{t}) \tag{4.10}$$

as $n \to \infty$ P_3-a.s. by the law of large numbers. By Fubini's theorem on the product probability space $\big(\Omega_1 \times \Omega_3, \mathcal{F}_1 \times \mathcal{F}_3, P_1 \times P_3\big)$, we see that there is an event $\Omega_3^{(1)} \in \mathcal{F}_3$ of full P_3-probability such that (4.10) holds P_1-a.s. for every $\omega_3 \in \Omega_3^{(1)}$.

Repeating the argument, we see that there is an event $\Omega_3^{(2)} \in \mathcal{F}_3$ of full P_3-probability such that for every $\omega_3 \in \Omega_3^{(2)}$,

$$\frac{1}{n}\sum_{j=1}^{n} f\big(Y(\mathbf{T}_j)\big) \to \int_A f\big(Y(\mathbf{t})\big)\, \lambda_d(d\mathbf{t}) \tag{4.11}$$

as $n \to \infty$ P_2-a.s.. The event $\Omega_3^{(1)} \cap \Omega_3^{(2)}$ has full P_3-probability, so its must contain a point ω_3, which we fix. This gives us a fixed sequence $\big(\mathbf{T}_j\big)$, and for this sequence the expressions in the left hand sides of (4.10) and (4.11) have the same distributions. Since we have convergence in both (4.10) and (4.11), the claim of the lemma follows. $\square$

Proof of Proposition 4.4.1 For part (i), let $A \in \mathbb{R}^d$ be a set of a finite positive Lebesgue measure. It follows from Lemma 4.4.2 applied to indicator functions of Borel sets and linear combinations of such indicator functions that

$$\big(\mu_{\mathbf{X},A}(B),\ B \text{ Borel}\big) \overset{d}{=} \big(\mu_{\mathbf{Y},A}(B),\ B \text{ Borel}\big) \tag{4.12}$$

in the sense of equality of the finite dimensional distributions. Suppose that $\big(X(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ has a local time over the set A. Then on an event of probability 1, the probability measure $\mu_{\mathbf{X},A}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. This implies that, on that event, for every $m_1 \geq 1$ there is $m_2 \geq 1$

such that for all $k = 1, 2, \ldots$ and rational numbers $\tau_1 < \tau_1' < \tau_2 < \tau_2' < \ldots < \tau_k < \tau_k'$ with $\sum_{-=1}^{k}(\tau_i' - \tau_i) < 1/m_2$ we have

$$\sum_{i=1}^{k} \mu_{\mathbf{X},A}\big((\tau_i, \tau_i']\big) < 1/m_1 \,.$$

Then (4.12) implies that the same is true for the probability measure $\mu_{\mathbf{Y},A}$, and so on an event of probability 1, the probability measure $\mu_{\mathbf{Y},A}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ as well; see e.g. Royden (1968). This means that the process $\big(Y(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ has a local time over the set A.

Next, suppose that that the process $\big(X(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ is defined on some probability space $\big(\Omega_1, \mathcal{F}_1, P_1\big)$, while the process $\big(Y(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ is defined on some other probability space $\big(\Omega_2, \mathcal{F}_2, P_2\big)$, and let $\Omega_i^{(1)} \in \mathcal{F}_1$, $i = 1, 2$ be events of full probabilities on which $\big(X(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ and $\big(Y(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ have local times over the set A. Let $S \subset \mathbb{R}$ be a Borel set of null Lebesgue measure such that for every $x \in S^c$, the relation (4.8) holds for P_1-almost every $\omega_1 \in \Omega_i^{(1)}$, and the version of (4.8) written for the process $\big(Y(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ holds for P_2-almost every $\omega_2 \in \Omega_i^{(2)}$. Then the fact that the finite dimensional distributions of $\big(l_{\mathbf{X},A}(x),\, x \in S^c\big)$ coincide with the finite dimensional distributions of $\big(l_{\mathbf{Y},A}(x),\, x \in S^c\big)$ follows from Lemma 4.4.2. This proves part (i) of the proposition.

For part (ii) of the proposition, the fact that the process $\big(Y(t),\, t \in \mathbb{R}\big)$ has a local time over each interval $[0, t]$ in the range $t \in [0, T]$ follows from part (i), while the equality of the finite dimensional distributions follows from (4.8) in the same way as the corresponding statement in part (i). $\Box$

The next, and basic, property of the local time follows from its definition.

Proposition 4.4.3 *(i) Suppose that a process $\big(X(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d\big)$ has a local time over a set A. Let $B \subset \mathbb{R}$ be a Borel set. If $\Omega_0 \in \mathcal{F}$ is a event such that for every $\omega \in \Omega_0$, $X(t) \in B^c$ for each $t \in A$, then for every $\omega \in \Omega_0$, $l_{\mathbf{X},A}(x) = 0$ for almost every $x \in B$.*

(ii) Let $t > 0$ and suppose that the local time $l_{\mathbf{X}}(\cdot, t)$ of the process $\big(X(t),\, t \in \mathbb{R}\big)$ over the interval $[0, t]$ exists. Let $y > 0$. If $\Omega_0 \in \mathcal{F}$ is a event such that for every $\omega \in \Omega_0$, $\sup_{s\in[0,t]} |X(s)| < y$, then for every $\omega \in \Omega_0$, $l_{\mathbf{X}}(x, t) = 0$ for almost every x with $|x| \geq y$.

Proof: Letting f to be the indicator function of the set B and appealing to (4.7) proves the first statement of the proposition. The second statement follows from the first one with $B = (-\infty, -y] \cup [y, \infty)$. $\Box$

When do local times exist? An easy to check criterion for existence of a local time is due to Berman (1969). It is based on the following classical result on characteristic functions of random vectors.

Lemma 4.4.4 *(i) Let* $\mathbf{X}$ *be a random vector, and let* $\varphi_{\mathbf{X}}(\boldsymbol{\theta}) = Ee^{i(\boldsymbol{\theta},\mathbf{X})}$, $\boldsymbol{\theta} \in \mathbb{R}^d$ *be its characteristic function. Then* $\mathbf{X}$ *has a square integrable density if and only if*

$$\int_{\mathbb{R}^d} |\varphi_{\mathbf{X}}(\boldsymbol{\theta})|^2 \, \lambda_d(d\boldsymbol{\theta}) < \infty .$$

(ii) If

$$\int_{\mathbb{R}^d} |\varphi_{\mathbf{X}}(\boldsymbol{\theta})| \, \lambda_d(d\boldsymbol{\theta}) < \infty ,$$

then $\mathbf{X}$ *has a bounded uniformly continuous density.*

Proof: The second part of the lemma appears in about every book in probability; see e.g. Corollary 2, p. 149 in Laha and Rohatgi (1979)). The statement of the first part of the lemma is less common in the probabilistic literature, so we include a proof.

Suppose that $\varphi_{\mathbf{X}}$ is square integrable. The general theory of the L^2 Fourier transforms tells us that the function

$$f(\mathbf{x}) = \lim_{h\uparrow\infty} \frac{1}{(2\pi)^{d/2}} \int_{\|\boldsymbol{\theta}\|\le h} e^{i(\boldsymbol{\theta},\mathbf{x})} \varphi_{\mathbf{X}}(\boldsymbol{\theta}) \, \lambda_d(d\boldsymbol{\theta}), \ \mathbf{x} \in \mathbb{R}^d$$

exists in $L^2(\lambda_d)$ and, moreover, the function

$$f_1(\mathbf{x}) = \int_0^{x_1} \cdots \int_0^{x_d} f(y_1, \ldots, y_d) \, dy_1 \ldots dy_d, \ \mathbf{x} = (x_1, \ldots, x_d) \in (0,\infty)^d$$

satisfies the relation

$$f_1(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi_{\mathbf{X}}(\boldsymbol{\theta}) \prod_{j=1}^d \frac{e^{-it_j\theta_j} - 1}{-it_j} \, \lambda_d(d\boldsymbol{\theta}), \ \mathbf{x} = (x_1, \ldots, x_d) \in (0,\infty)^d ;$$

see Section VI.2 in Yosida (1965). On the other hand, by the inversion theorem for the characteristic functions (see e.g. Theorem 3.3.3 in Laha and Rohatgi (1979)) we know that for every $\mathbf{x} = (x_1, \ldots, x_d) \in (0,\infty)^d$ such that each x_j is a continuity point of the marginal distribution of the jth component of $\mathbf{X}$,

$$\frac{1}{\pi^d} \int_{\mathbb{R}^d} \varphi_{\mathbf{X}}(\boldsymbol{\theta}) \prod_{j=1}^d \frac{e^{-it_j\theta_j} - 1}{-it_j} \, \lambda_d(d\boldsymbol{\theta}) = P\Big(\mathbf{X} \in \prod_{j=1}^d (0, x_j]^d\Big) .$$

We conclude that

$$P\Big(\mathbf{X} \in \prod_{j=1}^d (0, x_j]^d\Big) = \Big(\frac{\pi}{2}\Big)^{d/2} \int_0^{x_1} \cdots \int_0^{x_d} f(y_1, \ldots, y_d) \, dy_1 \ldots dy_d$$

a.e. on $(0,\infty)^d$. That means that the function f is real and nonnegative a.e. on $(0,\infty)^d$, the law of $\mathbf{X}$ is absolutely continuous on this set, and its density is square integrable. Since this argument can be repeated with only notational changes for other quadrants of $\mathbb{R}^d$, this proves the "if" part of the lemma. The other direction is easy since the usual Fourier transform of a function in $L^1(\lambda_d)\cap L^2(\lambda_d)$ is in $L^2(\lambda_d)$; see once again Section VI.2 in Yosida (1965). $\square$

The following proposition, due to Berman (1969), is an easy consequence of the lemma.

Proposition 4.4.5 *Let $\big(X(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}^d\big)$ be a measurable stochastic process. Let $A\in\mathbb{R}^d$ be a measurable set of a finite d-dimensional Lebesgue measure. A sufficient condition for the process to have a local time over the set A satisfying*

$$\int_{\mathbb{R}} l_{\mathbf{X},A}(x)^2\, dx < \infty \text{ with probability 1}$$

is

$$\int_{\mathbb{R}}\int_A\int_A E e^{i\theta(X(t)-X(s))}\,\lambda_d(dt)\,\lambda_d(ds)\,d\theta < \infty\,. \tag{4.13}$$

A sufficient condition for the process to have a bounded and uniformly continuous local time over the set A is

$$\int_{\mathbb{R}}\left(\int_A\int_A E e^{i\theta(X(t)-X(s))}\,\lambda_d(dt)\,\lambda_d(ds)\right)^{1/2} d\theta < \infty\,. \tag{4.14}$$

Proof: For a fixed $\omega\in\Omega$ consider $X = X(t)$, $t\in A$ as a random variable on the probability space A with the Borel σ-field restricted to A, and the probability measure $Q = (\lambda_d(A))^{-1}\lambda_d$. Then the occupation measure $\mu_{\mathbf{X},A}(\cdot)$ is, up to a constant, the probability law of X, and existence of a square integrable local time over the set A is equivalent to existence of a square integrable density of the probability law of X. Since the characteristic function of X at a point $\theta\in\mathbb{R}$ is given by

$$\frac{1}{\lambda_d(A)}\int_A e^{i\theta X(t)}\,\lambda_d(dt)\,,$$

by part (i) of Lemma 4.4.4 existence of such a square integrable density is equivalent to finiteness of the expression

$$\int_{\mathbb{R}}\left|\int_A e^{i\theta X(t)}\,\lambda_d(dt)\right|^2 d\theta\,.$$

The expectation of this expression coincides with the left hand side of (4.13), and if the expectation is finite, then the expression itself is finite on a set of probability 1. This proves the first statement.

Similarly, existence of a bounded and uniformly continuous local time over the set A is equivalent to existence of a bounded and uniformly continuous density of the probability law of X above. By part (ii) of Lemma 4.4.4 existence of such a density follows from a.s. finiteness of the integral

$$\int_{\mathbb{R}} \left| \int_A e^{i\theta X(t)} \, \lambda_d(dt) \right| \, d\theta \, .$$

Taking expectation and using the Cauchy-Scwartz inequality

$$E \left| \int_A e^{i\theta X(t)} \, \lambda_d(dt) \right| \leq \left(E \left| \int_A e^{i\theta X(t)} \, \lambda_d(dt) \right|^2 \right)^{1/2}$$

proves the second statement. □

Even the simple tools of Proposition 4.4.5 already guarantee existence and regularity of the local times of certain self-similar SαS process with stationary increments; see Exercise 4.6.3. Stronger results have been obtained for certain Gaussian processes using the theory of local nondeterminism introduced by Berman (1973), and later extended to non-Gaussian stable processes by Nolan (1982). The following proposition shows existence of jointly continuous local times for certain self-similar processes with stationary increments.

Proposition 4.4.6 *Let $\big(X(t),\, t \in \mathbb{R}\big)$ be a Fractional Brownian motion, or the real Harmonizable SαS motion with exponent self-similarity $0 < H < 1$, or a Linear Fractional SαS motion with $\alpha > 1$, $1/\alpha < H < 1$ and $c_2 = 0$. Then the process has a local time over every interval $[0,t]$, $t > 0$ and, moreover, there is a version of the local time that is jointly continuous in time and space. That is, there is a random field*

$$l_{\mathbf{X}}(x,t) = l_{\mathbf{X}}(x,t,\omega), \ 0 \leq t \leq T, \ x \in \mathbb{R}, \omega \in \Omega$$

such that every $\omega \in \Omega$, $l_{\mathbf{X}}(x,t)$ is jointly continuous in $x \in \mathbb{R}$ and $t \geq 0$ (with $l_{\mathbf{X}}(x,0) = 0$ for all $x \in \mathbb{R}$), and for each $t > 0$, $l_{\mathbf{X}}(x,t)$, $x \in \mathbb{R}$ is a version of the local time $l_{\mathbf{X},[0,t]}(x)$, $x \in \mathbb{R}$.

Proof: For the Fractional Brownian motion the claim follows from Section 7 in Pitt (1978) and Theorem 8.1 in Berman (1973). For the real Harmonizable SαS motion the claim follows from Theorem 4.11 in Nolan (1989). For the Linear Fractional SαS motion the claim follows from Ayache et al. (2008). □

Very precise estimates on the size of the time increments of the local of the Fractional Brownian motion are due to Xiao (1997). Some of them are summarized in the following proposition.

Proposition 4.4.7 *(i) Let $\big(l_{\mathbf{X}}(x,t),\, x \in \mathbb{R},\, t \geq 0\big)$ be the jointly continuous local time of a Fractional Brownian motion with exponent $0 < H < 1$ of self-similarity. Then the supremum*

$$\sup_{\substack{x\in\mathbb{R} \\ 0\leq s<t\leq 1/2}} \frac{l(x,t)-l(x,s)}{(t-s)^{1-H}\left(\log \frac{1}{t-s}\right)^{H}}$$

is a.s. finite, and has finite moments of all orders.

(ii) For every $t > 0$ and $p > 0$

$$E \sup_{x\in\mathbb{R}} l(x,t)^p < \infty\,.$$

Proof: The finiteness of the supremum in first part of the proposition follows from Corollary 1.1 in Xiao (1997). The finiteness of the moments is a very slight modification of the argument leading to the above corollary. The second part of the proposition follows from the first part by breaking the interval $[0,t]$ into parts of the length less than 1/2. $\square$

It is, perhaps, not surprising that certain properties of a stochastic process, such as self-similarity, stationarity and stationarity of the increments, are reflected in an appropriate way in the properties of the local time, assuming that the latter exists. In order to simplify the formulation of these relationships, we will assume that the local time is continuous.

Proposition 4.4.8 *Let $\big(X(t),\, t\in\mathbb{R}\big)$ be a measurable stochastic process, and assume that it has local time $\big(l_{\mathbf{X}}(x,t),\, x \in \mathbb{R},\, t \geq 0\big)$ that is jointly continuous in time and space.*

(i) If the process is self-similar with exponent H is self-similarity, then for every $c > 0$

$$\big(l_{\mathbf{X}}(c^H x, ct),\, x\in\mathbb{R},\, t\geq 0\big) \overset{d}{=} \big(c^{1-H} l_{\mathbf{X}}(x,t),\, x\in\mathbb{R},\, t\geq 0\big)\,. \tag{4.15}$$

(ii) If the process is stationary, then for every $h > 0$,

$$\big(l_{\mathbf{X}}(x,t+h) - l_{\mathbf{X}}(x,h),\, x\in\mathbb{R},\, t\geq 0\big) \overset{d}{=} \big(l_{\mathbf{X}}(x,t),\, x\in\mathbb{R},\, t\geq 0\big)\,. \tag{4.16}$$

(iii) Suppose that the process $\big(X(t),\, t\in\mathbb{R}\big)$ is defined on some probability space $\big(\Omega,\mathcal{F},P\big)$. Suppose that the process has stationary increments and sample paths that are bounded on compact intervals, satisfying

$$E \sup_{0\leq t\leq T} |X(t)| < \infty\,.$$

Then for every $h > 0$, the infinite "law" of

$$\big(l_{\mathbf{X}}(x+u,t+h)(\omega) - l_{\mathbf{X}}(x+u,h)(\omega),\, x\in\mathbb{R},\, t\geq 0\big)$$

under the infinite measure $P\times\lambda$ does not depend on the shift h.

Proof: Note, first of all, that when the local times are continuous, the exceptional set S in Proposition 4.4.1 may be taken to be the empty set, and we will do that throughout this proof.

For part (i), let $c > 0$, and define a new stochastic process by

$$Y(t) = c^{-H} X(ct),\, t \in \mathbb{R}\,.$$

By the self-similarity, the new process has the same finite-dimensional distributions as the original process $\big(X(t),\, t \in \mathbb{R}\big)$. Let f be a nonnegative measurable function on $\mathbb{R}$. For a $t > 0$ we change the variable of integration twice, using in between (4.7) for $A = [0, ct]$, to write

$$\int_0^t f(Y(s))\, ds = \int_0^t f\big(c^{-H} X(cs)\big)\, ds$$

$$= c^{-1} \int_0^{ct} f\big(c^{-H} X(s)\big)\, ds = c^{-1} \int_{\mathbb{R}} f\big(c^{-H} x\big)\, l_{\mathbf{X}}(x, ct)\, dx$$

$$= c^{H-1} \int_{\mathbb{R}} f\big(x\big)\, l_{\mathbf{X}}(c^H x, ct)\, dx\,.$$

Therefore, $\big(c^{H-1} l_{\mathbf{X}}(c^H x, ct),\, x \in \mathbb{R},\, t \geq 0\big)$ is a version of the local time $\big(l_{\mathbf{Y}}(x, t),\, x \in \mathbb{R},\, t \geq 0\big)$. By Proposition 4.4.1 the latter has the same finite-dimensional distributions as the local time $\big(l_{\mathbf{X}}(x, t),\, x \in \mathbb{R},\, t \geq 0\big)$, and this proves (4.15).

In a similar manner, for part (ii) we take $h > 0$, define a new stochastic process by $Y(t) = X(t + h)$, $t \in \mathbb{R}$, and write for a nonnegative measurable function f and $t > 0$

$$\int_0^t f(Y(s))\, ds = \int_0^t f\big(X(s + h)\big)\, ds$$

$$= \int_h^{t+h} f\big(X(s)\big)\, ds = \int_0^{t+h} f\big(X(s)\big)\, ds - \int_0^h f\big(X(s)\big)\, ds$$

$$= \int_{\mathbb{R}} f(x) l_{\mathbf{X}}(x, t + h)\, dx - \int_{\mathbb{R}} f(x) l_{\mathbf{X}}(x, h)\, dx$$

$$= \int_{\mathbb{R}} f(x) \big(l_{\mathbf{X}}(x, t + h) - l_{\mathbf{X}}(x, h)\big)\, dx\,,$$

so that

$$\big(l_{\mathbf{X}}(x, t + h) - l_{\mathbf{X}}(x, h),\, x \in \mathbb{R},\, t \geq 0\big)$$

is a version of the local time $\big(l_{\mathbf{Y}}(x, t),\, x \in \mathbb{R},\, t \geq 0\big)$. Now another appeal to Proposition 4.4.1 proves (4.16).

The proof of part (iii) of the proposition, which we now commence, has the same idea as the proof of part (ii), except that now we have to deal with infinite measures.

Fix $h>0$. Let f be a nonnegative measurable function. As before, there is an event of full probability such that, on this event, for every $u\in\mathbb{R}$ and $t>0$,

$$\int_0^t f\big(u+X(s+h)\big)\,ds=\int_{\mathbb{R}} f(x)\big(l_{\mathbf{X}}(x-u,t+h)-l_{\mathbf{X}}(x-u,h)\big)\,dx\,,$$

Applying this to a function $f=\mathbf{1}_{[x-\varepsilon,x+\varepsilon]}/2\varepsilon$ for $\varepsilon>0$ and using the continuity of the local times gives us

$$l_{\mathbf{X}}(x-u,t+h)-l_{\mathbf{X}}(x-u,h)=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{[x-\varepsilon,x+\varepsilon]}\big(u+X(s+h)\big)\,ds \tag{4.17}$$

for every $u\in\mathbb{R}$, $t>0$ and $x\in\mathbb{R}$.

Denote the expression in the left hand side of (4.17) by $A_h(x,t;u,\omega)$ and the expession under the limit in the right hand side of (4.17) by $A_{h,\varepsilon}(x,t;u,\omega)$. Choose pairs (x_j,t_j), $j=1,\dots,k$. Fix ω, and note that by Fubini's theorem, there is a measurable set $F\in(0,\infty)^k$ of full Lebesgue measure such that for all $(a_1,\dots,a_k)\in F$ we have

$$\mathbf{1}\Big(A_{h,\varepsilon}(x_j,t_j;u,\omega)>a_j,\ j=1,\dots,k\Big)\to\mathbf{1}\Big(A_h(x_j,t_j;u,\omega)>a_j,\ j=1,\dots,k\Big)$$

for almost every $u\in\mathbb{R}$. Let now $M>|x_1|+1+\sup_{h\le s\le t_1+h}|X(s)|$. We have, by the dominated convergence theorem,

$$\begin{aligned}&\lambda\big\{u\in\mathbb{R}:\ A_{h,\varepsilon}(x_j,t_j;u,\omega)>a_j,\ j=1,\dots,k\big\}\\ &=\lambda\big\{u\in[-M,M]:\ A_{h,\varepsilon}(x_j,t_j;u,\omega)>a_j,\ j=1,\dots,k\big\}\\ &\to\lambda\big\{u\in[-M,M]:\ A_h(x_j,t_j;u,\omega)>a_j,\ j=1,\dots,k\big\}\end{aligned}$$

for every $(a_1,\dots,a_k)\in F$.

Further, for every $\varepsilon>0$

$$\begin{aligned}&\lambda\big\{u:\ A_{h,\varepsilon}(x_j,t_j;u,\omega)>a_j,\ j=1,\dots,k\big\}\\ &\le\lambda\big\{u:\ A_{h,\varepsilon}(x_1,t_1;u,\omega)>a_1\big\}\\ &\le\frac{1}{2a_1\varepsilon}\int_{\mathbb{R}}\int_0^{t_1}\mathbf{1}_{[x_1-\varepsilon,x_1+\varepsilon]}\big(u+X(s+h)\big)\,ds\,du=\frac{t_1}{a_1}\,,\end{aligned}$$

where at the last step we used Fubini's theorem. Finally, using once again the dominated convergence theorem, we conclude that

$$\begin{aligned}&P\times\lambda\big\{(\omega,u)\in\Omega\times\mathbb{R}:\ A_{h,\varepsilon}(x_j,t_j;u,\omega)>a_j,\ j=1,\dots,k\big\} \qquad (4.18)\\ &\to P\times\lambda\big\{(\omega,u)\in\Omega\times\mathbb{R}:\ A_h(x_j,t_j;u,\omega)>a_j,\ j=1,\dots,k\big\}\end{aligned}$$

for every $(a_1, \ldots, a_k) \in F$. In particular, the "law" of $\big(A_h(x_j,t_j),\, j = 1,\ldots,k\big)$ under $P \times \lambda$ is σ-finite on $(0,\infty)^k$.

Suppose that we show that, for every $\varepsilon > 0$, the "law" of $\big(A_{h,\varepsilon}(x_j,t_j),\, j = 1,\ldots,k\big)$ under $P \times \lambda$ is independent of $h > 0$. Then for every $h_1, h_2 > 0$ we can find a subset of $(0,\infty)^k$ of full Lebesgue measure, such that (4.18) holds for h_1, h_2 and all $(a_1, \ldots, a_k)$ in that set. This means that for such $(a_1, \ldots, a_k)$

$$P \times \lambda\big\{(\omega,u) \in \Omega \times \mathbb{R} :\, A_{h_1}(x_j,t_j;u,\omega) > a_j,\, j = 1,\ldots,k\big\}$$

$$= P \times \lambda\big\{(\omega,u) \in \Omega \times \mathbb{R} :\, A_{h_2}(x_j,t_j;u,\omega) > a_j,\, j = 1,\ldots,k\big\}$$

and, hence, the equality holds for all $a_1 > 0, \ldots, a_k > 0$. This will establish the claim of part (iii) of the proposition.

It remains to show that for every $\varepsilon > 0$, the "law" of $\big(A_{h,\varepsilon}(x_j,t_j),\, j = 1,\ldots,k\big)$ under $P \times \lambda$ is independent of $h > 0$. It is, of course, enough to consider there laws restricted to the "punctured" set $[0,\infty)^d \setminus \{0\}$. Assume without loss of generality that $t_1 < t_2 < \ldots < t_d$, and notice that only those pairs (ω, u) for which

$$u + \inf_{0 \le s \le t_d} X(s+h) \le x + \varepsilon \text{ and } u + \sup_{0 \le s \le t_d} X(s+h) \ge x - \varepsilon$$

contribute to the values of $\big(A_{h,\varepsilon}(x_j,t_j),\, j = 1,\ldots,k\big)$ in the set $[0,\infty)^d \setminus \{0\}$. Call this set $\mathcal{V}_h$. It follows from the assumption that the supremum of the process over compact intervals is integrable that the measure $P \times \lambda$ restricted to $\mathcal{V}_h$ is finite. Moreover, it follows from Proposition 2.1.11 that the total mass of this restricted measure is independent of $h > 0$. Normalizing this restricted measure to be a probability measure, the required independence of $h > 0$ of the "law" of $\big(A_{h,\varepsilon}(x_j,t_j),\, j = 1,\ldots,k\big)$ follows from Proposition 2.1.11 and Lemma 4.4.2 applied to the linear combinations of $\big(A_{h,\varepsilon}(x_j,t_j),\, j = 1,\ldots,k\big)$. $\square$

4.5 Comments on Chapter 4

Comments on Section 4.2

The name "Borell-TIS" of the inequality in Theorem 4.2.3 is due to the fact that the version of (4.1) using the median of the supremum was proved at about the same time by Borell (1975) and Tsirelson et al. (1976).

Comments on Section 4.4

In the one-dimensional case the statement of Lemma 4.4.4 is in Problem 11, page 147 in Chung (1968).

Comments on Section 4.4

Theory of local times was originally developed for Markov processes, beginning with Lévy (1939). Extending the idea of local times to non-Markov processes is due to Berman (1969), who mostly considered Gaussian processes. Existence of "nice" local times requires certain roughness of the sample paths of a stochastic process, and the powerful idea of local nondeterminism introduced in Berman (1973) can be viewed as exploting this observation in the case of Gaussian processes. This approach was extended by Pitt (1978) to Gaussian random fields (with values in finite-dimensional Euclidian spaces), and to stable processes by Nolan (1982). Many details on local times of stochastic processed and random fields can be found in Geman and Horowitz (1980) and Kahane (1985).

Esimates similar to those in Proposition 4.4.7 (but with a slighly worse power of the logarithm) were also obtained in Csörgo et al. (1995).

4.6 Exercises to Chapter 4

Exercise 4.6.1 *Let ν and μ be two signed measure on $\big(S, \mathcal{S}\big)$. Construct the signed measure $\nu + \mu$ on $\big(S, \mathcal{S}\big)$. Is it true that the positive and negative parts of the new measure are equal to the sums of the corresponding parts of the original measure?*

Exercise 4.6.2 *Prove that if ν and μ are two signed measure on $\big(S, \mathcal{S}\big)$ and $f_i : S \to \mathbb{R}$, $i = 1, 2$ measurable functions such that (4.2) holds both with $f = f_1$ and $f = f_2$, then $f_1 = f_2$ a.e. with respect to the total variation measure $\|\nu\|$.*

Exercise 4.6.3 *Let $\big(X(t),\, t \geq 0\big)$ be an H-self-similar $S\alpha S$ process with stationary increments, $0 < \alpha \leq 2$ (a Fractional Brownian motion in the case $\alpha = 2$). Use Proposition 4.4.5 to show that over each compact interval the process has square integrable local time if $0 < H < 1$ and a bounded and uniformly continuous local time if $0 < H < 1/2$.*

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