The sign of affection [1]
Balance-theoretic models and incomplete signed digraphs

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Abstract

The initial structural models based on balance theory (structural balance and clusterability) dealt with incomplete signed digraphs. However, newer models (ranked clusters, transitivity, hierarchical $\overline{M}$-cliques) apply only to zero-one matrices representing either unsigned digraphs or complete signed digraphs. Since empirical networks of affective relations are signed but seldom complete — actors may have neutral feelings towards alters or affections may be unknown to the researcher — the latter models must be redefined if affective relations are to be analysed. It was found that each balance-theoretic model is characterised by a particular type of semicycle or path. Counts of the types of semicycles and paths suffice to identify the models. This approach is more general than triadic analysis because it handles complete as well as incomplete signed digraphs. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

If success is defined as theoretical progress, then balance theory is one of the success stories of the social sciences. In the course of the last 5 decades, Fritz Heider's initial theory on balanced sentiments (Heider, 1946, 1958) has been expanded to ever more

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general theories. The mathematical formalisation of Heider's theory by graph theorists and network analysts was central to this development.

Cartwright and Harary (1956) were the first to formalise Heider's theory on affective relations. They used a signed digraph to represent a structure of affective relations. In this network, the sign of an arc between two nodes corresponds to the sign of the affective relation from ego towards an alter: a positive arc expresses a positive sentiment, a negative arc expresses a negative sentiment, and no arc means the affective relation is either absent or neutral. Since absent and neutral relations may be treated alike according to Morrissette and Jahnke (1967), who did not find a psychological distinction between them, it is necessary to allow for three values or weights for the relationship: positive, negative, and zero. Cartwright and Harary (1956) and, subsequently, Davis (1967) showed that semicycles of arcs can be analysed in order to detect structural balance or clusterability in a network.

When structural balance and clusterability were reduced to models allowing for ranking (Holland and Leinhardt, 1971; Davis and Leinhardt, 1972), the sign of the arcs was dropped. Network analysts shifted their attention from affective relations towards sociometric data, e.g., friendship choices (Wasserman and Faust, 1994, p. 242; Manhart, 1995, pp. 205–206). Usually, sociometric data are dichotomous — relations are either present or absent — so they are modelled as unsigned digraphs. The triad has become the basic unit of analysis and each model is identified by a unique set of permitted and forbidden triads. Unfortunately, dichotomies are not suitable to represent affective relations since they cannot distinguish between positive affects, negative affects, and neutral or absent affects. One of three categories had to be ruled out so, initially, the neutral or absent category was omitted (e.g., Holland and Leinhardt, 1971, p. 115). This implies that all affective relations within a group are known and are either positive or negative. In working with empirical data on affective relations, both assumptions are untenable. Thus, the evolution of balance theories entails a restriction or specialisation: the newer models cannot be applied to the affective relations that Heider focused on.

In order to reunite balance theory and affective relations, the models incorporating ranking must be adapted so they can handle incomplete data: absent or neutral affective relations along with positive and negative relations. In this article, a method is presented that handles incomplete as well as complete signed digraphs. First, I will present a micro-model based on paths and semicycles. Next, I will show that all hierarchical models developed for triads can be defined in terms of this micro-model. Finally, I will apply this model to an incomplete signed digraph in order to demonstrate that it yields other results than triadic analysis of the dichotomised graph. The translation of an incomplete signed digraph into an unsigned digraph seriously alters the digraph's structure. Therefore, the model proposed here is a useful extension to classical models based on triads.

2. Model

In this section I will define a model that connects a group's microstructure, i.e., patterns of relations between its members, to its macrostructure: ranks and clusters.

2.1. Preliminary definitions

Let $X = \{1, 2, \ldots, n\}$ represent a finite set of social units or nodes, that are related by binary relations P (positive judgement) and N (negative judgement). Let $(i, j) \in P$ indicate that i evaluates j positively, and let $(i, j) \in N$ indicate that i evaluates j negatively, where $i, j, \in X$. $W = \{P, N\}$ is a multiple network. Let us assume that W does not contain loops. We can visualise W as a signed digraph, where $(i, j) \in P$ is represented by a positive or solid arc and $(i, j) \in N$ by a negative or dashed arc (compare Pattison, 1993, p. 258).

Definition 1. Let binary relation O be a partial order on a subset of X indicating (social) ranking, such that iff $(i, j) \in P$ or $(j, i) \in N$ and $(i, j) \in O$, then $i \leq j$.

Positive affects are directed towards nodes at higher ranks or towards nodes at the same rank. Negative affects point towards nodes at a lower rank or towards nodes at the same rank. This is the heart of the micro-model proposed here. It is an alternative to the model introduced by Davis and Leinhardt (1972), which is based on (mutual positive, asymmetric and mutual non-positive) dyads. However, in Section 3 I will show that the new micro-model yields the same results when applied to triads.

O captures the ranking of points, which is an equivalence relation. Therefore, all nodes under O can be partitioned into ranks. O does not necessarily operate on all nodes since the ranking of (some) nodes may be unknown, and multiple rankings may occur.

Definition 2. Let $C_1 C_2 \ldots C_m$ indicate a compound relation of length m on the nodes in X, where C_h is either P or the converse of N (indicated by N') for $h = 1, 2, \ldots, m$.

Let us call a semipath in W that corresponds to a labelled path in the compound relation $C_1 C_2 \ldots C_m$ a *counterpositive path*, and let us call a semicycle in W corresponding to a closed labelled path in this relation a *counterpositive cycle*. Note that labelled paths in a compound relation may correspond to walks in stead of paths. However, I intentionally reserve the expression *counterpositive path* to semipaths in stead of semiwalks in W.

2.2. Theorems

Theorem 3. *Two nodes i and j belong to the same rank iff they are connected by one or more interlocking counterpositive cycles, where all ordered pairs on the counterpositive cycles are elements of O.*

Proof: It is easily seen that each labelled path from i to j in the compound relation $C_1 C_2 \ldots C_m$ indicates that $i \leq j$, since the definition of a compound relation implies that iff $(i, j) \in C_1 C_2 \ldots C_m$ then there exists a sequence $i = k_0, k_1, \ldots, k_m = j$ of elements of X, such that $(k_{h-1}, k_h) \in P$ or $(k_{h-1}, k_h) \in N'$; $h = 1, 2, \ldots, m$ (Pattison, 1993,

p. 41). From Definitions 1 and 2 it follows that $k_{h-1} \leq k_h$ for each $(k_{h-1}, k_h) \in P$ as well as for each $(k_{h-1}, k_h) \in N'$ where $h = 1, 2, \ldots, m$. In consequence, $i \leq j$. Of course, this presupposes that all ordered pairs are elements of O. According to the Axiom of Quality (Boorman and White, 1976) $i = j$ iff $i \leq j$ and $i \geq j$. Therefore, $i = j$ iff $(i, j) \in C_1 C_2 \ldots C_m$ and $(j, i) \in C_1 C_2 \ldots C_m$. If there are counterpositive paths between i and j in both ways, either there is no node k other than i and j where the paths intersect. In this case there is a counterpositive cycle containing i as well as j. Or there is at least one node k other than i and j, where the counterpositive paths intersect, dividing the paths into two or more counterpositive cycles connected by k.

A rank is a set of nodes that cannot be ranked (Axiom of Quality). A node must belong to a counterpositive cycle in order to be assigned to a rank. In stead of analysing connected counterpositive cycles it suffices to analyse each counterpositive cycle on its own and apply the transitive property of the relation: if $i = k$ and $k = j$ then $i = j$.

Definition 4a. All nodes on counterpositive cycles that are connected by one or more positive semipaths belong to the same cluster.

Definition 4b. A cluster without nodes on an unclusterable counterpositive cycle is a regular cluster.

Definition 4c. A cluster with nodes on at least one unclusterable counterpositive cycle is a hierarchical cluster, and each negative arc within the cluster points from a node on a higher position to a node on a lower position within the cluster.

A rank may consist of regular and/or hierarchical clusters. If all connected counter-positive cycles are clusterable, i.e., no counterpositive cycle contains exactly one negative arc, a rank consists of one or more regular clusters, which are similar to Davis' plussets (Davis, 1967). A regular cluster contains nodes connected by positive arcs only, whereas negative arcs connect nodes of different clusters. This is the type of cluster associated with the models of balance, clusterability, ranked clusters (RC) and transitivity. The clusterable counterpositive cycle identifies regular clustering. Fig. 1 shows a clusterable digraph consisting of four nodes and three clusters (dotted).

On the other hand, the unclusterable counterpositive cycle identifies a rank with at least one hierarchical cluster. In Section 3 we will see that the hierarchical cluster cycle identifies Johnsen's hierarchical $\overline{M}$-cliques (Johnsen, 1985). Following Johnsen, I propose to interpret the counterpositive cycle with exactly one negative arc as a set of nodes belonging to a cluster that is ranked internally: a hierarchical cluster. Orwell's ''all animals are equal but some are more equal'' comes to mind. I will refer to this type of semicycle as a hierarchical cluster cycle. Positive arcs on this semicycle indicate group affiliation. Negative arcs within a hierarchical cluster indicate ranking: the sender of the negative affect occupies a higher position than the receiver, although they are members of the same cluster. It represents internal differentiation within a cluster, which should be a partial order to make sense. Therefore, ranking within a hierarchical cluster
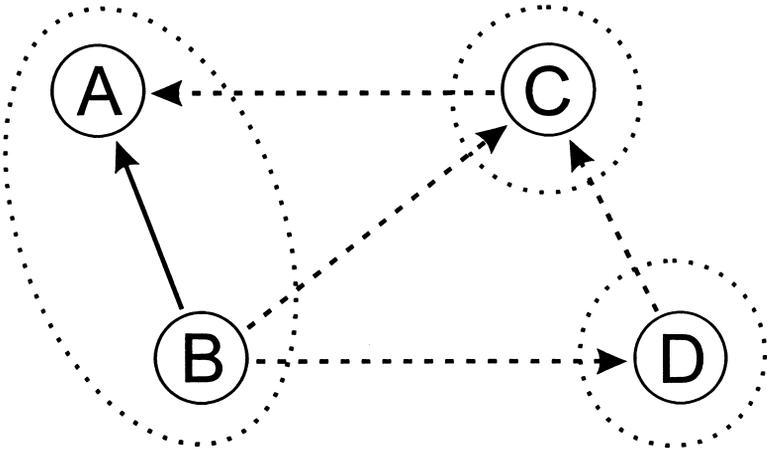
Fig. 1. Regular clusters.

must be antisymmetric. Fig. 2 shows a hierarchical cluster in which *C* outranks *D*. Note the new meaning of a negative arc in the context of a hierarchical cluster.

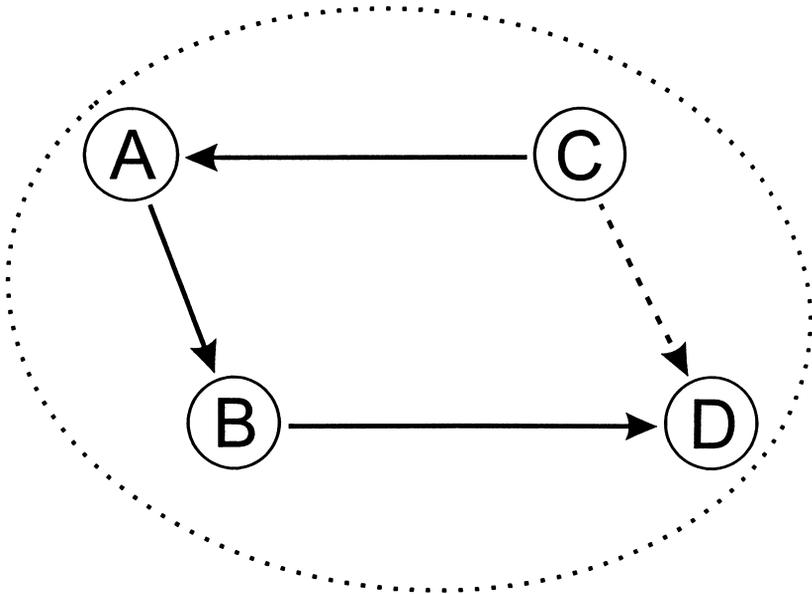**Definition 5**. A rank may contain all nodes that can be divided into regular or hierarchical clusters.

Fig. 2. Hierarchical cluster.

This definition is necessary in order to determine which nodes belong to different ranks. Note that this axiom cannot be derived from Theorem 3, because nodes that do not necessarily belong to the same rank, may still belong to the same rank; their ranking is unknown or ambiguous.

Theorem 6. $i < j$ iff there are at least two counterpositive paths from i to j, and the subgraph induced by the counterpositive paths contains at least one unclusterable semicycle, where all ordered pairs on both paths are elements of O.

Proof: $(i < j)$ is equivalent to the expression: $(i \leq j)$ and $(i \neq j)$. In order to satisfy the first condition $(i \leq j)$ there must be a counterpositive path from i to j. In order to satisfy the second condition $(i \neq j)$ this counterpositive path must contain at least one pair of nodes (k, l) that cannot be assigned to regular or hierarchical clusters within a rank. Davis (1967) proved that a signed graph has a clustering iff it does not contain a semicycle with exactly one negative arc. So all nodes are clusterable, except nodes on an unclusterable semicycle.

However, not all pairs of nodes on an unclusterable semicycle need to be ranked. Actually, there is just one pair of nodes on one type of unclusterable semicycle that must be ranked, as I will show now. Let k be the first node on the counterpositive path from i to j that is part of the unclusterable semicycle, and let l be the last node on the counterpositive path belonging to the same unclusterable semicycle. Because k is antecedent to l on the counterpositive path, it is easy to see that $(k \leq l)$. Since k and l are part of an unclusterable semicycle, there is at least one more semipath from k to l. This semipath conforms to one of the following descriptions.

1. It is not a counterpositive path. In consequence it contains at least one node m, such that neither $(k \leq m$ and $m \leq l)$ is true, nor $(l \leq m$ and $m \leq k)$ is true. Therefore, $(m \neq k)$ or $(m \neq l)$ may be true if $(k = l)$. So $(k \neq l)$ or $(m \neq k)$ or $(m \neq l)$ is true, and it is not possible to decide which pair of nodes must be ordered.

2. It is a counterpositive path from l to k. In this case, the unclusterable semicycle is a counterpositive cycle, therefore k and l are part of a hierarchical cluster and they belong to one rank (Definition 4c).

3. It is a counterpositive path from k to l, in which case for all nodes m on both counterpositive paths $(k \leq m$ and $m \leq l)$ is true. If $(k = l)$ is true, then $(k = m = l)$ is true for all nodes m on the counterpositive paths between k and l. However at least one pair of nodes must be ranked because of the unclusterable semicycle, therefore $(k = l)$ cannot be true. If $(k \neq l)$ and both counterpositive paths from k to l on the unclusterable semicycle imply that $(k \leq l)$, then $(k < l)$ is true. The ranking of m with respect to k and l remains unknown: if $(k \neq l)$ is true, then $(k \leq m$ and $m \leq l)$ is still true. Therefore, (k, l) is the only pair of nodes on the semicycle that must be ranked.

Since k and l are part of the counterpositive path from i to j, $(i \leq k)$ and $(l \leq j)$ are true. Combined with $(k < l)$ we can easily deduce $(i < l)$, $(k < j)$ and $(i < j)$. Of course, this proof assumes all ordered pairs on both paths to be elements of O. Finally, because there are two counterpositive paths on the unclusterable semicycle from k to l, and there

is at least one counterpositive path from i to k, as well as from l to j, there are at least two counterpositive paths from i to j.

Node j is ranked over node i iff there are two counterpositive paths from i to j, that, combined, contain at least one unclusterable semicycle, provided that all nodes on both paths are part of the same partial order. Let us call this kind of semicycle an RC cycle. Fig. 3 contains one RC cycle {$(B, A), (A, D), (D, B)$} ranking D over B, because it encompasses two counterpositive paths from B to D. Because of the positive arc from C to B, there are two counterpositive paths from C to D, too, so D is also ranked over C. However, C does not need to belong to the same rank as B; it may occupy a lower rank. Also, A's rank is unknown, since A may belong to B's rank, D's rank or a rank in between.

An RC cycle is a necessary condition for (regular) ranking. It identifies the model of RCs, as we will see in Section 3. If an RC cycle is part of longer counterpositive paths, all nodes on counterpositive paths up to and including the first node on the unclusterable semicycle must be ranked under the node where the unclusterable semicycle ends, and all nodes behind it on the counterpositive paths. Of course, this is true only if all nodes are subjected to the same partial order.

Theorem 7a and 7b. *A multiple graph {P, N} containing at least one pair of nodes (i, j), where (i = j) and (i < j) are true under the assumption that all ordered pairs are elements of no more than one partial order O:*

(Theorem 7a) contains at least two partial orders O_h that cannot be integrated into one partial order.

(Theorem 7b) contains at least one pair of nodes (k, l) such that:
  – k and l are ranked because they are part of an RC cycle, and
  – there is at least one counterpositive path from l to k, and
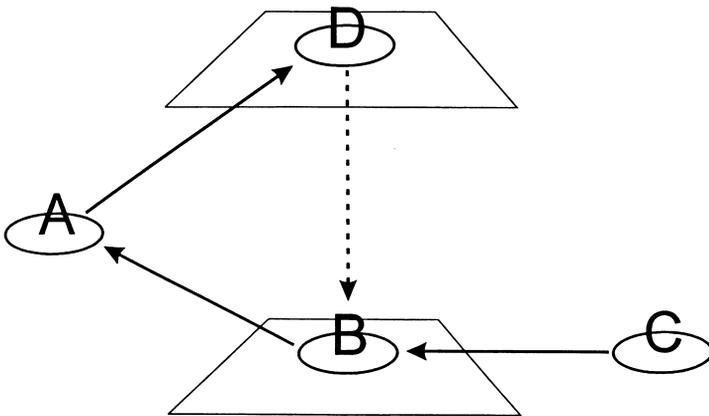 *– each counterpositive path from l to k contains at least two negative arcs (N'-labels).*

Fig. 3. Ranked clusters.

Proof: Within partial order O, it is logical impossible that two nodes are ranked $(i < j)$ and equals $(i = j)$ at the same time, since $(i < j)$ implies (not $i \geq j$), whereas $(i = j)$ implies $(i \leq j)$ and $(i \geq j)$, where $(i, j) \in O$. Therefore not all nodes on the counterpositive paths that cause ranking and equality, belong to the same partial order. Hence, the multiple graph contains at least two partial orders that are connected (because of the paths), but cannot be integrated. This proofs Theorem 7a.

Let us assume that the graph contains just one partial order. If $(i < j)$ is true, there are at least two counterpositive paths from i to j. The subgraph induced by these paths contains at least one RC cycle (Theorem 6). Let k be the node on both counterpositive paths that starts an RC cycle, and let l be the node that ends it. According to Theorem 6, k and l are ranked. Then, there are two counterpositive paths from k to l, viz. one positive path and one counterpositive path containing exactly one negative arc. Since $(i = j)$, there is at least one counterpositive path from j to i according to the proof of Theorem 3. Also, there are counterpositive paths from i to k, and from l to j, because k and l are part of the counterpositive paths from i to j. So, there is a counterpositive path from l to k. [2] This counterpositive path cannot contain positive arcs only, because a positive path would make up a hierarchical cluster cycle in combination with the counterpositive path from k to l containing one negative arc. In this case, k and l would belong to one rank according to Definition 4c. For the same reason, the counterpositive path from l to k cannot contain one negative arc, since it would form a hierarchical cluster cycle combined with the positive path from k to l. Therefore, any counterpositive path from l to k must include two or more negative arcs. This completes the proof of Theorem 7b.

Fig. 4 shows an example of two partial orders, separated by a cleavage: $\{A, B, D\}$ and $\{C, E\}$. Let us call a counterpositive path (with two or more negative arcs) starting at a node that outranks the node where it ends, a cleavage path, e.g., $\{(C, D), (D, E)\}$. The cleavage path, as well as a pair of nodes that are ranked $(i < j)$ and equal $(i = j)$ at the same time, identifies a model that resembles the transitivity model, as I will show in Section 3. Since negative arcs on a cleavage path point from a lower rank towards a higher rank, they clearly point in the wrong direction. Either the ranking is incorrect, or the interpretation that all nodes on the counterpositive path belong to the same partial order. I propose to choose for the latter option in Definition 8.

Definition 8. At least two negative arcs on a cleavage path connect nodes from different partial orders.

I choose the negative arcs to connect nodes in different partial orders, because this fits the interpretation of a social cleavage in a group, e.g., a sex-cleavage: boys disliking girls regardless of their ranking, and vice versa. Let us call them *cleavage arcs*. Also, negative arcs are the most likely candidates to cross a cleavage, because they are always present on a cleavage path, whereas positive arcs may be absent (cf. Fig. 4). Since a

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[2] There is a counterpositive walk from l to k that includes a counterpositive path, which may or may not contain i and j.
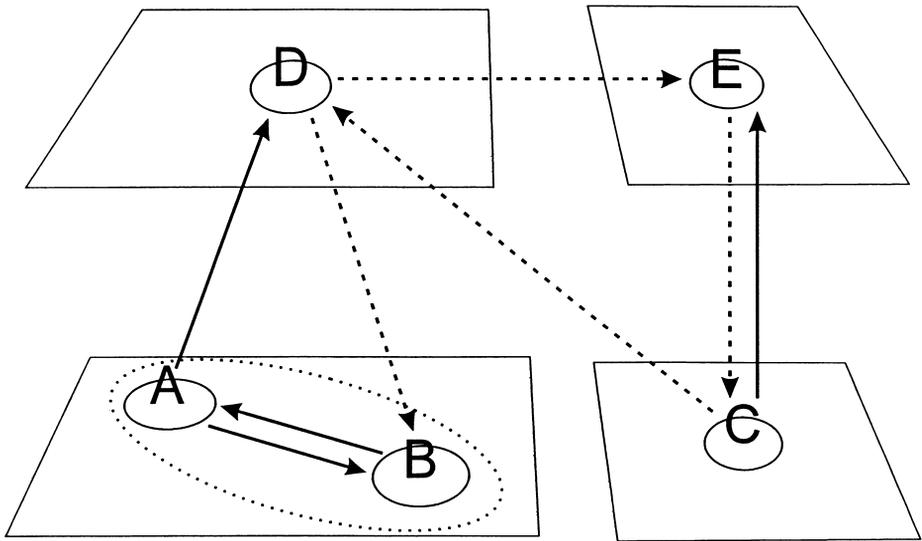
Fig. 4. A cleavage between two partial orders.

cleavage path always returns to the partial order where it starts, at least two negative arcs cross a cleavage.

If the nodes on a cleavage path do not belong to the same partial order, a condition of Definition 1 is not satisfied. Therefore, a cleavage path from i to j, which is a counterpositive path from j to i by definition, combined with a counterpositive path from i to j does not imply that its nodes belong to the same rank (Theorem 3). This resolves the contradiction implied by $(i < j)$ and $(i = j)$, since $(i = j)$ is no longer true. For the same reason, Theorem 6 cannot apply to a cycle containing one or more cleavage arcs. Ranking deduced from RCs cycles containing cleavage arcs must be cancelled.

3. Application to triads

As has been shown before (Holland and Leinhardt, 1971; Davis and Leinhardt, 1972; Johnsen, 1985) in a complete signed digraph, each balance-theoretic model can be defined as a set of permitted triads. In this section, I will show that semicycles and paths within triads also offer the information needed to select the right model. As a result, all balance-theoretic models in complete digraphs can be identified by means of semicycles and paths.

3.1. Triads 300, 102, and 003: balance and clusterability

The initial models of structural balance and clusterability only deal with clustering. Positive arcs occur within clusters; negative arcs occur between clusters. Both models
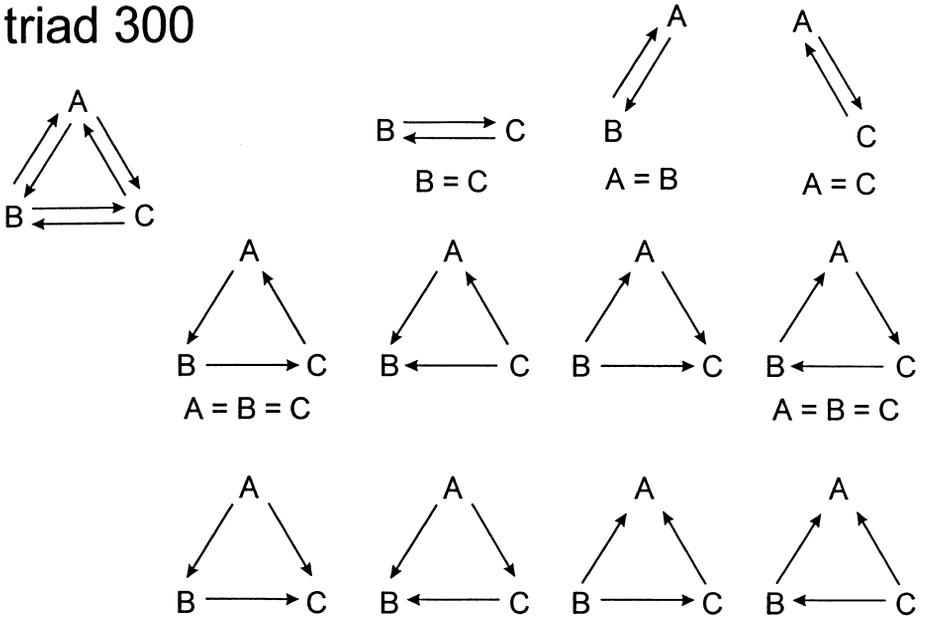
Fig. 5. Semicycles within triad 300.

exclude ranking of nodes. A graph conforming to these models does not contain semicycles with exactly one negative arc (Cartwright and Harary, 1956; Davis, 1967). For example, triad 300 contains balanced counterpositive cycles (Fig. 5). According to Theorem 3, the nodes must belong to one rank. This triad contains no ordered pairs, which is true for triads 102 and 003 as well.

Note, however, that not all balanced and clusterable semicycles identify regular clustering if ranked models are available too. Balanced and clusterable semicycles that are not counterpositive cycles, contain nodes that may be ranked. On its own, semicycle $\{(B, C), (D, C), (B, D)\}$ in Fig. 1, which is not a counterpositive cycle, may mean that B is ranked over C. It is the counterpositive cycle $\{(B, D), (D, C), (C, A), (B, A)\}$ that assigns the nodes to one rank. Balanced and clusterable semicycles that are not counterpositive cycles, do not favour the models of balance or clusterability over a ranked model. Therefore, the choice of these models should depend only on counterpositive cycles. This reduces the number of semicycles to be analysed within a signed digraph significantly.

3.2. Triads 120D, 120U, 021D, 021U, 030T: ranked clusters

Davis and Leinhardt (1972) introduced the first model incorporating ranking, viz., RCs. In this model, pairs of nodes either belong to clusters at the same rank or they belong to different ranks. Since ranking is allowed, RC cycles are permitted, but ranked pairs of nodes $(i < j)$ may not belong to the same rank $(i = j)$ simultaneously, nor are

they allowed to be ranked in reverse order ($j < i$) at the same time. As we will see, triads 120D, 120U, 021D, 021U, and 030T satisfy these criteria.

Triad 120D (Fig. 6) contains one counterpositive semicycle joining nodes B and C in a cluster at the same rank. Also, there are six semicycles with exactly one negative arc (RC cycles) which repeatedly show that node A must be on a lower rank than nodes B and C, because they can be split into two counterpositive paths starting at A and ending in B or C. Finally, four semicycles that yield no information on the ranking of the nodes. This triad contains an antisymmetric order, and no ranked pair is assigned to clusters at one rank at the same time.

In triad 120U, the rank order is reversed: node A now belongs to a higher rank than nodes B and C. With triads 021D and 021U, we obtain the same results, except that nodes B and C belong to different clusters at the same rank. The semicycles within triad 030T yield a hierarchy in which node C is at a higher rank than node A, which is at a higher rank than node B. No nodes have to belong to the same rank and the rank order is antisymmetric. As we will see, all triads not dealt with until now either contain symmetric ranking or ranked pairs that belong to the same rank as well.

Triads 120D, 120U, 021D, and 021U contain balanced counterpositive cycles. Of course, clustering is allowed in a model that ranks clusters. The model of RCs is less restrictive than the models of balance and clusterability. The restrictions can be identified by means of triads as well as types of semicycles that are allowed to occur.

Johnsen distinguishes between two models: ranked clusters (RC model) and ranked 2-clusters of M-cliques (R2C model) (Johnsen, 1985). The RC model applies to all
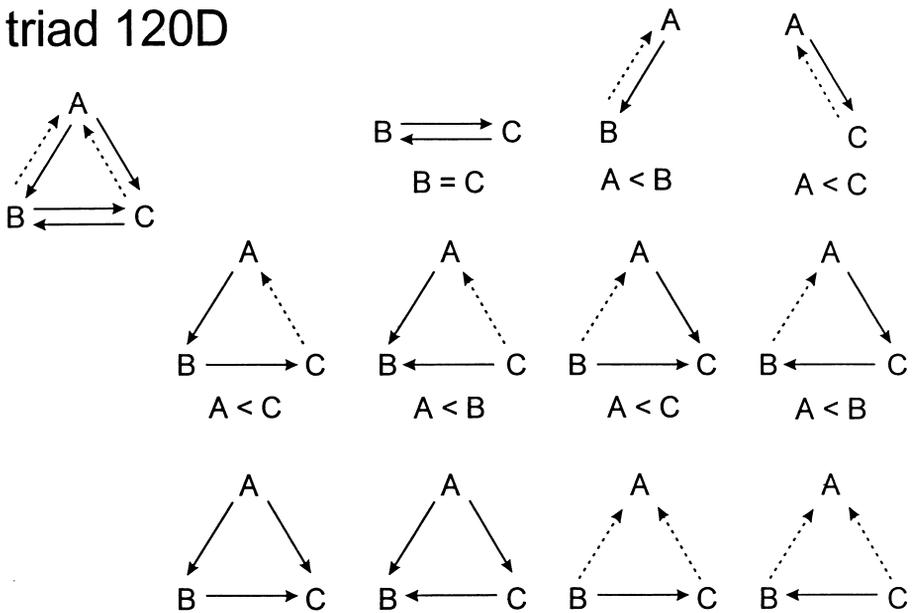

Fig. 6. Semicycles within triad 120D.

hierarchies with at least one rank that contains more than two clusters. The R2C model is reserved for a hierarchy of ranks with no more than two clusters (polarisation) at each rank. If a graph with RCs contains triad 003, the RC model applies. If not, the R2C model applies. The method I propose distinguishes between these two models in as simple a manner. In a graph containing RCs, the absence of clusterable semicycles, i.e., the semicycle characteristic of clusterability containing an uneven number of negative arcs (but more than one), is sufficient to choose for ranked 2-clusters.

3.3. Triad 012: transitivity

The transitivity model was proposed as a generalisation of the RC model (Holland and Leinhardt, 1971). It includes triad 012, which is forbidden under the RC model. I will now show that this model is characterised by a cleavage as defined in Theorem 7.

The triad identifying transitivity (triad 012) contains a pair that at first glance has to belong to the same rank and to different ranks simultaneously: pair (B, C) (see Fig. 7). In triad 012, node A is separated from nodes B and C by a cleavage. Especially the negative arcs from node B to node A and from node A to node C contradict the rank order of nodes B and C. At least two negative arcs on a cleavage path, pointing in the wrong direction, have been called cleavage arcs; they connect units from different partial orders (see Definition 8). Therefore, Theorem 3 concerning nodes that must belong to the same rank does not apply to (interlocking) counterpositive cycles containing these
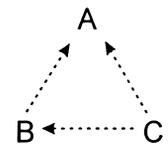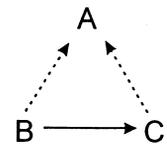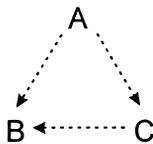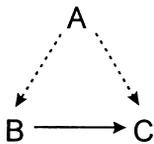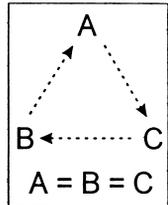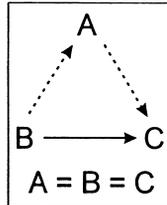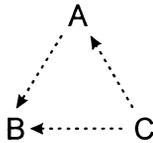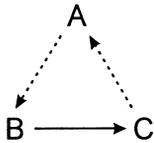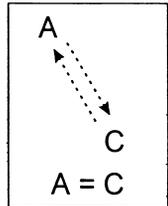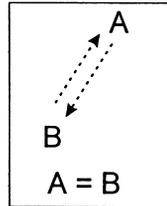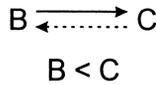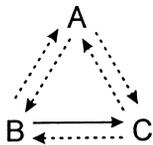

Fig. 7. Semicycles within triad 012.

arcs. Since all four counterpositive cycles of triad 012 (in Fig. 7, they are displayed within a box) contain at least one such arc, under the transitivity model, no nodes of this triad have to belong to the same rank. This, in fact, resolves the problem that nodes are ranked and equal in rank at the same time.

The transitivity model is less restrictive than the model of RCs. It allows for balanced and clusterable semicycles, as well as RC cycles. In addition, cleavage paths are allowed under the transitivity model. The model of RCs forbids them.

Unlike complete digraphs, incomplete digraphs may also contain independent hierarchies: several unconnected systems of ranks. The systems of ranks are unconnected because there is insufficient information on the affect relations between all units, or the affect relations concerned are neutral. It is important to note that independent hierarchies differ from split hierarchies in the transitivity model. Independent hierarchies can be linked into a single order, whereas a cleavage separates ranks that cannot be integrated into a single order. Transitivity points to incompatible systems of ranks, not to unconnected ranks. This distinction is irrelevant if the model is applied to dichotomous data, e.g., sociometric data, that are complete by definition.

The argumentation suggests a small adjustment to the balance-theoretic concept of transitivity with respect to affect relations. In an incomplete signed digraph, positive affect ties within a triple do not need to be transitive in the sense that each positive path of length two in a triple is closed by a positive arc from the starting node to the end node of the path. Transitivity only demands that it *can* be closed by such an arc. It does
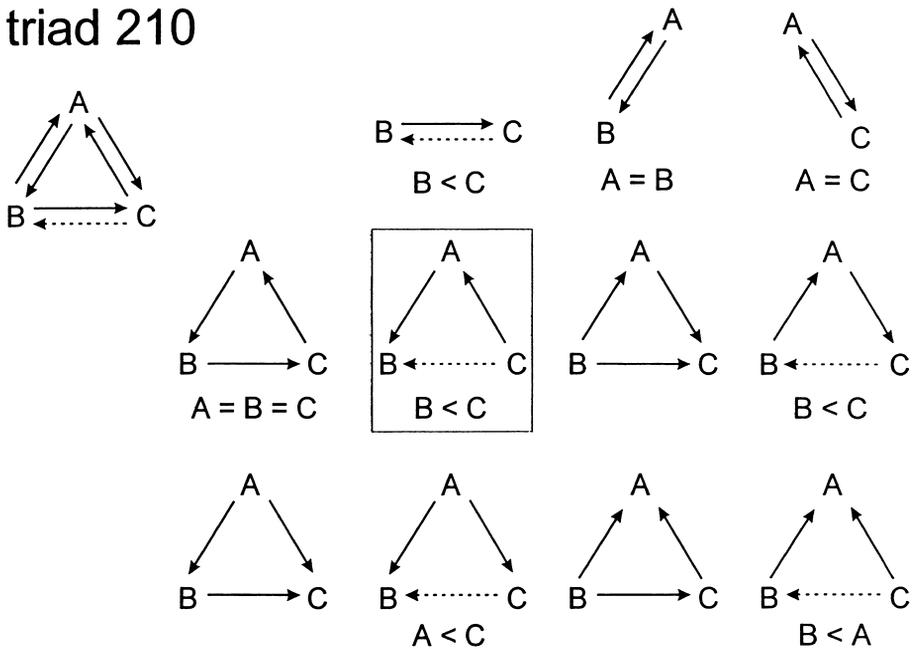

Fig. 8. Semicycles within triad 210.

not allow closure by a negative arc in this direction as that would yield the 'forbidden' counterpositive cycle with exactly one negative arc. This applies not only to triples, but to semicycles of all lengths. In short, I propose to define a structure of affective relations as transitive if it has, or can have a transitive closure. In essence, this means that people tend to show positive or neutral affect for their friends' friends (etc.). They avoid showing negative affect.

3.4. Triads 210 and 120C: hierarchical clusters

In 1985, Johnsen defined the latest balance-theoretic model, which he called hierarchical $\overline{M}$-cliques. This model does not allow for symmetric rank ordering, but it accepts the semicycle forbidden under transitivity: each triad peculiar to the model of hierarchical $\overline{M}$-cliques (triads 210 and 120C) contains one or more counterpositive cycles with exactly one negative arc (see the boxes in Figs. 8 and 9).

Since *A*, *B*, and *C* constitute a hierarchical cluster, the only ranking to be considered in triad 210 is connected to the negative arc from node *C* to node *B*, indicating that node *C* outranks node *B*. Triad 120C, the other triad that contains hierarchical cluster cycles (see boxes in Fig. 9), has two negative arcs implying that node *C* outranks node *A*, which in turn outranks node *B*. Triad 120C conforms to the criterion that the hierarchy within the graph is antisymmetric. I propose to include this triad in the model. Johnsen, however, only allows this triad in the 39 + model for larger groups. He rejects
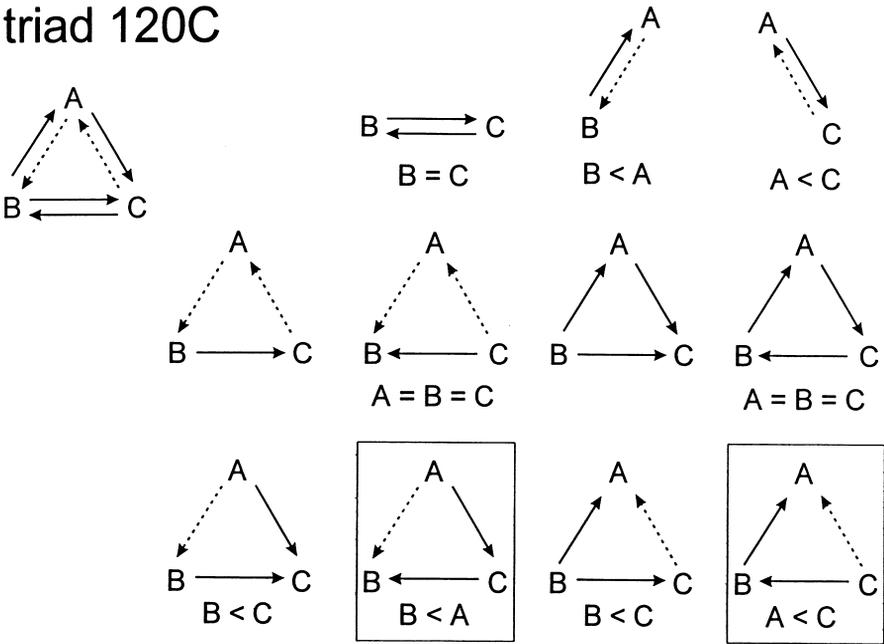

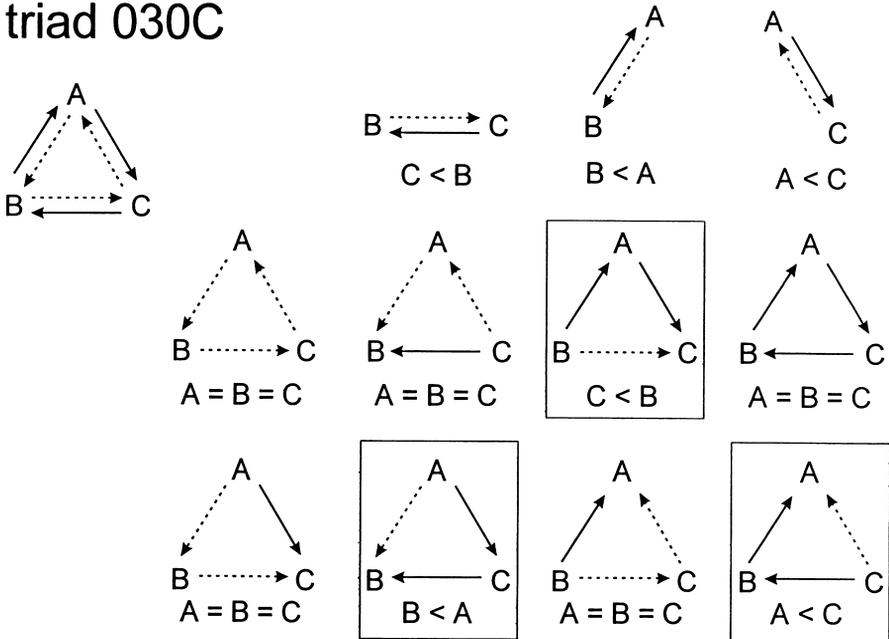Fig. 9. Semicycles within triad 120C.

triad 030C

Fig. 10. Semicycles within triad 030C.

the triad because the antisymmetric dyads are not transitive and therefore the hierarchy is not transitive. The cycle approach shows that this triad does contain a transitive hierarchy. Hierarchy is not connected to antisymmetric dyads only. Therefore the model of hierarchical clusters should also allow for this triad. The micro-model proposed here turns out to be more parsimonious: there is no need to distinguish between small and large groups.

3.5. Triads 021C, 111D, 111U, 201, 030C: symmetric ranking

The remaining five triads 021C, 111D, 111U, 201, 030C contain hierarchical clusters with symmetric ranking. No balance-theoretic model allows this, since symmetric ranking contradicts the concept of a partial order. For example, in all five triads node *A* is assigned to a higher rank than node *C* as well as to a lower rank. In the two cyclic triads (021C and 030C), this inconsistency results from the transitivity of a hierarchical relation: since node *A* outranks node *B*, which in turn outranks node *C*, node *A* outranks node *C* in triad 030C (cf. Fig. 10).

4. Dropping the sign: changing the structure

Section 3 showed that the micro-model proposed in this article, identifies the same balance-theoretic models in complete signed digraphs as triadic analysis. However, the

micro-model is not meant to be applied to complete signed digraphs. The semicycle approach is conceived to analyse data that triadic analysis cannot handle, viz. incomplete signed digraphs.

Until now, researchers have turned signed digraphs into unsigned digraphs (zero-one matrices) in order to subject them to triadic analysis. Either negative arcs and absent relations have been lumped into a single category of zero-relations, or negative arcs have been treated as zero-relations and positive as well as absent relations have been treated as one-relations. However, both operations change the structure of a digraph profoundly as well as the balance-theoretic models that fit it, as I will illustrate now.

Let us have another look at Fig. 3, representing a digraph that perfectly matches the model of RCs according to the micro-model proposed here. If we change all negative arcs into absent relations, we obtain the unsigned digraph and triads depicted in Fig. 11. The 'forbidden' triads 021C indicate that no balance-theoretic model applies. If we ignore them, we opt for the model of transitivity, because there are two 012 triads. Since incomplete digraphs use to have a lot of absent relations, this type of dichotomization is likely to produce a lot of 012 triads. A high degree of transitivity judged by the number of 012 triads may well be an artefact that is produced when a low density signed digraph is dichotomised.

The other dichotomization, considering the negative arcs to be the only zero-relations in the digraph (Fig. 12), yields another result. Two triads (210) characterise the model of hierarchical $\overline{M}$-cliques and another two are balanced (300). Both dichotomizations display other balance-theoretic models than the model suggested by the semicycle approach, which is RC.

The example shows that a digraph's structure may change radically when an incomplete signed digraph is turned into a complete unsigned digraph. This is due to the
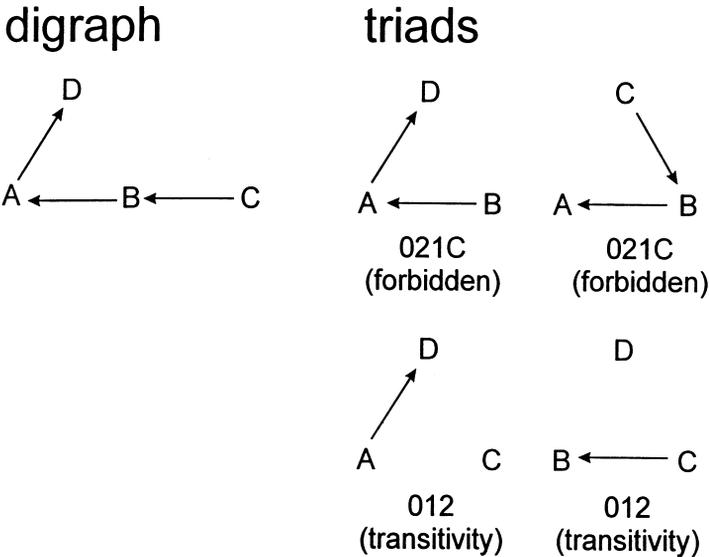

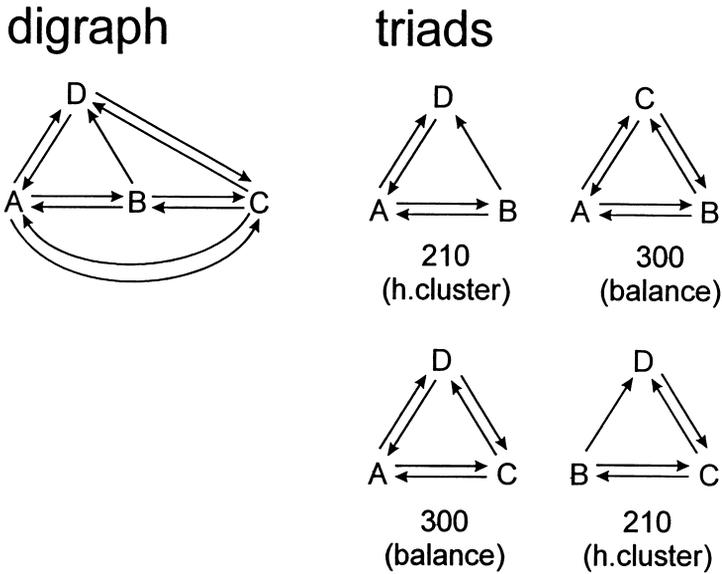Fig. 11. Ranked clusters digraph: negative arcs dropped.

Fig. 12. Ranked clusters digraph: negative arcs as the only absent relations.

fact that dichotomization assumes all absent arcs in an incomplete signed digraph to be the same: either absent (negative), or present (positive). The semicycle approach allows absent relations to be negative or positive, e.g., (D, A) in Fig. 3. If D displays negative affect towards A, D is ranked over A. A positive affective relation would group A and D in a cluster. Finally, a neutral or unknown affection produces the ambiguous situation of A depicted in Fig. 3.

The semicycle approach yields other results than triadic analysis. The former method is to be preferred when a digraph contains relations that have three meaningful values (positive, negative and neutral or absent), because it does not reduce one value to another arbitrarily.

5. Conclusion

The micro-model proposed here allows balance-theoretic models to be applied to incomplete signed digraphs. Since complete signed digraphs and unsigned digraphs may be treated as special cases, the micro-model is more general than models based on dyads and triads. It is not suggested to abandon triadic analysis in favour of the semicycle approach. With respect to complete signed digraphs and unsigned digraphs, triadic analysis is more efficient. However, the semicycle approach is the preferred method to analyse relations that may be positive, negative, and neutral or absent, because the rather arbitrary dichotomization of the data required by triadic analysis changes the graph's structure.

Affections, sentiments, as well as attitudes towards norms and values, are important classes of social relations that are either positive, negative, or neutral. They represent the cultural dimension of social structure. Balance-theoretic models may well serve to trace cultural processes contributing to the formation of social groups and hierarchies. Artistic hierarchies and factions, e.g., artistic prestige and classifications, are a case in point (cf. De Nooy, 1999). The method proposed in this article is able to deal with incomplete signed digraphs, which are indispensable for modelling networks of affective relations. Balance-theoretic models may once more take into consideration the sign of affection.

References

Boorman, S.A., White, H.C., 1976. Social structures from multiple networks: II. Role structures. Am. J. Sociol. 81, 1384–1446.

Cartwright, D., Harary, F., 1956. Structural balance: a generalisation of Heider's theory. Psychol. Rev. 63, 277–293.

Davis, J.A., 1967. Clustering and structural balance in graphs. Hum. Relations 20, 181–187.

Davis, J.A., Leinhardt, S., 1972. The structure of positive interpersonal relations in small groups. In: Berger, J. (Ed.), Sociological Theories in Progress. Houghton Mifflin, Boston, pp. 218–251.

De Nooy, W., 1999. A literary playground. Literary criticism and balance theory. Poetics 26, 385–404.

Heider, F., 1946. Attitudes and cognitive organization. J. Psychol. 21, 107–112.

Heider, F., 1958. The Psychology of Interpersonal Relations. Wiley, New York.

Holland, P.W., Leinhardt, S., 1971. Transitivity in structural models of small groups. Comp. Group Stud. 2, 107–124.

Johnsen, E.C., 1985. Network macrostructure models for the Davis–Leinhardt set of empirical sociomatrices. Social Networks 7, 203–224.

Manhart, K., 1995. Forschungsprogramm und Evolution der Balancetheorien. Eine diachrone Theorienanalyse auf Basis der strukturalistischen Wissenschaftstheorie (Research-program and evolution of balance theories. A diachronic theory analysis based on the structuralist view of theories). Zeitschrift für Sozialpsychologie 26, 194–220.

Morrissette, J.O., Jahnke, J.C., 1967. No relations and relations of strength zero in the theory of structural balance. Hum. Relations 20, 189–195.

Pattison, P. 1993. Algebraic Models for Social Networks. Cambridge Univ. Press, Cambridge.

Wasserman, S., Faust, K., 1994. Social Network Analysis: Methods and Applications. Cambridge Univ. Press, Cambridge.