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Reliability and Lifetime Data Analysis

On Uniformly Optimal Networks: A Reversal of Fortune?

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In this article, the general problem of comparing the performance of two communication networks is examined. The standard approach, using stochastic ordering as a metric, is reviewed, as are the mixed results on the existence of uniformly optimal networks (UONs) which have emerged from this approach. While UONs have been shown to exist for certain classes of networks, it has also been shown that no UON network exists for other classes. Results to date beg the question: Is the problem of identifying a Uniformly Optimal Network (UON) of a given size dead or alive? We reframe the investigation into UONs in terms of network signatures *and the alternative metric of* stochastic precedence. *While the endeavor has been dead, or at least dormant, for some 20 years, the findings in the present article suggest that the question above is by no means settled. Specifically, we examine a class of networks of a particular size for which it was shown that no individual network was uniformly optimal relative to the standard metric (the uniform ordering of reliability polynomials), and we show, using the aforementioned alternative metric, that this class is totally ordered and that a uniformly optimal network exists after all. Optimality with respect to "performance per unit cost" type metrics is also discussed.*

Keywords Networks; Network Signatures; Reliability; Stochastic ordering; Stochastic precedence; Total ordering; Uniform optimality.

Mathematics Subject Clssification Primary 68M10, 90B25; Secondary 60E15, 65K10.

1. Introduction

Communication networks have become pervasive in modern society, and the study of their performance (in terms of the persistence of connectivity of a desired type) has received increasing attention in recent years. Important applications abound. Larger-than-life examples include the World Wide Web, telephone networks, and transportation networks for

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air and land travel in the continental U.S. Moderate-size applications include local area networks centered around a dedicated server and a bus service connecting a collection of locations in a medium-sized city. Rivera and Young (2009) provide both an explanation of this growing interest and motivation for further research in the area, as it relates to military applications, as evidenced in the following remark: "The recent National Research Council Report on Network Science identified the need to develop fundamental knowledge about large, complex networks that will enable a better understanding of how to apply technology to the Army's network-centric operations. Two key topics of our research are the basic issues of the fundamental capacity of MANETs (Mobile *Ad Hoc* Networks) and the connectivity of nodes in a MANET." In an edited Proceedings volume from a conference of the Army Research Laboratory's Collaborative Alliance in Communications and Networks, Gowens et al. (2009) feature 70 research papers on a multiplicity of subjects dealing with the design and performance characteristics of communications networks, studied with a view toward their assessment relative to a variety of metrics including speed, reliability, security, survivability, and performance per unit cost. Among the themes receiving special emphasis were optimality issues, the development of secure, scalable, reliable communications in dynamic environments and the performance of networks as a function of their topologies (or designs).

The general problem of comparing the performance of two communications networks has been investigated in various ways. We will review a particular approach, one involving the comparison of network signatures as described in Boland et al. (2003). Our discussion of the concept of network signatures leads naturally to an examination of the question of primary interest here: is the problem of identifying a Uniformly Optimal Network (UON) dead or alive? The latter problem has an interesting, if somewhat rocky, history. We will review the early successes in finding UONs among networks of a given size, and we will revisit the stunning, infinite array of counterexamples of Myrvold et al. (1991) showing that UONs do not exist among networks of certain specific sizes. The latter article essentially dashed the hopes of network researchers seeking to develop a general methodology for finding UONs. This type of endeavor has been dead, or at least dormant, for some 20 years. The findings in the present article suggest that the question above is by no means settled. Specifically, we will examine a class of networks of a particular size for which it was shown that no individual network was universally optimal relative to a standard metric (the uniform ordering of reliability polynomials), and we will show, using an alternative metric, that this class is totally ordered and that a uniformly optimal network exists after all.

Before proceeding, we will set some basic definitions and briefly discuss the needed background on network signatures and on two approaches to ordering random variables that will be central to the comparisons that lie ahead. We will follow the standard practice of representing a communications network as an undirected graph. Such graphs are completely specified by a collection of vertices (or nodes) and a set of edges joining selected pairs of vertices. The so-called Wheatstone Bridge, a well-known network with four vertices and five edges, is displayed in Fig. 1 and is utilized later in the section to illustrate the computation of the "signature" of a network.

The family of networks with v vertices and n edges will be denoted by $G(v, n)$. Following the standard convention (see Colbourn, 1987), we assume that vertices function with certainty, while the edges in a network are subject to failure. We will restrict attention to "coherent" networks, defined as follows.

Definition 1.1. A network is *coherent* is every edge is relevant and the network's functioning cannot be diminished when a failed edge is replaced by a working edge.

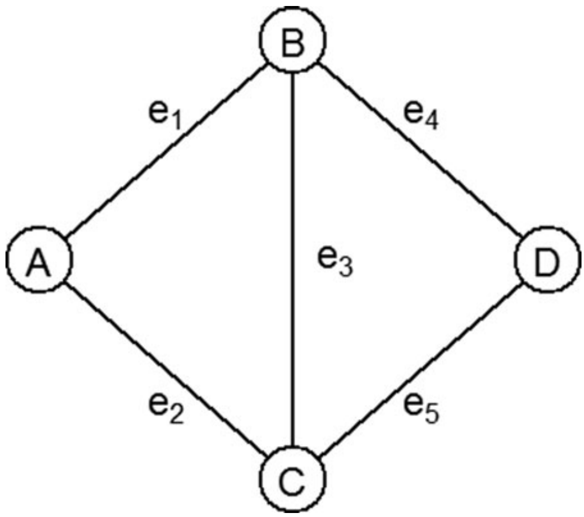

Figure 1. The Wheatstone Bridge.

The main quality of interest in a communications network is *connectivity*. Different types of connectivity may be relevant at a given time or for a particular purpose. We will distinguish among the following options: "2-terminal connectivity" (that is, there exists a working path—a series of adjacent working edges—linking two distinguished vertices), "k-terminal connectivity" (for $2 < k < v$, there exists a working path linking any pair in a set of k distinguished vertices), and "all-terminal connectivity" (there exists a working path linking any pair of vertices). Physically, the coherence of a network simply ensures certain desirable performance properties of the network, since the inclusion of each edge within the network can only serve to enhance a given connectivity objective.

The reliability of a network is defined, simply, as the probability that the network meets its connectivity goal. If T represents the time at which the network's connectivity fails, then the reliability function of the network is represented by the continuous function $\overline{F}_T(t) = P(T > t)$. The reliability of a network is of course a function of the reliability of its edges. A variety of options exist for the modeling of edge reliabilities. In what follows, we will use the term "lifetime" when referring to the time at which connectivity of an edge or network fails, that is, we will routinely use the term "lifetime" in place of "failure time." Consider a network in the family $G(v, n)$. A completely general model for edge reliabilities would posit that the edge lifetimes $X_1, \ldots, X_n$ have a continuous multivariate distribution which allows for possible dependencies and singularities. Rather little work has been done at this level of generality, partly because of the paucity of tractable multivariate models for random vectors with positive elements and partly because of the complexity of characterizing the precise dependencies that might be present in a given application. One viable option that should be mentioned is the Marshall–Olkin (1967) multivariate exponential model, which can be derived as a shock model and gives rise to edge lifetimes that are positively correlated. However, the fact that the marginal distributions of edge lifetimes have exponential distributions limits the model's applicability. A less general but reasonably tractable alternative to an unconstrained multivariate model is a model which posits independent but not identically distributed (i.n.i.d.) edge lifetimes. If X_i is the lifetime of the ith edge, then the X_i are independent, with $X_i \sim F_i$, $i = 1, \ldots, n$, with each F_i being a continuous function on $(0, \infty)$. The probability that edge i is working at time t_0 is $p_i = \overline{F}_i(t_0)$, where $\overline{F}_i = 1 - F_i$. This intermediate type of stochastic model makes it possible to represent the reliability of a network at a fixed time t_0 as a multinomial expression of the edge reliabilities $\{p_1, \ldots, p_n\}$ that is linear in each p_i. While work does exist under the i.n.i.d. assumption, the model does not readily lend itself to the comparison of network performance.

The most commonly encountered model in network reliability studies is the following important special case: Edge lifetimes are assumed to be independent and to have identical distributions, that is, $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} F$. At a fixed time t_0, the probability that any given edge is working is $p = \overline{F}(t_0)$. In this case, the reliability at time t_0 of a network in the $G(v, n)$ class may be written as an nth degree polynomial in the argument p. The monograph by Colbourn (1987), as well as much of the published work on the comparison of the reliability of various networks, including research work on UONs, restrict attention to the i.i.d. framework. The i.i.d. assumption can be defended on several levels, and we put forward just a brief justification here. First, it will become readily apparent that the simplifying effect of the i.i.d. assumption makes network analysis analytically feasible, and it serves, as well, as a useful guidepost regarding network performance under more general conditions. This is no doubt why the assumption is commonly used in the literature on network reliability. It is also fair to say that the assumption reasonably approximates the stochastic behavior of edges in certain networks used in wired or even wireless communications. There are many instances in which all the edges of a network are reasonably thought to be equally vulnerable to failure and actually fail in similar but unrelated ways. Further, if the independence of edge lifetimes is deemed a reasonable assumption, then the study of network reliability under the additional assumption of a common edge reliability p may serve as a helpful way of bounding the reliability of the network. Specifically, if p may reasonably be assumed to be a lower bound on all $p_i = \overline{F}_i(t_0)$, or if $\overline{F}_i(t) \geq \overline{F}(t)$ is assumed for all t, then the network's reliability is bounded below by the reliability of the network based on the i.i.d. assumption with $p_i \equiv p$ or $\overline{F}_i \equiv \overline{F}$. Finally, the i.i.d. framework "levels the playing field" when comparing two networks. It is clear that a poorly-designed network with highly reliable edges will outperform a well-designed network with quite unreliable edges. Further, under the i.i.d. assumption on edge lifetimes, the differences between network designs can be characterized through distribution-free summaries like "network signatures." We now turn to a discussion of network signatures and some of their properties.

Consider a network in the $G(v, n)$ class, and assume that its n edges have lifetimes $\{X_i\}$ which are modeled as $X_1, X_2, \ldots, X_n \overset{iid}{\sim} F$. Suppose that a given type of connectivity is of interest. Let T be the time at which connectivity of the network fails. The failure of connectivity necessarily coincides with a particular edge failure. The signature of the network, given its specific connectivity goal, is defined as follows.

Definition 1.2. The *signature* **s** of a $G(v, n)$ network is an n-dimensional probability vector whose ith element is $s_i = P(T = X_{i:n})$, where $X_{i:n}$ is the ith smallest X among the sample of edge lifetimes $X_1, X_2, \ldots, X_n \overset{iid}{\sim} F$, or, alternatively, the ith order statistic in the random sample $X_1, X_2, \ldots, X_n$ of edge lifetimes.

The signature vector is the probability distribution of the discrete random variable Y defined as the index of the ordered edge failure that causes the network to fail. The interpretive value of the signature is thus related to the size of its elements; a network having a signature vector whose mass is concentrated on larger values of Y may be identified as a network with a longer lifetime than networks without this property. The computation of signatures generally requires combinatorial reasoning, as illustrated in the following examples. Consider again the Wheatstone Bridge Network shown in Fig. 1. We will compute the signature for two types of connectivity.

Example 1.1. Suppose 2-terminal connectivity (between vertices A and D) is of interest. The "minimal cut sets" for this network's connectivity are

$$\{1, 2\}, \quad \{4, 5\}, \quad \{1, 3, 5\}, \quad \{2, 3, 4\}.$$

Under the i.i.d. assumption, the 120 permutations of edge lifetimes X_1, X_2, X_3, X_4, and X_5 have equal likelihood. We may verify that the signature vector $\mathbf{s} = (0,\ 1/5,\ 3/5,\ 1/5,\ 0)$ as follows.

- It is clear from inspection that $P(T = X_{1:5}) = 0 = P(T = X_{5:5})$.
- Since $T = X_{2:5}$ if and only if permutations of the forms

  $$(1, 2, \_, \_, \_), \quad (2, 1, \_, \_, \_), \quad (4, 5, \_, \_, \_) \quad \text{or} \quad (5, 4, \_, \_, \_)$$

  occur, we obtain that $P(T = X_{2:5}) = 24/120 = 1/5$.
- Since $T = X_{4:5}$ if and only if permutations of the forms

  $$(\_, \_, \_, 2, 5), \quad (\_, \_, \_, 5, 2), \quad (\_, \_, \_, 1, 4) \quad \text{or} \quad (\_, \_, \_, 4, 1)$$

  occur, we obtain that $P(T = X_{2:5}) = 1/5$.
- It follows that $P(T = X_{3:5}) = 3/5$.

Example 1.2. Now, suppose that all-terminal connectivity in the Wheatstone bridge is of interest. The "minimal cut sets" for this network connectivity are

$$\{1, 2\}, \quad \{4, 5\}, \quad \{1, 3, 4\}, \quad \{1, 3, 5\}, \quad \{2, 3, 4\}, \quad \{2, 3, 5\}.$$

Under the i.i.d. assumption, the 120 permutations of edge lifetimes X_1, X_2, X_3, X_4, and X_5 have equal likelihood. We may verify that $\mathbf{s} = (0, 1/5, 4/5, 0, 0)$ as follows.

- It is clear from inspection that $P(T = X_{1:5}) = 0 = P(T = X_{5:5})$.
- Since $T = X_{2:5}$ if and only if permutations of the forms

  $$(1, 2, \_, \_, \_), \quad (2, 1, \_, \_, \_), \quad (4, 5, \_, \_, \_) \quad \text{or} \quad (5, 4, \_, \_, \_)$$

  occur, we have $P(T = X_{2:5}) = 1/5$.
- Since it is not possible to connect 4 vertices with just 2 edges, connectivity will fail at or before the 3rd edge failure. Thus, $P(T = X_{4:5}) = 0$ and $P(T = X_{3:5}) = 4/5$.

Network signatures have proven useful in the analysis of network performance and in comparisons between and among different network designs. Although the definition of a network's signature, as given above, involves the assumption of i.i.d. edge lifetimes, the signature vector is in fact a distribution-free topological invariant which may be used as an index of the network's design. The following representation theorem shows that, under the i.i.d. assumption, the distribution of the lifetimes of a network is solely a function of its signature vector and the underlying common distribution F of the lifetimes of its edges.

Theorem 1.1. (Samaniego, 1985). *Consider a network in the $G(v, n)$ class. Assume that the lifetimes of its n edges are i.i.d. with common distribution* F. *Let* **s** *be its signature vector. Then the survival function of the network's lifetime is given by*

$$\overline{F}_T(t) = \sum_{j=0}^{n-1} \left(\sum_{i=j+1}^{n} s_i \right) \binom{n}{j} (F(t))^j (\overline{F}(t))^{n-j}. \tag{1.1}$$

For extensions of the representation theorem above to the reliability function of a network with heterogeneous components, see Navarro et al. (2011).

In addition to "representation results" such as Theorem 1.1 above, reliability analysts are often also interested in "preservation theorems" which show that certain characteristics of an index of a class of networks are inherited by the networks themselves. Such results are often essential tools in studying the comparative performance of networks. The result below shows that several types of stochastic relationships enjoyed by pairs of network signatures are preserved by the lifetimes of the corresponding networks. The three most commonly used criteria for comparing the relative sizes of two random variables are defined below. These orderings apply to comparisons between continuous variables as well as to comparisons between discrete variables. For more detail, see Shaked and Shanthikumar (2007).

Definition 1.3. Given two independent random variables X and Y, discrete or continuous: (i) X is smaller than Y in the *stochastic ordering* (denoted by $X \leq_{\mathrm{st}} Y$) if and only if their respective survival functions satisfy the inequality $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all t; (ii) X is smaller than Y in the *hazard rate ordering* (denoted by $X \leq_{\mathrm{hr}} Y$) if and only if the ratio of survival functions $\overline{F}_Y(t)/\overline{F}_X(t)$ is increasing in t; and (iii) X is smaller than Y in the *likelihood ratio ordering* (denoted by $X \leq_{\mathrm{lr}} Y$) if and only if the ratio $f_Y(t)/f_X(t)$ is non-decreasing in t, where f_X and f_Y represent the densities or probability mass functions of X and Y, respectively. For each of these orderings (denoted generically by "*"), the notations $X \leq_* Y$ and $F_X \leq_* F_Y$ are used interchangeably.

In the following preservation theorem, proven by Kochar et al. (1999), signature vectors are seen as the distributions of discrete variables (namely, the index r of the ordered component failure time $X_{r:n}$ which is fatal to the network).

Theorem 1.2 (Kochar et al., 1999). *Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the signatures of the two networks, both containing n edges whose lifetimes are i.i.d. with common distribution F. Let T_1 and T_2 be their corresponding network lifetimes. The following preservation results hold:*

(a) if $\mathbf{s}_1 \leq_{\mathrm{st}} \mathbf{s}_2$, then $T_1 \leq_{\mathrm{st}} T_2$;
(b) if $\mathbf{s}_1 \leq_{\mathrm{hr}} \mathbf{s}_2$, then $T_1 \leq_{\mathrm{hr}} T_2$; and
(c) if $\mathbf{s}_1 \leq_{\mathrm{lr}} \mathbf{s}_2$ and F is absolutely continuous, then $T_1 \leq_{\mathrm{lr}} T_2$.

The result above makes it clear that one may compare the reliability of two networks by examining properties of the corresponding signature vectors.

In making stochastic comparisons among networks in the sections that follow, we will examine in detail two specific types of orderings between random variables. The first of these, "stochastic ordering," is defined above. We note that stochastic ordering applies to both discrete and continuous variables, and it is known to be a weaker ordering than hazard-rate and likelihood-ratio ordering. When all three orderings are well-defined, it is well known that lr $\Rightarrow$ hr $\Rightarrow$ st. An alternative concept capturing the notion that the random variable X is smaller than the random variable Y is that of "stochastic precedence." Arcones, Kvam and Samaniego (JASA, 2002) studied this ordering in a reliability context. Since it will play an important role in the sequel, we include a formal definition here.

Definition 1.4. Two independent random variables X and Y are ordered in *stochastic precedence* (denoted by $X \leq_{\text{sp}} Y$) if and only if $P(X < Y) \geq P(X > Y)$; the variables are *equivalent in stochastic precedence* (i.e., $X =_{\text{sp}} Y$) if and only if $P(X < Y) = P(X > Y)$.

We note that the "sp" ordering applies to both discrete or continuous (X, Y). For independent random variables X and Y, the sp ordering is weaker than stochastic ordering, that is, st $\Rightarrow$ sp. The sp ordering has a natural interpretation when comparing the (continuous) time to failure of two competing networks: If network lifetimes T_1 and T_2 satisfy $T_1 <_{\text{sp}} T_2$, then $P(T_1 < T_2) > 0.5$, that is, the chances are that network 2 will last longer than network 1.

2. Comparing Two $G(v, n)$ Networks

When all edges work independently of each other and have a common probability p of working, the reliability of a network with n edges can be written as an nth degree polynomial. The reliability polynomial of the network can be expressed, in standard form, as

$$h(p) = \sum_{r=1}^{n} d_r p^r \ . \tag{2.1}$$

Satyanarayana and Prabhakar (1978) provided an efficient technique for computing the "signed dominations" $\{d_r\}$ in the reliability polynomial. This polynomial provides a closed form expression for the probability that the network will retain its connectivity goal when all n edges operate independently and each works with probability p. The reliability of the network at a fixed time t is given by $h(\overline{F}(t))$.

The survival function of a network's lifetime T can also be written as a function of its signature $\mathbf{s}$ and the common component distribution F. At a fixed time t, where it is assumed that $P(X_j > t) = p$ for all j, the representation in Theorem 1.1 reduces to the reliability polynomial in "pq-form," where $q = 1 - p$:

$$\begin{aligned} h(p) &= \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} s_i \right) \binom{n}{j} p^j q^{n-j} \\ &= \sum_{j=1}^{n} a_j \binom{n}{j} p^j q^{n-j} \ , \end{aligned} \tag{2.2}$$

where $a_j = \sum_{i=n-j+1}^{n} s_i$ for $j = 1, \ldots, n$. The vectors $\mathbf{a}$ and $\mathbf{s}$ are linearly related. We will express this relationship as $\mathbf{a} = \mathbf{P}\mathbf{s}$, where,

$$P_{uv} = \begin{cases} 0 & \text{if } u + v \leq n \\ 1 & \text{if } u + v > n \end{cases} \ .$$

Writing $q^{n-j} = (1-p)^{n-j}$ as a binomial expansion, we may identify each domination d_i as a linear combination of the elements $a_1, \ldots, a_n$. More specifically, we may write $\mathbf{d} = \mathbf{Ma}$, where

$$\mathbf{M} = \begin{pmatrix} \binom{n}{1}\binom{n-1}{0} & 0 & 0 & \cdots & 0 \\ -\binom{n}{1}\binom{n-1}{1} & \binom{n}{2}\binom{n-2}{0} & 0 & \cdots & 0 \\ \binom{n}{1}\binom{n-1}{2} & -\binom{n}{2}\binom{n-2}{1} & \binom{n}{3}\binom{n-3}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pm\binom{n}{1}\binom{n-1}{n-1} & \mp\binom{n}{2}\binom{n-2}{n-2} & \pm\binom{n}{3}\binom{n-3}{n-3} & \cdots & \binom{n}{n}\binom{n-n}{n-n} \end{pmatrix}.$$

Since $\mathbf{d} = \mathbf{MPs}$, we may express the relationship of interest to us as

$$\mathbf{s} = \mathbf{P}^{-1}\mathbf{M}^{-1}\mathbf{d}. \tag{2.3}$$

We will use the symbol $(k)_j$ for the number of permutations of k items taken j at a time, that is, $(k)_j = k(k-1)\cdots(k-j+1)$. It is shown by Boland et al. (2003) that $\mathbf{M}^{-1}$ is the matrix $\mathbf{M}^*$ whose ith row is given by

$$(m^*_{i1}, \cdots, m^*_{ii}, 0, \cdots, 0) = \left(\underbrace{\frac{(i)_1}{(n)_1}, \frac{(i)_2}{(n)_2}, \cdots, \frac{(i)_i}{(n)_i}}_{i \text{ slots}}, \underbrace{0, \cdots, 0}_{n-i \text{ slots}} \right).$$

Theorem 2.1. *Let* **d** *and* **s** *denote the domination and signature vectors for a given network of order n (Boland et al., 2003). Then for* $i = 1, \ldots, n$, *we have*

$$s_i = \sum_{j=1}^{n-i} \frac{(n-i+1)_j - (n-i)_j}{(n)_j} d_j + \frac{(n-i+1)_{n-i+1}}{(n)_{n-i+1}} d_{n-i+1}. \tag{2.4}$$

Having the relationship $\mathbf{s} = f(\mathbf{d})$ in hand enables us to exploit both the computational advantages of dominations and the interpretive value of signatures.

Example 2.1. Consider the comparison between the two $G(9, 27)$ networks pictured in Fig 2. It is difficult to determine which is better by a visual inspection. The reliability polynomials of these two networks (see Satyarananaya and Prabhakar (1978)) are displayed below:

$$\begin{aligned} h_{G_1}(p) = {} & 419904p^{27} - 6021144p^{26} + 41705280p^{25} - 18489826p^{24} \\ & + 586821717p^{23} - 1413876060p^{25} + 2677774329p^{21} \\ & - 4074363810p^{20} + 5048856414p^{19} - 5135792742p^{18} \\ & + 4303029693p^{17} - 2967712776p^{16} + 1676975886p^{15} \\ & - 769265910p^{14} - 282176568p^{13} + 80853282p^{12} \end{aligned}$$

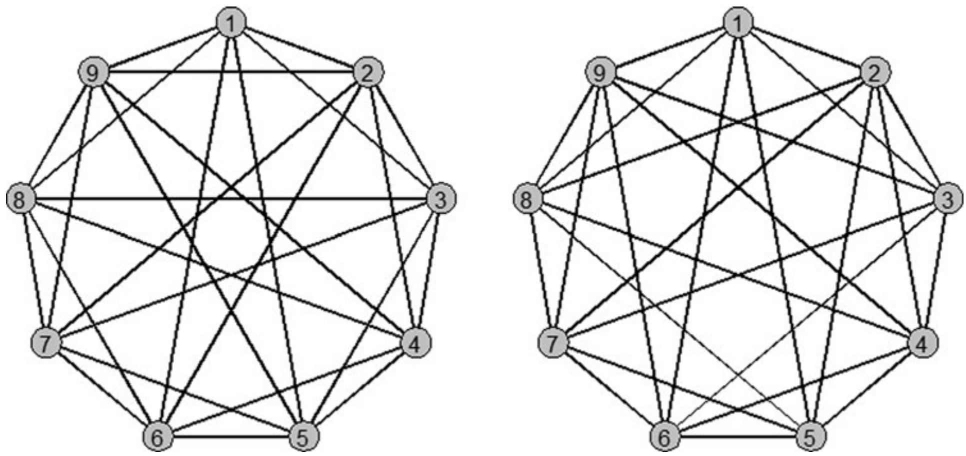

Figure 2. Networks G_1 and G_2.

$$+ 17445456p^{11} - 2667060p^{10} + 257634p^{9} - 11828p^{8}$$

$$\begin{aligned} h_{G_2}(p) = {} & 414720p^{27} - 5934288p^{26} + 41015964p^{25} - 181453380p^{24} \\ & + 574666025p^{23} - 1381692972p^{22} + 2611463517p^{21} \\ & - 3965536554p^{20} - 4904464002p^{19} + 4979513718p^{18} \\ & + 4164454729p^{17} - 2867022480p^{16} + 1617256842p^{15} \\ & - 740601350p^{14} - 271201476p^{13} + 77576922p^{12} \\ & + 16709916p^{11} - 2550156p^{10} + 245898p^{9} - 11268p^{8}. \end{aligned}$$

Since the difference polynomial $h_{G_1}(p) - h_{G_2}(p)$ has alternating signs, the identification of the better performing network by this means is cumbersome. But a comparison of the signatures of these networks readily yields an answer.

From the second and third columns of Table 1, we see that $\mathbf{s}_{G_1} \geq_{\text{st}} \mathbf{s}_{G_2}$, an inequality that immediately implies that $h_{G_1}(p) \geq h_{G_2}(p)$ for $p \in (0, 1)$. Further, $\mathbf{s}_{G_1} \geq_{\text{hr}} \mathbf{s}_{G_2}$, a conclusion that is not possible to obtain from an analysis of the polynomials $h_{G_1}(p)$ and $h_{G_2}(p)$ alone. This additional fact establishes that network G_1 is not only better than network G_2, it is actually better in quite a strong sense. More importantly, the example above illustrates well the potential utility of network signatures in the comparative analysis of network prerformance.

3. Uniformly Optimal Networks

Consider a class of networks of the same size, that is, with the same number of vertices (v) and the same number of edges (n). Suppose that, for any member of the $G(v, n)$ class, the lifetimes of the n edges are independent and have a common distribution F. The "traditional" approach to the problem of identifying a uniformly optimal network (UON) in the $G(v, n)$ class is to find, if possible, the $G(v, n)$ network (or equivalent group of networks) for which the time T^* to failure of connectivity (of a predetermined type) has a reliability function $\overline{F}_{T^*}(t)$ that is greater than or equal to the reliability function of every other network in

Table 1
Signature tail probabilities $S(x) = \sum_{i=x}^{27} s_i$ and their ratios

x	$S_{G_1}(x)$	$S_{G_2}(x)$	$S_{G_1}(x)/S_{G_2}(x)$
1	1.0	1.0	1.0
2	1.0	1.0	1.0
3	1.0	1.0	1.0
4	1.0	1.0	1.0
5	1.0	1.0	1.0
6	1.0	1.0	1.0
7	0.999970	0.999970	1.0
8	0.999787	0.999787	1.0
9	0.999149	0.999149	1.0
10	0.997367	0.997367	1.0
11	0.993612	0.993612	1.0
12	0.985922	0.985922	1.0
13	0.971744	0.971743	1.0000005
14	0.947220	0.947214	1.0000063
15	0.906907	0.906867	1.0000442
16	0.843421	0.843240	1.0002148
17	0.747317	0.746717	1.0008024
18	0.607883	0.606416	1.0024183
19	0.417560	0.415077	1.0059834
20	0.189140	0.186804	1.0125000
21	0.0	0.0	—
22	0.0	0.0	—
23	0.0	0.0	—
24	0.0	0.0	—
25	0.0	0.0	—
26	0.0	0.0	—
27	0.0	0.0	—

the class. Letting $p = \overline{F}(t)$, where F is the common lifetime distribution of the network's edges, the UON G^* (with lifetime T^*) satisfies

$$h_{G^*}(p) \geq h_G(p) \text{ for all } p \in [0, 1]$$

or, equivalently,

$$P_{G^*}(T^* > t) \geq P_G(T > t) \text{ for all } t \in [0, \infty)$$

for any network G (with lifetime T) in the class of interest. These inequalities are equivalent to the statement that $T \leq_{\text{st}} T^*$ for all network lifetimes T corresponding to a network in the class $G(v, n)$. The search for Uniformly Optimal Networks (UONs) among networks $G(v, n)$ of a given size includes work by Boesch et al. (1991), who, for example, identified the unique UON among networks in the $G(v, v-1)$, $G(v, v)$, $G(v, v+1)$, and $G(v, v+2)$ classes (see also Boesch and Suffel, 1984; Boesch, 1986). The UON in the $G(v, v+3)$ class

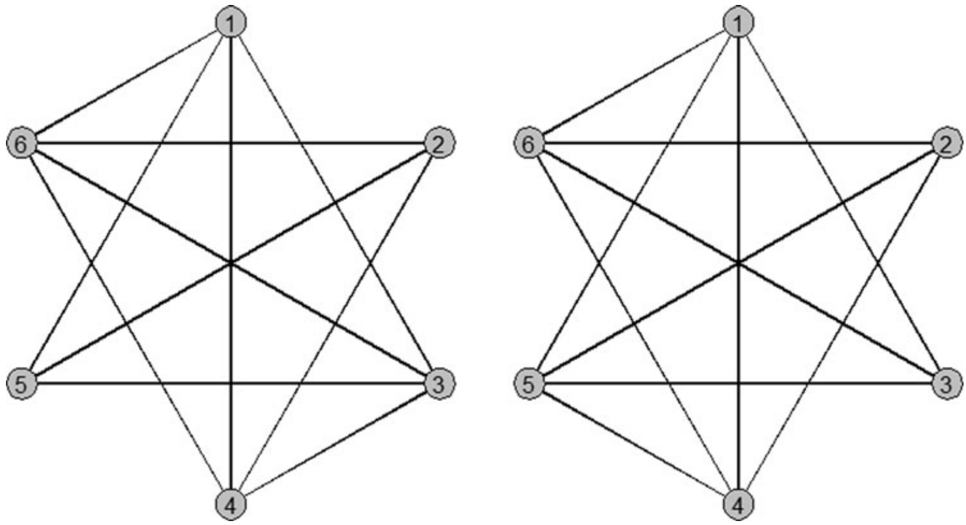

Figure 3. Networks G_8 and G_9.

was later identified by Wang (1994). This work appeared, at the time, to be the beginning of a major surge in the study of methods and results associated with identifying UONs.

Around this same time, a group of researchers based in Victoria, British Columbia, and Raleigh, North Carolina, had some doubts about the potential for success in these endeavors. In a stunning article, they demonstrated quite dramatically that such searches for UONs might well be for naught. Specifically, Myrvold et al. (1991) showed that for some classes of networks, e.g., the class $G(v, \binom{v}{2} - \frac{v}{2} - 1)$ for any even $v \geq 6$, a UON does not exist. They presented a similar collection of network classes with an odd number of vertices which, likewise, contained no UON. They proved the existence of a network in each such class which dominated every other network in the class for p sufficiently large, but was inferior to an alternative network if p is suitably small. The reliability polynomials of the two $G(6, 11)$ networks pictured in Fig. 3 have precisely the crossing property alluded to above.

Myrvold et al. all but squelched the vigorous research that had focused on the identification of UONs. Apparently, any further research in the area would need to face the fact that a given search might come up empty. Further, the problem of characterizing the classes of networks for which a UON does exist remained an imposing open problem.

This leads us to the main theme of the present paper. Is the search for UONs truly fraught with peril? Is this area of research essentially dead, or are there other formulations of the UON problem that hold some real promise? The question of interest to us is: Could it be that stochastic ordering is too strong a criterion to expect uniform optimality of a single member of a class $G(v, n)$? The following empirical study of $G(6, 11)$ networks represents, to us, a "reversal of fortune" in the study of optimality questions for communications networks. The $G(6, 11)$ class is, of course, one of the classes featured by Myrvold et al. as a class for which no UON exists relative to the stochastic ordering criterion.

The $G(6, 11)$ class contains $\binom{15}{11} = 1365$ possible network designs. Suppose we are interested in all-terminal connectivity. When all edge probabilities p_i are equal to p, we can compute the signatures of each of these networks. As shown in Table 2, there are precisely nine distinct signatures.

Table 2
The nine possible signatures of $G(6, 11)$ networks

$$\mathbf{s}_1 = (0.0909,\ 0.0909,\ 0.0909,\ 0.1061,\ 0.1407,\ 0.2100,\ 0.2706,\ 0,\ 0,\ 0,\ 0)$$
$$\mathbf{s}_2 = (0.0000,\ 0.0182,\ 0.0485,\ 0.0939,\ 0.1662,\ 0.2835,\ 0.3896,\ 0,\ 0,\ 0,\ 0)$$
$$\mathbf{s}_3 = (0.0000,\ 0.0182,\ 0.0424,\ 0.0848,\ 0.1619,\ 0.2922,\ 0.4004,\ 0,\ 0,\ 0,\ 0)$$
$$\mathbf{s}_4 = (0.0000,\ 0.0182,\ 0.0364,\ 0.0758,\ 0.1489,\ 0.2879,\ 0.4329,\ 0,\ 0,\ 0,\ 0)$$
$$\mathbf{s}_5 = (0.0000,\ 0.0000,\ 0.0242,\ 0.0788,\ 0.1697,\ 0.3117,\ 0.4156,\ 0,\ 0,\ 0,\ 0)$$
$$\mathbf{s}_6 = (0.0000,\ 0.0000,\ 0.0182,\ 0.0636,\ 0.1541,\ 0.3117,\ 0.4524,\ 0,\ 0,\ 0,\ 0)$$
$$\mathbf{s}_7 = (0.0000,\ 0.0000,\ 0.0182,\ 0.0606,\ 0.1485,\ 0.3052,\ 0.4675,\ 0,\ 0,\ 0,\ 0)$$
$$\mathbf{s}_8 = (0.0000,\ 0.0000,\ 0.0121,\ 0.0515,\ 0.1398,\ 0.3095,\ 0.4870,\ 0,\ 0,\ 0,\ 0)$$
$$\mathbf{s}_9 = (0.0000,\ 0.0000,\ 0.0121,\ 0.0485,\ 0.1385,\ 0.3160,\ 0.4848,\ 0,\ 0,\ 0,\ 0)$$

Networks G_8 and G_9 pictured in Fig. 3 have signatures $\mathbf{s}_8$ and $\mathbf{s}_9$, respectively. The signatures $\mathbf{s}_8$ and $\mathbf{s}_9$ are not comparable relative to stochastic ordering. Further, it is clear that there are many $G(6, 11)$ networks that are "equivalent" (under the assumption of i.i.d. edge lifetimes) to a given network with any one of the signatures above. For example, there are 180 networks in the $G(6, 11)$ class that have $\mathbf{s}_9$ as a signature vector.

The following claims about $G(6, 11)$ networks (with i.i.d. edge reliabilities) are easily confirmed.

- The signatures of the 1365 possible $G(6, 11)$ networks are totally ordered in stochastic precedence, and the nine distinct $G(6, 11)$ network signatures shown above are strictly sp-ordered:

$$\mathbf{s}_1 <_{\text{sp}} \mathbf{s}_2 <_{\text{sp}} \mathbf{s}_3 <_{\text{sp}} \mathbf{s}_4 <_{\text{sp}} \mathbf{s}_5 <_{\text{sp}} \mathbf{s}_6 <_{\text{sp}} \mathbf{s}_7 <_{\text{sp}} \mathbf{s}_8 <_{\text{sp}} \mathbf{s}_9$$

- Further, the following preservation result holds for all $G(6, 11)$ networks under stochastic precedence: if $\mathbf{s}_i \leq_{\text{sp}} \mathbf{s}_j$, then $T_i \leq_{\text{sp}} T_j$, where T_k represents the time of connectivity failure for a network of type k, with $k = 1, \ldots, 9$.
- The following comparisons show that the network G_9 is the Uniformly Optimal Network relative to the stochastic precedence ordering:

$$P(T_9 > T_8) = 0.501,\ P(T_9 > T_7) = 0.510,\ P(T_9 > T_6) = 0.514$$
$$P(T_9 > T_5) = 0.528,\ P(T_9 > T_4) = 0.534,\ P(T_9 > T_3) = 0.546$$
$$P(T_9 > T_2) = 0.553,\ P(T_9 > T_1) = 0.659$$

So there does exist a uniformly optimal $G(6, 11)$ network after all! It is clear that the criterion used in comparing networks makes a critical difference in both determining the existence of a UON and in identifying it.

4. Reliability-Economics Analysis of Network Designs

Relative to the sp criterion, one is able to identify G_9 as the Universally Optimal Network within the $G(6, 11)$ class. This network, and those with the same signature, have the uniformly best performance among all $G(6, 11)$ networks. Now, suppose that network costs are taken into account. Consider the criterion function

$$m_r(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \sum_{i=1}^{n} a_i s_i \Big/ \left(\sum_{i=1}^{n} c_i s_i\right)^r, \tag{4.1}$$

where the vectors $\mathbf{a}$ and $\mathbf{c}$ can be chosen arbitrarily within the context of two natural constraints: $0 < a_1 < \cdots < a_n$ and $0 < c_1 < \cdots < c_n$; the constant $r > 0$ is a calibration parameter that places more or less weight on costs depending on whether $r > 1$ or $r < 1$. The function $m_r(\mathbf{s}, \mathbf{a}, \mathbf{c})$ in (4.1) represents, when $r = 1$, one reasonable way of measuring performance per unit cost; its natural variants (for $r \neq 1$) may serve as criterion functions for identifying optimal networks when either performance or cost is deemed to merit greater weight than the other. The criterion was utilized in Dugas and Samaniego (2007) in identifying optimal networks of a given size (see also Samaniego, 2007). Since the numerical measure m results in a total ordering of networks, the existence of an optimal network (or networks) is guaranteed, and the identification of optimal networks is reduced to a tractable minimization problem.

Before proceeding further, we interject a brief word which provides some motivation for the criterion in (4.1). Suppose that the coefficients $\{a_i,\ i = 1, \ldots, n\}$ are chosen to be $a_i = EX_{i:n}$ for $i = 1, \ldots, n$. In this case, the numerator of $m_r(\mathbf{s}, \mathbf{a}, \mathbf{c})$ is simply ET, the expected lifetime of the network. On the other hand, the linear form of the denominator of $m_r(\mathbf{s}, \mathbf{a}, \mathbf{c})$ arises, for example, in the "salvage model" for a wired network which yields an expected cost of the network equal to

$$EC = \sum_{i=1}^{n} (C_f + n(A - B) + Bi)s_i,$$

where C_f is the fixed cost of manufacturing the networks of interest, A is the cost of an individual edge and B is the salvage value of an edge that is used but working when the network fails. We should add, however, that the salvage model represents merely one example of the cost criterion in the denominator of $m_r(\mathbf{s}, \mathbf{a}, \mathbf{c})$. There is considerable flexibility in the choice of the vector $\mathbf{c}$, as the monotonicity $c_1 < \cdots < c_n$, by itself, may be viewed as simply quantifying the fact that systems with stronger designs (with elements s_i large when i is large) tend to be more costly to produce. An engineer's assessment of such costs may lead to the most appropriate selection of the vector $\mathbf{c}$ in specific applications.

Example 4.1. Suppose the lifetimes of edges in a $G(6, 11)$ network are i.i.d. exponential variables with a mean life of 100 hr. Taking $a_i = EX_{i:11}$ for $i = 1, 2, \ldots, 11$, we may calculate the vector $\mathbf{a}$ as

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
9.1	19.1	30.2	42.7	57.0	73.7	93.7	118.7	152.0	202.0	302.0

Suppose we use the cost factors $c_i = 1 + 0.5 \times i$ for $i = 1, 2, \ldots, 11$. For $r = 1$, $m_1(\mathbf{s})$ is maximized at $\mathbf{s}_9$. Setting $r = 2$, the criterion function m becomes

$$m_2(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \sum_{i=1}^{11} a_i s_i \Big/ \left(\sum_{i=1}^{11} c_i s_i\right)^2, \tag{4.2}$$

and we obtain the following results for the criterion function $m_2(\mathbf{s}) = m_2(\mathbf{s}, \mathbf{a}, \mathbf{c})$ for the 9 distinct signatures of $G(6, 11)$ networks:

$m(\mathbf{s_1})$	$m(\mathbf{s_2})$	$m(\mathbf{s_3})$	$m(\mathbf{s_4})$	$m(\mathbf{s_5})$	$m(\mathbf{s_6})$	$m(\mathbf{s_7})$	$m(\mathbf{s_8})$	$m(\mathbf{s_9})$
5.034	4.757	4.748	4.740	4.710	4.698	4.697	4.687	4.686

From this, we see that any $G(6, 11)$ network with signature $\mathbf{s}_1$ is optimal on the basis of this performance vs. cost analysis, with $r = 2$.

We now report on a numerical search for optimal $G(6, 11)$ networks relative to $m_r(\mathbf{s}, \mathbf{a}, \mathbf{c})$ for values of r and c located in a grid. Specifically, given this criterion function with $c_i = U + Vi$ and a_i as above, we varied r from 1 to 10, and for each r we varied both U and V independently from 1 to 100. At each pair (U, V), we evaluated the criterion function for each of the nine signatures, and noted which signature maximizes the criterion function. For each r, the relative frequency distribution for the optimal signature over the grid of (U, V) pairs is shown in Table 3. When $r = 1$, signature $\mathbf{s}_9$ is optimal at all 10,000 (U, V) pairs. For $r > 1$, optimality is distributed among signatures 1, 4, 8, and 9 over the grid, with signature $\mathbf{s}_1$ dominating.

An intriguing feature of Table 3 is the fact that, among the 100,000 outcomes for which the optimal $G(6, 11)$ network signature relative to the metric $m_2(\mathbf{s}, \mathbf{a}, \mathbf{c})$ was recorded, the networks G_2, G_3, G_5, G_6 and G_7 never surfaced as optimal. This suggests that a certain "discontinuity" exists in the metric $m_2(\mathbf{s}, \mathbf{a}, \mathbf{c})$ as a function of the index of the network signatures ordered by their "sp ranking."

5. Discussion

Our examination of networks in the $G(6, 11)$ class is striking in a variety of different ways. First, it affirms that, relative to the stochastic precedence criterion, the class does contain a group of equivalent networks that are uniformly optimal, that is, better than all others at every possible value of the common edge reliability p. This suggests that the stochastic ordering criterion, which is stronger and more restrictive than stochastic precedence, is too blunt a tool for treating the delicate question of optimality among communication networks. Secondly, we note that, in the developments above, the sp criterion induces a special structure among

Table 3
Relative frequency of optimality

r	$\mathbf{s}_1$	$\mathbf{s}_4$	$\mathbf{s}_8$	$\mathbf{s}_9$
1	0.0000	0.0000	0.0000	1.0000
2	0.8718	0.0000	0.0353	0.0929
3	0.9450	0.0015	0.0118	0.0417
4	0.9657	0.0019	0.0069	0.0255
5	0.9761	0.0013	0.0046	0.0180
6	0.9820	0.0012	0.0033	0.0135
7	0.9859	0.0009	0.0032	0.0100
8	0.9887	0.0011	0.0019	0.0083
9	0.9906	0.0008	0.0019	0.0067
10	0.9921	0.0008	0.0021	0.0050

$G(6, 11)$ networks; all 1365 networks in the class are totally ordered, satisfying the reflexive, anti-symmetric, transitive, and trichotomy properties which characterize "order relations." It is well known that stochastic precedence need not be transitive in general (see, for example, Blyth, 1972). However, the present study of coherent $G(6, 11)$ networks includes three intriguing findings: the fact that transitivity in the sp criterion holds for network signatures and network lifetimes, the fact that stochastic precedence among the network signatures is preserved in the lifetimes of the networks themselves, and the fact that the $G(6, 11)$ class does indeed contain an equivalence class of networks that are uniformly optimal relative to the stochastic precedence criterion. These findings lead naturally to the question: how widely applicable are these tools and ideas in the general problem of comparing the performance of communications networks and in the search for uniform optimality? For now, the question of the existence of uniformly optimal networks has clearly been reopened for investigation. It is of course of considerable interest to determine the extent to which our optimality results for $G(6, 11)$ networks under the stochastic precedence ordering can be replicated for networks of different sizes under a variety of connectivity goals. Further, under the performance-per-unit cost metric in (4.1), it is of interest to determine if the optimal network design can be identified analytically. Analytical solutions to optimal system design problems in a reliability-economics context are presented in Samaniego (2007, Ch. 7).

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