A Comonotonic Image of Independence for Additive Risk Measures

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Background

- A variety of risk measures;

- Axiomatic characterizations of risk measures;

- Additivity axioms:

  $\pi[X+Y] = \pi[X] + \pi[Y]$, when X and Y are comonotonic;

  $\pi[X+Y] = \pi[X] + \pi[Y]$, when X and Y are independent;

  $\pi[X+Y] \leq \pi[X] + \pi[Y]$.

Outline

1. Exponential premium and Esscher premium;
2. The four axioms;
3. A comonotonic image;
4. The representation theorem;
5. Properties;
6. Connections with financial no arbitrage pricing principles.

Exponential principle

$$\varphi_X(t) = \begin{cases} \frac{1}{t}\log \mathbb{E}[e^{tX}], & t \neq 0; \\ \mathbb{E}[X], & t = 0. \end{cases}$$

Esscher principle

For the cdf $F_X(\cdot)$ we define by

$$dF_X^{(t)}(x) = \frac{e^{tx} dF_X(x)}{\mathbb{E}[e^{tX}]}, \qquad t \in \mathbb{R},$$

its *Esscher* transform. Note that $\frac{e^{tX}}{\mathbb{E}[e^{tX}]} > 0$ and $\mathbb{E}\left[\frac{e^{tX}}{\mathbb{E}[e^{tX}]}\right] = 1$.

$$\psi_X(t) = \int_{(-\infty,+\infty)} x dF_X^{(t)}(x) = \frac{\mathbb{E}[Xe^{tX}]}{\mathbb{E}[e^{tX}]}.$$

Since $\frac{d}{dt}\Big(t\varphi_X(t)\Big) = \frac{d}{dt} \log \mathbb{E}[e^{tX}] = \psi_X(t)$ when $t \neq 0$, it follows that

$$\varphi_X(t) = \frac{1}{t} \int_0^t \psi_X(s) ds, \qquad t \neq 0.$$

Properties of the principles [1]

Both the Esscher premium and the exponential premium increase with their parameters:

$$\frac{d}{dt}\psi_X(t) = \frac{\mathbb{E}[X^2 e^{tX}]}{\mathbb{E}[e^{tX}]} - \left(\frac{\mathbb{E}[Xe^{tX}]}{\mathbb{E}[e^{tX}]}\right)^2 .$$

$$\begin{aligned}\frac{d}{dt}\varphi_X(t) &= \frac{1}{t}\left(\frac{\mathbb{E}[Xe^{tX}]}{\mathbb{E}[e^{tX}]} - \frac{1}{t}\log \mathbb{E}[e^{tX}]\right)\\ &= \frac{1}{t}\left(\frac{\mathbb{E}[Xe^{tX}]}{\mathbb{E}[e^{tX}]} - \frac{1}{t}\int_0^t \frac{\mathbb{E}[Xe^{sX}]}{\mathbb{E}[e^{sX}]}ds\right).\end{aligned}$$

Properties of the principles [2]

If

$$\min[X] = \inf\left(x \mid \mathbb{P}[x \le X \le x + \varepsilon] > 0, \forall \varepsilon > 0\right),$$

$$\max[X] = \sup\left(x \mid \mathbb{P}[x - \varepsilon \le X \le x] > 0, \forall \varepsilon > 0\right).$$

Then

$$\lim_{t \to -\infty} \varphi_X(t) = \min[X] = \lim_{t \to -\infty} \psi_X(t),$$

$$\lim_{t \to +\infty} \varphi_X(t) = \max[X] = \lim_{t \to +\infty} \psi_X(t).$$

The axioms

We introduce the set $\mathbb{S}$ of axioms that a risk measure $\pi[\cdot]$ must satisfy:

A1. If $\mathbb{E}[e^{tX}] \leq \mathbb{E}[e^{tY}]$ for all $t \geq 0$, and $\mathbb{E}[e^{tX}] \geq \mathbb{E}[e^{tY}]$ for all $t \leq 0$, then $\pi[X] \leq \pi[Y]$;

A2. $\pi[c] = c$, for all real c;

A3. $\pi[X + Y] = \pi[X] + \pi[Y]$ when X and Y are independent;

A4. If X_n converges weakly to X, with $\min[X_n] \to \min[X]$ and $\max[X_n] \to \max[X]$, then $\lim_{n\to+\infty} \pi[X_n] = \pi[X]$.

Previous work by Gerber & Goovaerts (1981)

B1. If $\frac{\mathbb{E}[Xe^{tX}]}{\mathbb{E}[e^{tX}]} \leq \frac{\mathbb{E}[Ye^{tY}]}{\mathbb{E}[e^{tY}]}$ for all t, then $\pi[X] \leq \pi[Y]$;

B2. $\pi[c] = c$, for all real c;

B3. $\pi[X+Y] = \pi[X] + \pi[Y]$ when X and Y are independent;

B4. Continuity imposed tacitly.

A random parameter

T_0: defective and continuous with a strictly increasing cdf $F_{T_0}(\cdot)$, supported on $[-\infty, +\infty]$ and having positive jumps at both $-\infty$ and $+\infty$.

$\varphi_X(T_0)$: the exponential premium with random parameter T_0.

Comonotonicity

A random vector $(X_1, \ldots, X_n)$ is *comonotonic* if there exists a r.v. T and non-decreasing functions f_i, $i = 1, \ldots, n$, such that

$$(X_1, \ldots, X_n) = (f_1(T), \ldots, f_n(T)), \qquad \textit{in distribution}.$$

A functional related to $\pi[\cdot]$

Let

$$\Phi_{T_0} = \{\varphi_X(T_0) | X \text{ a bounded r.v.}\}.$$

Then $\rho_{T_0} : \Phi_{T_0} \to \mathbb{R}$ is defined by

$$\rho_{T_0}[\varphi_X(T_0)] = \pi[X].$$

The axioms restated

If (and only if) $\pi[\cdot]$ satisfies the set $\mathbb{S}$ of axioms, the functional $\rho_{T_0}[\cdot]$ satisfies the following set $\mathbb{S}$' of axioms:

A1'. If $\varphi_X(T_0) \leq \varphi_Y(T_0)$ a.s., then $\rho_{T_0}[\varphi_X(T_0)] \leq \rho_{T_0}[\varphi_Y(T_0)]$;

A2'. $\rho_{T_0}[\varphi_c(T_0)] = c$, for all real c;

A3'. $\rho_{T_0}[\varphi_X(T_0) + \varphi_Y(T_0)] = \rho_{T_0}[\varphi_X(T_0)] + \rho_{T_0}[\varphi_Y(T_0)]$;

A4'. If $\varphi_{X_n}(T_0)$ converges a.s. to $\varphi_X(T_0)$, then $\lim_{n\to+\infty} \rho_{T_0}[\varphi_{X_n}(T_0)] = \rho_{T_0}[\varphi_X(T_0)]$.

Verification of A4'

For a given sequence $\{X_n\}$ of bounded r.v.'s and a bounded (limit) r.v. X, it holds that X_n converges weakly to X, with min$[X_n] \to$ min$[X]$ and max$[X_n] \to$ max$[X]$, if and only if $\varphi_{X_n}(T_0)$ converges a.s. to $\varphi_X(T_0)$.

Only if: by dominated convergence theorem.

If: by Continuity Theorem of Moment Generating Functions.

A class extension

Recall that $\rho_{T_0} : \Phi_{T_0} \to \mathbb{R}$. Any $\varphi_X(T_0) \in \Phi_{T_0}$ has a strictly increasing cdf with same support as X.

Let $U(T_0)$ be uniformly distributed on $(0, 1)$.

Let

$$\Theta_{T_0} = \{F_V^{-1}(U(T_0))|V \text{ a bounded r.v.}\}.$$

Note that $\Phi_{T_0} \subset \Theta_{T_0}$.

The axioms restated

We impose that $\rho_{T_0}[\cdot] : \Theta_{T_0} \to \mathbb{R}$ satisfies the set $\mathbb{S}$'' of axioms, which is the analog of $\mathbb{S}$', given by

A1''. **(Monotonicity)** If $F_V^{-1}(U(T_0)) \leq F_W^{-1}(U(T_0))$ a.s., then $\rho_{T_0}[F_V^{-1}(U(T_0))] \leq \rho_{T_0}[F_W^{-1}(U(T_0))]$;

A2''. **(Certainty Equivalence)** $\rho_{T_0}[c] = c$, for all real c;

A3''. **(Comonotonic Additivity)** $\rho_{T_0}[F_V^{-1}(U(T_0))+F_W^{-1}(U(T_0))] = \rho_{T_0}[F_V^{-1}(U(T_0))] + \rho_{T_0}[F_W^{-1}(U(T_0))]$;

A4''. **(Continuity)** If $F_{V_n}^{-1}(U(T_0))$ converges a.s. to $F_V^{-1}(U(T_0))$, then $\lim_{n\to+\infty} \rho_{T_0}[F_{V_n}^{-1}(U(T_0))] = \rho_{T_0}[F_V^{-1}(U(T_0))]$.

The representation theorem

Theorem 1 *The functional $\rho_{T_0}[\cdot]$ satisfies the set $\mathbb{S}$" of axioms if and only if there exists some non-decreasing function G : $[-\infty, +\infty] \to [0, 1]$ such that*

$$\rho_{T_0}[F_V^{-1}(U(T_0))] = \int_{[-\infty,+\infty]} F_V^{-1}(F_{T_0}(t))dG(t).$$

On the subclass Φ_{T_0}, the functional $\rho_{T_0}[\cdot]$ (and consequently the risk measure $\pi[\cdot]$) can be represented by a mixture of exponential premiums, i.e.,

$$\pi[X] = \int_{[-\infty,+\infty]} \varphi_X(t)dG(t).$$

A sketch of the proof

Under the set $\mathbb{S}$" of axioms, the functional $\rho_{T_0}[\cdot]$ can be represented by

$$\rho_{T_0}[F_V^{-1}(U(T_0))] = \int_{(-\infty,+\infty)} v d\left(1 - w(1 - F_{F_V^{-1}(U(T_0))}(v))\right),$$

in which the function $w(\cdot) : [0,1] \to [0,1]$ is non-decreasing, right continuous and satisfies $w(0) = 0$ and $w(1) = 1$.

A note on the representation theorem

The set of axioms A1'' to A4'' is more restrictive than the set of axioms A1' to A4' (and hence also more restrictive than the original set of axioms A1 to A4).

Consider $\Delta_{T_0} = \Theta_{T_0} \backslash \Phi_{T_0}$.

The class Δ_{T_0} includes all non-trivial discrete random variables $F_V^{-1}(U(T_0))$ with V a bounded random variable.

Do there exist axioms other than A1'' to A4'' that impose the same conditions on the class Φ_{T_0} but different conditions on the class Δ_{T_0}, characterizing a different functional?

An equivalent representation

The mixture of exponential premiums can also be expressed as a *unimodal* mixture of Esscher premiums, i.e., there exists some non-decreasing function $H : [-\infty, +\infty] \to [0, 1]$, concave on $(0, +\infty)$ and convex on $(-\infty, 0)$ such that

$$\pi[X] = \int_{[-\infty,+\infty]} \psi_X(t) dH(t).$$

The mixed Esscher transform

Notice that $\pi[\cdot]$ can be expressed as $\pi[X] = \mathbb{E}^*[X]$, where the expectation is calculated using the differential

$$dF_X^{(H(\cdot))}(x) = \left(\int_{t\in[-\infty,+\infty]} \frac{e^{tx}dH(t)}{\mathbb{E}[e^{tX}]}\right) dF_X(x).$$

Connections with financial no arbitrage pricing [1]

General equilibrium models with negative exponential utility functions:

- Bühlmann (1980), one-period;
- Iwaki, Kijima & Morimoto (2001), multi-period.

Connections with financial no arbitrage pricing [2]

Pure no arbitrage models:

- Gerber & Shiu (1996): Lévy processes;
- Delbaen & Haezendonck (1989): compound Poisson processes;
- Bühlmann, Delbaen, Embrechts & Shiryaev (1996): conditional Esscher transforms for semi-martingales.

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