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A signature-based approach to comparisons among multiple systems

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Abstract

Consider a coherent system whose n components have independent, identically distributed (i.i.d.) lifetimes. The signature of the system is an n-dimensional vector $\mathbf{s} = (s_1, \ldots, s_n)$ representing the probability distribution of the index of the ordered component whose failure causes the system to fail. A brief review of the theory and applications of system signatures is given. The notions of "stochastic ordering" and "stochastic precedence," and the way these notions are applied in comparing the performance of two system designs, are discussed. Some of the limitations of pairwise comparisons using these orderings are noted. A new metric (the "maximum lifetime ordering") is proposed for selecting the "best" system from among k systems: if the probability $P(T_i = \max\{T_1, \ldots, T_k\})$ is a maximum when $i = r$, then system r is selected as the preferred system. The interpretation and computation of this metric are discussed and its use is illustrated in an example in which this metric provides meaningful comparisons while pairwise comparisons fail to do so. Justification is provided for the recommendation that a specific stepwise process be employed when selecting a system for use.

Key words: *Coherent systems, multiple comparisons, stochastic order, stochastic precedence, system signatures.*

1 Introduction

Consider a coherent system of order n whose components have i.i.d. lifetimes. As is well known (see, e.g., Barlow and Proschan (1981)), coherent systems are characterized by the properties of component relevance and the monotonicity of system performance as a function of component performance. In this paper, we investigate a problem in structural reliability dealing with the comparison of multiple systems and the selection of the "best" system for use. Since our treatment will involve the application of the theory of system signatures and the use of stochastic notions of ordering among random variables or among their distributions, we will begin with a brief overview of these ideas and some comments about their utility in studying the reliability of engineered systems. A well-known tool in distinguishing between and among n-component coherent systems is the structure function, a function $\varphi : \{0,1\}^n \to \{0,1\}$ which characterizes the deterministic relationship between the system's functioning and the functioning of its components (see Barlow and Proschan (1981)). While there is a one-to-one correspondence between coherent systems and their structure functions, these functions are somewhat awkward algebraic objects that do not easily serve as an index for coherent systems. Samaniego (1985) introduced a characteristic of system designs which, while being less general than the structure function, has the virtues of being more interpretable, more easily computed and more useful in many applications. Most importantly, the so-called "system signature" of a coherent system of order n is of fixed dimension n and is distribution-free, justifying its use as an index for coherent systems of any size. A formal definition follows.

Definition 1.1. Consider a coherent system of order n. Assume that the system's n components are independent and identically distributed (i.i.d.) according to the (continuous) lifetime distribution F. The *signature* of the system, denoted by $\mathbf{s}$, is an n-dimensional probability vector whose ith element is given by $s_i = P(T = X_{i:n})$ for $i = 1, 2, \ldots, n$, where T is the lifetime of the system and the variables $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ are the order statistics corresponding to the failure times $X_1, X_2, \ldots, X_n$ of the system's components.

Some comments on the i.i.d. assumption on component lifetimes might be helpful. One of the important uses of signatures in reliability analysis has been the comparison of system designs. It is evident that a comparison between two systems with quite different component characteristics can often be misleading or inconclusive. For example, a series system in n highly reliable components may well outperform a parallel system with n relatively poor components. But it is clear that a parallel structure is a better system design than a series structure with the same number of components. Once the i.i.d. assumption is made, any remaining differences in system performance must be attributable to the system's design. In that sense, the i.i.d. assumption "levels the playing field" so that one has a basis for comparing the designs themselves. From an analytical point of view, signatures, as defined above, make available for use the tools of combinatorial mathematics and the well-known distribution theory for the order statistics of an i.i.d. sample for studying the performance of a particular system. It should, of course, be acknowledged that specific applications of signature-related results in non-i.i.d. settings must be done, if at all, with considerable caution. This caveat notwithstanding, signature-based results should shed light on models in some "neighborhood" of the i.i.d. setting that can be studied via system signatures. The five distinct coherent systems in three components are easily seen to have signatures $(1, 0, 0)$, $(1/3, 2/3, 0)$, $(0, 1, 0)$, $(0, 2/3, 1/3)$ and $(0, 0, 1)$. In terms of reliability, it is clear that these five systems would be increasingly reliable as one goes from first (series) to last (parallel).

Samaniego (1985) showed that the distribution of the system lifetime T, given i.i.d. components lifetimes $\sim F$, can be written in terms of $\mathbf{s}$ and F alone:

Theorem 1.1. *Let F be a continuous distribution on $(0, \infty)$, $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} F$ be the component lifetimes of a coherent system of order n, and T be the system lifetime. Then*

$$\overline{F}_T(t) = P(T > t) = \sum_{j=0}^{n-1} \left(\sum_{i=j+1}^{n} s_i \right) \binom{n}{i} (F(t))^i (\overline{F}(t))^{n-i} \,. \tag{1.1}$$

In addition to "representation results" such as Theorem 1.1, reliability analysts are often also interested in "preservation theorems" which show that certain characteristics of an index of a class of systems are inherited by the systems themselves. Such results are often essential tools in studying the comparative performance of systems. The result below shows that several types of stochastic relationships enjoyed by pairs of system signatures are preserved by the lifetimes of the corresponding systems. The three most commonly used criteria for comparing the relative sizes of two random variables are defined below. These orderings apply to comparisons between continuous variables and between discrete variables. For more detail, see Shaked and Shanthikumar (2007).

Definition 1.2. Given two independent random variables X and Y, (i) X is smaller than Y in the *stochastic ordering* (denoted by $X \leq_{\mathrm{st}} Y$) if and only if their respective survival functions satisfy the inequality $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all t, (ii) X is smaller than Y in the *hazard rate ordering* (denoted by $X \leq_{\mathrm{hr}} Y$) if and only if the ratio of survival functions $\overline{F}_X(t)/\overline{F}_Y(t)$ is increasing in t, and (iii) X is smaller than Y in the *likelihood ratio ordering* (denoted by $X \leq_{\mathrm{lr}} Y$) if and only if the ratio $f_Y(t)/f_X(t)$ is non-decreasing in t, where f_X and f_Y represent the densities or probability mass functions of X and Y, respectively. For each of these orderings (denoted generally by "$\leq_*$"), the notation $X \leq_* Y$ and $F_X \leq_* F_Y$ are used interchangeably.

In the following preservation theorem, proven by Kochar, Mukerjee and Samaniego (1999), signature vectors are seen as the distributions of discrete variables (namely, the index r of the ordered component failure time $X_{r:n}$ which is fatal to the system).

Theorem 1.2. *Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be the signatures of the two systems of order n, both based on components with i.i.d. lifetimes with common distribution F. Let T_1 and T_2 be their respective lifetimes. The following preservation results hold:*

(a) if $\boldsymbol{s}_1 \leq_{\mathrm{st}} \boldsymbol{s}_2$, then $T_1 \leq_{\mathrm{st}} T_2$,

(b) if $\boldsymbol{s}_1 \leq_{\mathrm{hr}} \boldsymbol{s}_2$*, then* $T_1 \leq_{\mathrm{hr}} T_2$*, and*

(c) if $\boldsymbol{s}_1 \leq_{\mathrm{lr}} \boldsymbol{s}_2$ *and* F *is absolutely continuous, then* $T_1 \leq_{\mathrm{lr}} T_2$*.*

The conditions on the system signatures in Theorem 1.2 are sufficient but not necessary conditions for the similar ordering to hold for system lifetimes. Block, Dugas and Samaniego (2006) generalized the result above, giving explicit signature-based necessary and sufficient conditions for the inequalities of the system lifetimes to hold in each of the stochastic orders considered above.

While the stochastic relationships above between system signatures are clearly useful tools in the comparison of competing systems, they provide only a partial rather than a total ordering among systems. Kochar, Mukerjee and Samaniego (1999) give examples of systems that are non-comparable according to the standard orderings. There is, however, a metric that does induce a complete ordering among systems. Arcones, Kvam and Samaniego (2002) treated the notion of "stochastic precedence," an alternative way to quantify the fact that one random variable is smaller than another. The "sp" relationship may be defined as follows.

Definition 1.3. Let X and Y be independent random variables with respective distributions F_1 and F_2. Then X is said to *stochastically precede* Y (written $X \leq_{\mathrm{sp}} Y$) if and only if $P(X < Y) \geq P(X > Y)$. The variables are *equivalent in the sp ordering* if $P(X < Y) = P(X > Y)$. Continuous variables X and Y are sp-equivalent if and only if $P(X \leq Y) = 1/2$.

Suppose that T_1 and T_2 are the lifetimes of two coherent systems of arbitrary sizes. Even without any restrictions on the joint distribution of the components of each of the two systems, one will always be able to classify the second system as better than, equivalent to or worse than the first system according to whether $P(T_1 \leq T_2)$ is less than, equal to or greater than 1/2. Thus, stochastic precedence provides a total ordering among all coherent systems of a given size and, in fact, among any arbitrary collection of coherent systems. Hollander and Samaniego (2008) provide an explicit formula for calculating the probability $P(T_1 \leq T_2)$ when all components have i.i.d. lifetimes with common continuous distribution F. Theorem 1.3 is stated in the more general setting in which the component lifetime distributions of the two systems may differ.

Theorem 1.3. *Consider coherent system 1 with components having lifetimes* $X_1, \ldots, X_n \overset{\text{iid}}{\sim} F_1$ *and coherent system 2 with components having lifetimes* $Y_1, \ldots, Y_m \overset{\text{iid}}{\sim} F_2$. *Denote the systems' signatures by* $\boldsymbol{s}_1$ *and* $\boldsymbol{s}_2$ *and their respective lifetimes by* T_1 *and* T_2. *Then*

$$P(T_1 \leq T_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} s_{1i} s_{2j} P(X_{i:n} \leq Y_{j:m}) \ . \tag{1.2}$$

Hollander and Samaniego (2008) provide a combinatorial argument which establishes an explicit expression for the probability $P(X_{i:n} \leq Y_{j:m})$ in (1.2) under the assumption that all component lifetimes are i.i.d. with common distribution F. In this case, $P(X_{i:n} \leq Y_{j:m})$ is seen to be a constant independent of F. Substituting this expression for $P(X_{i:n} \leq Y_{j:m})$ in Theorem 1.3 yields:

Theorem 1.4. *Under the conditions of Theorem 1.3, with* $F_1 = F_2$,

$$P(T_1 \leq T_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} s_{1i} s_{2j} \sum_{s=i}^{n} \frac{\binom{n}{s}\binom{m}{j}}{\binom{n+m}{s+j}} \cdot \frac{j}{s+j} \ . \tag{1.3}$$

Example 1.1. It is easy to verify that the two four-component systems with respective minimal cut sets $\{\{1\},\{2,3,4\}\}$ and $\{\{1,2\},\{1,3\},\{1,4\},\{2,3\}\}$ are non-comparable in the "st" sense (or, of course, also in the stronger "hr" or "lr" senses). The first of these systems has signature $\mathbf{s}_1 = (1/4, 1/4, 1/2, 0)$ and the second has signature $\mathbf{s}_2 = (0, 2/3, 1/3, 0)$. Using Theorem 1.4, $P(T_1 \leq T_2)$ may be computed as

$$(1/6)(11/14) + (1/12)(13/14) + (1/6)(1/2) + (1/12)(53/70) \\ + (1/3)(17/70) + (1/6)(1/2),$$

a sum that reduces to the fraction $109/210 = 0.519$. From this we conclude that the second system will last longer than the first slightly more than half the time. Thus, in the "sp" ordering, the second system is to be preferred to the first. When comparing larger or more complex systems, the direct calculation using formula (1.3) can be a tiresome exercise; however, since this formula is easily programmed, even seemingly cumbersome pairwise comparisons can be done in a few milliseconds. ■

The expression in (1.3) is clearly distribution-free, immediately yielding signature-based necessary and sufficient conditions for stochastic precedence between two systems of arbitrary order with i.i.d. component lifetimes.

Theorem 1.5. *Let T_1 and T_2 represent the lifetimes of two systems of orders n and m based on two independent i.i.d. samples of sizes n and m from a common continuous distribution F. Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be their respective signatures, and let W be the function of these signatures given by*

$$W(\boldsymbol{s}_1, \boldsymbol{s}_2) = \sum_{i=1}^{n}\sum_{j=1}^{m} s_{1i}s_{2j} \sum_{s=i}^{n} \frac{\binom{n}{s}\binom{m}{j}}{\binom{n+m}{s+j}} \cdot \frac{j}{s+j}\,. \tag{1.4}$$

Then

$$P(T_1 \leq T_2) > 1/2 \text{ if and only if } W(\boldsymbol{s}_1, \boldsymbol{s}_2) > 1/2, \tag{1.5}$$

$$P(T_1 \leq T_2) = 1/2 \text{ if and only if } W(\boldsymbol{s}_1, \boldsymbol{s}_2) = 1/2, \tag{1.6}$$

$$P(T_1 \leq T_2) < 1/2 \text{ if and only if } W(\boldsymbol{s}_1, \boldsymbol{s}_2) < 1/2, \tag{1.7}$$

A comprehensive account of the theory and applications of system signatures is presented in the recent monograph by Samaniego (2007). Covered there are the topics of signature-based closure and preservation theorems in reliability, applications of signatures to special system designs (e.g., direct and indirect majority systems and consecutive k-out-of-n systems), system comparisons based on "stochastic precedence," applications of signatures to the comparison of communication networks and optimality results in a Reliability Economics framework (i.e., based on criteria which depend on both performance and cost). Recent extensions of the signature concept, and new applications of signatures in engineering reliability, include the extension of system signatures to systems whose components have exchangeable lifetimes

(Navarro et al. (2008)), the development of "dynamic signatures" and their applications to the comparisons of new and used systems and to the engineering practice of "burn-in" (Samaniego et al. (2009)), statistical inference about the common lifetime distribution of components based on system failure time data (Bhattacharya and Samaniego (2010)), the joint signature of systems with shared components (Navarro et al. (2010)) and applications of signatures to systems with heterogeneous components (Navarro et al. (2011)).

2 Comparisons among three or more systems

Our discussion of the comparison of systems in this section will touch upon the notion of "mixed systems." In brief, a mixed system, as defined in Boland and Samaniego (2004), is simply a stochastic mixture of two or more coherent systems of a given size n, each of which has components with i.i.d. lifetimes with common distribution F. Such systems may be viewed as being selected from the class of all systems of size n by a randomization process which chooses a coherent system for use according to a fixed probability distribution. If one randomly selects from among k systems with signature vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$ according to the probability distribution $\mathbf{p} = (p_1, p_2, \ldots, p_k)$, the mixed system is easily shown to have the signature vector $\mathbf{s} = \sum_{i=1}^{k} p_i \mathbf{s}_i$. In practice, mixed systems would be used in situations in which a set of coherent systems are to be chosen sequentially according to a fixed distribution $\mathbf{p}$. The extension from coherent systems to mixed systems broadens the space of possible signature vectors from a large, discrete space (whose size is not, in general, known) to a larger but more manageable space, the simplex of n-dimensional probability vectors. Dugas and Samaniego (2007) demonstrated that, in a large class of problems, the optimal system of size n, relative to criterion functions which account for both performance and cost, will be a mixed system rather than an individual coherent system.

Let us consider the comparison of three or more systems using the pairwise stochastic precedence criteria discussed in Section 1.

Example 2.1. Suppose that we have a choice among three mixed systems of order 6, the components of which are assumed to have independent lifetimes with common distribution F. Suppose these three systems have the following signature vectors:

$$\begin{aligned} \mathbf{s}_1 &= (0.2, 0.2, 0.2, 0.0, 0.2, 0.2) \\ \mathbf{s}_2 &= (0.2, 0.1, 0.2, 0.2, 0.2, 0.1) \\ \mathbf{s}_3 &= (0.3, 0.1, 0.1, 0.1, 0.2, 0.2) \end{aligned} \tag{2.1}$$

It is not difficult to confirm that these signatures are not ordered in stochastic precedence; specifically, denoting as X_1, X_2 and X_3 the random variables on the integers 1, 2, 3, 4, 5, 6 with the signatures $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$ above as their probability mass functions, then $X_1 =_{\text{sp}} X_2$ while $X_3 <_{\text{sp}} X_1$ and $X_2 <_{\text{sp}} X_3$. However, the fact that these signature vectors are not sp ordered does not preclude the possibility that the corresponding system lifetimes T_1, T_2 and T_3 are ordered in stochastic precedence. That determination requires an application of Theorem 1.4. For the systems with signatures in (2.1), we may obtain, using formula (1.3), that

$$\begin{aligned} P(T_1 < T_2) &= 0.5006, \\ P(T_2 < T_3) &= 0.5001, \quad \text{and} \\ P(T_1 < T_3) &= 0.4975, \end{aligned} \tag{2.2}$$

that is, $T_1 <_{\text{sp}} T_2$ and $T_2 <_{\text{sp}} T_3$ but $T_3 <_{\text{sp}} T_1$. Thus, we see that on the basis of direct pairwise comparisons, no system can be declared better than the other two in stochastic precedence. Since stochastic ordering implies stochastic precedence, it is evident that the system lifetimes T_1, T_2 and T_3 are not stochastically ordered either. ■

It is apparent that pairwise comparisons using either the st or sp metric might well fail to give helpful guidance about which of several systems stands to provide better performance. Examples such as the one above lead us to suggest an alternative approach to the comparison of multiple systems.

We will define a general criterion for comparing k systems, and we will illustrate the approach concretely in the comparison of three systems. In particular, we shall see that the ambiguity reflected in the example leading to (2.2) can be ameliorated by the application of the proposed criterion. We note that the MLO criterion defined below reduces to a pairwise comparison relative to stochastic precedence when only two systems are being compared.

Definition 2.1. Consider k systems of possibly varying sizes $n_1, n_2, \ldots, n_k$. Assume that the components of each system have i.i.d. lifetimes with common distribution F. Let the signature vectors of the k systems be denoted by $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$, and denote the corresponding system lifetimes by $T_1, T_2, \ldots, T_k$. Then, system r is said to be optimal in the *maximal lifetime ordering* (MLO) for the k systems of interest if

$$P(T_r = \max_{1 \le j \le k}\{T_j\}) \ge P(T_i = \max_{1 \le j \le k}\{T_j\}) \text{ for all } i \in \{1, \ldots, k,\ i \ne r\}\ . \tag{2.3}$$

The criterion in Definition 2.1 has a characteristic that is immediately appealing, namely, that it constitutes a total ordering of the k systems. Any pair of systems among the k can be compared by this criterion, and all k systems can be ranked from best to worst. The criterion also has a natural interpretation. The system that is declared optimal among these k systems is the system that has the highest probability of lasting the longest if all k systems began operation simultaneously. We propose the use of this criterion only in circumstances in which lower-order (e.g., pairwise) comparisons fail to identify the best system. If a system dominates a set of competitors in pairwise (st or sp) comparisons, one would have no real motivation to study other metrics involving the joint behavior of the systems of interest. We will comment further on the interpretation and potential utility of our proposed optimality criterion in the concluding section. We note that Blyth (1972) gives examples of paradoxical behavior that can occur when using the maximum as in (2.3). His example in the case when the variables are independent uses a discrete distribution that puts all its mass on one point.

Such models are of course inapplicable in any discussion of system lifetimes. Thus, Blyth's examples do not undermine the value of the maximum lifetime ordering in the setting studied here. We turn now to a result which provides a formula that facilitates its computation.

Theorem 2.1. *Consider k systems, with the ith system having n_i components. Assume that the components of each system have i.i.d. lifetimes with common distribution F. Let the k system signature vectors be denoted by $\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_k$, and denote the corresponding system lifetimes by $T_1, T_2, \ldots, T_k$. Then*

$$P(T_r = \max_{1\le j\le k}\{T_j\}) = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_k=1}^{n_k} s_{1i_1}s_{2i_2}\cdots s_{ki_k} \times P\left(X_{r,i_r:n_r} > X_{j,i_j:n_j}\ \forall\, j \in \{1,\ldots,k\}, j \neq r\right), \tag{2.4}$$

where $X_{u,v:w}$ is the vth order statistic among the w lifetimes of the components of system u.

Proof. By the Law of Total Probability, we have

$$P(T_r = \max_{1\le j\le k}\{T_j\}) = \sum_{i_1=1}^{n_1}\cdots\sum_{i_k=1}^{n_k}\prod_{j=1}^{k} P(T_j = X_{j,i_j:n_j}) \times P\left(T_r > T_j\ \forall\, j \neq r | T_j = X_{j,i_j:n_j}, j = 1,\ldots,k\right).$$

By the definition of the signature vector, we may rewrite this representation as

$$P(T_r = \max_{1\le j\le k}\{T_j\}) = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_k=1}^{n_k} s_{1i_1}s_{2i_2}\cdots s_{ki_k} \times P\left(T_r > T_j\ \forall\, j \neq r | T_j = X_{j,i_j:n_j}, j = 1,\ldots,k\right).$$

It follows from the independence assumption of all component lifetimes that this latter identity may be rewritten as (2.4). ∎

Calculation of the metric in (2.4) is thus seen to require the computability of the probability $P\left(X_{r,i_r:n_r} > X_{j,i_j:n_j} \ \forall\, j \in \{1,\ldots,k\}, j \neq r\right)$ for $r = 1,2,\ldots,k$. These probabilities depend, in turn, on the probabilities that k independent order statistics (obtained from k independent samples) have a particular ordering. A formula for probabilities of this latter type is given in Kvam and Samaniego (1993). However, for $k \geq 3$, the formula contains a minor typographical error and is not correct as written. For the purpose of this paper, it will suffice to establish the correct formula for the case $k = 3$. This enables us to return to the example involving three 6-component systems treated earlier in this section. Our further treatment of this example will illustrate the use and utility of the maximal lifetime ordering metric. We prove the result below using a different approach than in the calculus-based proof given in Kvam and Samaniego (1993).

Theorem 2.2. *Let $X_1,\ldots,X_n$, $Y_1,\ldots,Y_m$ and $Z_1,\ldots,Z_p$ be three independent i.i.d. samples from a common continuous distribution F, and let $X_{i:n}$, $Y_{j:m}$ and $Z_{k:p}$ represent, respectively, the ith, jth and kth order statistic of the first, second and third sample. Then*

$$P(X_{i:n} < Y_{j:m} < Z_{k:p}) = \sum_{s=i}^{n} \sum_{t=j+s}^{n+m} \frac{\binom{n}{s}\binom{m}{j}\binom{m+n}{t}\binom{p}{k}}{\binom{n+m}{j+s}\binom{m+n+p}{k+t}} \cdot \frac{j}{s+j} \cdot \frac{k}{k+t}\,. \qquad (2.5)$$

Proof. We will establish the identity in (2.5) by the following combinatorial argument. Let us imagine the Xs, Ys and Zs above as colored balls in an urn, with the Xs being red, the Ys being white and the Zs being blue. We will view each ball selected as being the next smallest of its type, that is, the smallest possible of its type that is not already drawn. The event "$X_{i:n} < Y_{j:m} < Z_{k:p}$" will occur if and only if, (1) for some $s \geq i$, there are, among the first $s+j$ non-blue balls drawn, precisely s red balls and j white balls, (2) taking (1) as given, the last ball drawn among these $s+j$ balls is red, (3) taking (1) and (2) as given, for some $t \geq s+j$, there are k blue balls and t non-blue balls drawn among the first $t+k$ balls drawn, and (4), taking (1), (2) and (3) as given, the last of these $t+k$ balls is blue. In validating the above accounting, note that steps (1) and (2) do not specify where these $s+j$ non-blue balls lie relative to the blue balls. The ordering of blue and non-blue balls is determined only in steps (3) and (4). Furthermore, given (1) and (2), it is implicit that the t non-blue balls selected in step (3), thought of there as the first t non-blue balls to be chosen, automatically include the s red balls and j white balls drawn in step (1). This is because $t \geq s+j$, and the $s+j$ balls selected in step (1) represent by definition the smallest s Xs and the smallest j Ys, with all s Xs smaller than the jth Y; thus, the selection of the smallest t non-blue balls would necessarily include them. We now proceed to the calculation of the probability that the four events in steps (1)–(4) happen simultaneously. Given that all of the balls in the urn at any given time are equally likely to be drawn, the probability of selecting s red balls and j white balls in these $s+j$ draws from the urn is the hypergeometric probability

$$\frac{\binom{n}{s}\binom{m}{j}}{\binom{n+m}{j+s}} \,. \tag{2.6}$$

Further, given (1), the probability that the last ball drawn is white is equal to

$$\frac{j}{s+j} \,. \tag{2.7}$$

Now, given (1) and (2), the probability of selecting k blue balls and t non-blue balls as the first $k+t$ draws from the urn is the hypergeometric probability

$$\frac{\binom{m+n}{t}\binom{p}{k}}{\binom{m+n+p}{k+t}} \,. \tag{2.8}$$

Finally, the probability that the last of these $k+t$ balls is blue is

$$\frac{k}{t+k} \,. \tag{2.9}$$

The product of the unconditional probability in (2.6) and the three successive conditional probabilities in (2.7), (2.8) and (2.9) is

$$\frac{\binom{n}{s}\binom{m}{j}\binom{m+n}{t}\binom{p}{k}}{\binom{n+m}{j+s}\binom{m+n+p}{k+t}} \cdot \frac{j}{s+j} \cdot \frac{k}{t+k} \,,$$

a value which represents the probability of having s Xs before the jth Y and having the jth Y before the kth Z. Since s can range from i to n and, given s, t can range from $s+j$ to $n+m$, the Law of Total Probability yields the identity in (2.5). ■

Example 2.1 (continued)**.** Consider the three systems of order 6 whose signature vectors are displayed in (2.1). Under an i.i.d. assumption on the lifetime of the components in each of the three systems, the pairwise comparisons of these systems under stochastic ordering and under stochastic precedence have both been shown to be inconclusive. Let us now execute the comparisons using the maximal lifetime ordering criterion. Under the conditions of Theorem 2.1, we write $P(T_3 = \max_{1\leq j\leq 3}\{T_j\}) = P(T_1 < T_2 < T_3) + P(T_2 < T_1 < T_3)$. Now, with system sizes $n_1 = n$, $n_2 = m$ and $n_3 = p$, we have

$$P(T_1 < T_2 < T_3) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{m}\sum_{i_3=1}^{p} s_{1i_1}s_{2i_2}s_{3i_3} \sum_{s=i_1}^{n}\sum_{t=i_2+s}^{n+m} \frac{\binom{n}{s}\binom{m}{i_2}\binom{m+n}{t}\binom{p}{i_3}}{\binom{n+m}{i_2+s}\binom{m+n+p}{i_3+t}} \frac{i_2}{s+i_2}\frac{i_3}{t+i_3}. \quad (2.10)$$

Since among the signature vectors in (2.1) there is only one zero element, the triple sum in (2.10) must account for 180 non-zero products $s_{1i_1}s_{2i_2}s_{3i_3}$. Computing the combinatorial portion of formula (2.10) is also tedious. Thus, to compare the three systems with signatures in (2.1) simultaneously via the MLO criterion, we have programmed formula (2.10) and obtained from it the required probabilities:

$$\begin{aligned} P(T_1 = \max\{T_1, T_2, T_3\}) &= 0.3369, \\ P(T_2 = \max\{T_1, T_2, T_3\}) &= 0.3169, \quad \text{and} \\ P(T_3 = \max\{T_1, T_2, T_3\}) &= 0.3462. \end{aligned} \quad (2.11)$$

Thus, by the maximal lifetime ordering criterion, system 3 is optimal among the three systems under consideration. ■

The result in Example 2.1 may be interpreted as follows. If the three systems with signatures in (2.1) were started at the same time and run in parallel until all three of them failed, the probability is 0.3462 that the third system would be the last to fail. As this probability is larger than the probabilities that either of the other two systems will be the last to fail, the third system is identified as optimal according to the MLO criterion.

3 Conclusions

As has been mentioned in the foregoing discussion, there are circumstances in which the standard methods for the pairwise comparisons of multiple systems of interest fail to yield a definitive ranking among them. In this paper, we propose an ordering that is applicable to the comparison of k systems, where k is an arbitrary integer greater than 2. The maximal lifetime ordering has the positive attribute of providing a total ordering of the systems being compared. Further, the sense in which optimality is claimed has an easily understood interpretation, namely, that the optimal system has the highest probability of lasting the longest if all the systems of interest were started at the same time. A general signature-based computational formula is given for calculating the relevant probabilities. For the special case of the comparison of three systems, the formula required for computing these probabilities explicitly is established. The latter result is applied in a specific example in which pairwise comparisons are inconclusive, and the optimal system in the sense of maximal lifetime ordering is identified.

The recommended ordering is seen as a reasonable alternative when one needs to choose among several available systems and pairwise comparisons are unable to identify an optimal system. A word of caution is warranted regarding the possible routine usage of the criterion introduced in this paper. It is possible for a system to be judged optimal in the presence of several systems but to be judged inferior to one of these competing systems in a direct comparison. The example above serves to illustrate this apparent dilemma. Our comparison of the three systems simultaneously leads to the judgment that system 3 is the best, that is, is optimal with respect to the maximal lifetime ordering. Recall, however, that in (2.2), the pairwise comparisons of the signatures of the three systems treated in Example 2.1 reveal that system 1 is better than system 3 in the stochastic precedence order. These two results together indicate that if system 3 is tested simultaneously with systems 1 and 2, system 3 will last the longest 34.62% of the time, a higher percentage than can be associated with either system 1 or system 2. However, if system 3 and system 1 are tested simultaneously, system 1 will last the longest 50.25% of the time, that is, system 1 is better than system 3. While these two conclusions seem contradictory, they can be reconciled by noting the fact that at the pairwise level, there is no clear winner. The comparisons in (2.2) show that in pairwise tests, while system 1 is superior to system 3, system 2 is superior to system 1 and system 3 is superior to system 2. Since no system dominates the others in pairwise comparisons, the conclusion drawn from the 3-way comparison remains as the only one of the comparisons that is unambiguous.

Our recommendation regarding the use of the method of multiple comparisons of systems presented here is to perform these comparisons in a stepwise manner. If one system can be declared best in a class of k systems based on a collection of pairwise comparisons, then this system should be selected for use, as further comparisons are unnecessary. As one might suspect, when the pairwise comparisons of three systems result in transitivity among the three system lifetimes relative to stochastic precedence, the maximum lifetime ordering of the three systems will usually lead to the same conclusion regarding optimality. This circumstance is illustrated in the following example.

Example 3.1. Suppose that we have a choice among three mixed systems of order 6, the components of which are assumed to have independent lifetimes with common distribution F. Suppose these three systems have the following signature vectors:

$$\begin{aligned} \mathbf{s}_1 &= (0.2, 0, 0, 0.8, 0, 0) \\ \mathbf{s}_2 &= (0, 0, 0.7, 0, 0, 0.3) \\ \mathbf{s}_3 &= (0, 0.3, 0, 0, 0.7, 0) \end{aligned} \tag{3.1}$$

For the systems with signatures in (2.12), we may use formula (1.3) to obtain pairwise comparisons among the system lifetimes T1, T2 and T3, obtaining

$$\begin{aligned} P(T_1 < T_2) &= 0.5642, \\ P(T_2 < T_3) &= 0.5363, \quad \text{and} \\ P(T_1 < T_3) &= 0.6217. \end{aligned} \tag{3.2}$$

Thus, we have that $T_1 <_{\text{sp}} T_2$, $T_2 <_{\text{sp}} T_3$ and $T_1 <_{\text{sp}} T_3$. We conclude from this that system 3 is optimal. For the comparison of the three system lifetimes simultaneously, we obtain from formula (2.10) that

$$\begin{aligned} P(T_1 = \max\{T_1, T_2, T_3\}) &= 0.2272, \\ P(T_2 = \max\{T_1, T_2, T_3\}) &= 0.3522, \quad \text{and} \\ P(T_3 = \max\{T_1, T_2, T_3\}) &= 0.4206. \end{aligned} \tag{3.3}$$

We see that for these three systems, system 3 would be identified as optimal by the MLO criterion as well. ■

One might conjecture that, when a particular system is optimal in pairwise sp comparisons among k systems, it will also be optimal when the k systems are compared via the MLO criterion. The following example shows that this need not be the case.

Example 3.2. Suppose that we have a choice among three mixed systems of order 6, the components of which are assumed to have independent lifetimes with common distribution F. Suppose these three systems have the following signature vectors:

$$\begin{aligned} \mathbf{s}_1 &= (0.3, 0.3, 0.1, 0, 0, 0.3) \\ \mathbf{s}_2 &= (0.1, 0.2, 0.3, 0, 0.3, 0.1) \\ \mathbf{s}_3 &= (0.1, 0.2, 0.2, 0.2, 0.2, 0.1) \end{aligned} \tag{3.4}$$

For the systems with signatures in (3.4), we may use formula (1.3) to obtain pairwise comparisons among the system lifetimes T_1, T_2 and T_3, obtaining

$$\begin{aligned} P(T_1 < T_2) &= 0.5770, \\ P(T_2 < T_3) &= 0.5004, \quad \text{and} \\ P(T_1 < T_3) &= 0.5785. \end{aligned} \tag{3.5}$$

Thus, we have that $T_1 <_{\text{sp}} T_2$, $T_2 <_{\text{sp}} T_3$ and $T_1 <_{\text{sp}} T_3$. We conclude from this that system 3 is optimal. However, for the comparison of the three system lifetimes simultaneously, we obtain from formula (2.10) that

$$\begin{aligned} P(T_1 = \max\{T_1, T_2, T_3\}) &= 0.2953, \\ P(T_2 = \max\{T_1, T_2, T_3\}) &= 0.3538, \quad \text{and} \\ P(T_3 = \max\{T_1, T_2, T_3\}) &= 0.3509. \end{aligned} \tag{3.6}$$

We see that for these three systems, system 2 would be identified as optimal by the MLO criterion. ■

Example 3.2 shows that it is possible that 2-way, 3-way, . . . , and k-way comparisons will not all lead to the same conclusion regarding the optimality of a given system. In this particular example, one can see that the 3-way comparisons of the systems with signatures in (3.4) are extremely close, and that systems 2 and 3 are nearly equivalent in the MLO ordering. Indeed, they are nearly equivalent, as well, in a pairwise comparison. So choosing system 2 over system 3 would not make much of a difference in a practical sense. It is nonetheless true that the pairwise and 3-way comparisons disagree. This provides further motivation for searching for an optimal system in a stepwise fashion. The stepwise strategy relieves the experimenter of the additional labor required to execute higher-order comparisons when lower-order comparisons are definitive. It avoids the unnecessary redundancy seen in Example 3.1, where lower-order and higher-order comparisons lead to the same conclusion. More importantly, it leads the experimenter to a sound conclusion when lower-order comparisons are definitive, leading to a particular conclusion while higher-order comparisons lead to a different conclusion.

A definitive ranking among k systems can always be achieved using the MLO criterion in k-way comparisons. But when a lower-order comparison yields a definitive result regarding optimality, it trumps all higher order comparisons. This is evident in the case of Example 3.2. In pairwise comparisons, it is seen that system 3 tends to last longer than either of the other two systems, quite a strong form of optimality. On the other hand, when all three systems are compared together, system 2 has the highest probability of failing last. Were we to select system 2 for use, we would do so knowing that system 3 will tend to last longer than either system 1 or 2. It is clear that choosing a system that will tend to outperform each of the available alternative systems is best.

When the results of pairwise comparisons are ambiguous, with no system dominating all others, then a higher-order comparison can be used to obtain a definitive answer. The k-way comparison of k systems with respect to the maximal lifetime ordering will always identify one or more systems that are optimal in the MLO sense. In general, we suggest that one employ a k-way comparison of k systems only when all lower-order comparisons are inconclusive.

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