Chapter 6
Appendix: Proofs

Proof of Proposition 4.2.

The proof proceeds by induction with respect to the number of periods N.

For $N = 1$ the cost of a strategy is given by $F(S_1) - \gamma_0(S_1 - S_0)$ for some real number $\gamma_0 = g_0(S_0)$, so it depends continuously on S_1. Since S_1 is restricted to an interval and since continuous functions map intervals to intervals, I^g must be an interval.

Next, assume that the proposition is true for models with fewer than N steps, and consider the total cost range I^g in an N-step model for some fixed strategy g. First consider the costs of price paths $\{S_0, \ldots, S_N\}$ with $S_N = S_{N-1}$. It follows from the induction hypothesis that the cost range of strategy g over these paths forms an interval, say, I'. Take $p \in I^g$ and let $\{S_0, \ldots, S_{N-1}, S_N\}$ be the corresponding path. Consider the paths $\{S_0, \ldots, S_{N-1}, \alpha S_N + (1-\alpha)S_{N-1}\}$ for $0 \le \alpha \le 1$. Since the corresponding costs depend continuously on α, they form an interval that contains p and that also contains at least one point of I'. Therefore, the set I^g may be written as a union of intervals that all have at least one point in common with the interval I', and so I^g is itself an interval.

If a strategy is continuous, then the cost function associated to it is continuous in the price paths. Because the set $\mathbb{I}^{u,d} \subset \mathbb{R}^{N+1}$ is compact, the cost function then achieves both its maximum and its minimum value on $\mathbb{I}^{u,d}$. □

The proof of Theorem 4.5 requires the following two technical lemmas.

Lemma 6.1. *Let u and d be such that $d < 1 < u$. If $h : (0,\infty) \mapsto \mathbb{R}$ is convex, then the function $\tilde{h}(x)$ defined for $x > 0$ by*

$$\tilde{h}(x) = \min_{\gamma \in \mathbb{R}} \max_{dx \le y \le ux} [h(y) - \gamma(y - x)] \tag{6.1}$$

is convex as well.

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Proof. Since $h(y) - \gamma(y-x)$ is convex as a function of y, the maximum in (6.1) must be taken at the boundary of the interval $[dx, ux]$, so

$$\tilde{h}(x) = \min_{\gamma} \max[h(dx) + \gamma(1-d)x, h(ux) - \gamma(u-1)x].$$

Since the first argument in the "max" operator is increasing in γ and the second is decreasing, the minimum is achieved when both are equal, that is to say, when γ is given by

$$\gamma = \frac{h(ux) - h(dx)}{(u-d)x}.$$

Therefore, we have the following explicit expression for $\tilde{h}$ in terms of h:

$$\tilde{h}(x) = \frac{1-d}{u-d} h(ux) + \frac{u-1}{u-d} h(dx).$$

Since the property of convexity is preserved under scaling and under positive linear combinations, it is seen from the preceding expression that the function $\tilde{h}$ is convex. □

Lemma 6.2. *Let $h(\cdot)$ be a convex function, and let u and d be such that $d < 1 < u$. Then we have*

$$\max_{\gamma \in \mathbb{R}} \min_{dx \le y \le ux} [h(y) - \gamma(y-x)] = h(x).$$

Proof. We obviously have

$$\min_{dx \le y \le ux} [h(y) - \gamma(y-x)] \le h(x)$$

for all γ since the value on the right-hand side is achieved on the left-hand side for $y = x$. So to complete the proof it suffices to show that there exists γ such that

$$h(y) \ge h(x) + \gamma(y-x)$$

for all y. Clearly, any subgradient of h at x has this property. □

Proof of Theorem 4.5.

1. The value function for the problem of minimizing worst-case costs is given by

$$V(S,j) = \min \max_{S_j = S} \left[F(S_N) - \sum_{k=j}^{N-1} \gamma_k (S_{k+1} - S_k) \right],$$

where the minimum is taken over all strategies and the maximum is taken over all paths in the given interval model that satisfy $S_j = S$. The value function satisfies the recursion

$$V(S, j-1) = \min_{\gamma} \max_{dS \le S' \le uS} [V(S', j) - \gamma(S' - S)],$$

and of course we have

$$V(S, N) = F(S).$$

It follows from Lemma 6.1 that the functions $V(\cdot, j)$ are convex for all j. Therefore, the strategy that minimizes the maximum costs is the same as the minmax strategy for the binomial tree model with parameters u and d, and the corresponding worst-case paths are the paths of this tree model.

2. The proof is *mutatis mutandis* the same as above; use Lemma 6.2 rather than Lemma 6.1.
3. This is by definition a consequence of items 1 and 2. □

Proof of Theorem 4.7.

Items 2 and 3 are clear from Theorem 4.5. One part of item 1 follows easily from the characterization of the consistent price interval as the intersection of all cost intervals. Indeed, if $\mathbf{Q}$ is a martingale measure, then $E_{\mathbf{Q}} F(S_N)$ is in the cost interval I^g for any strategy g since the expected result from any trading strategy under the martingale measure is zero. Thus $E_{\mathbf{Q}} F(S_N)$ is in the intersection of all cost intervals. To show that every such premium can be obtained as an expected value under some martingale measure, let $\mathbf{Q}^\alpha$ denote the martingale measure associated to the binomial tree $\mathbb{B}^{u,d}$ with parameters $u_\alpha := 1 + \alpha(u-1)$ and $d_\alpha := 1/u_\alpha$. For $0 \le \alpha \le 1$ the measure $\mathbf{Q}^\alpha$ is also a martingale measure on $\mathbb{I}^{u,d}$. The expected option value $f_\alpha := E_{\mathbf{Q}^\alpha} F(S_N)$ is continuous in α; moreover, $f_\alpha = f_{\min}$ for $\alpha = 0$ and $f_\alpha = f_{\max}$ for $\alpha = 1$. Hence every price $f \in [f_{\min}, f_{\max}]$ occurs as an expected option value under some martingale measure. □

Proof of Proposition 4.10.

For $N = 1$ it is obvious that worst cases are at the boundary of $S_1 = [dS_0, uS_0]$ with $u = u_\tau$ and $d = \frac{1}{u}$ and that these worst-case costs are convex in the initial price.

Similar to the proof of Theorem 4.5, it can be proved by induction that worst cases have extreme jumps and remain convex in the initial price for any number of time steps. □

Proof of Results Algorithm 4.9.

We prove that $H^{\min}(j,S_j)$ and $H^{\max}(j,S_j)$ are respectively the minimal and maximal current hedge costs $\Sigma_{k=0}^{j-1} - g_k(S_k)(S_{k+1}-S_k)$ over all paths in $\mathbb{I}^{u,d}$ that start in S_0 and end in S_j. This is obvious for $j=0$ and $j=1$. For $j>1$, observe that I_j denotes the range of all prices at S_j that are compatible with given end points S_0 and S_{j+1} in $\mathbb{I}^{u,d}$. From the continuity of the strategy it follows that the minimum (4.16) and the maximum in the definition of $H^{\max}$ are well defined and indeed denote respectively the minimal and maximal current hedge costs. In particular, $H^{\min}(N,S_N)$ and $H^{\max}(N,S_N)$ denote respectively the minimal and maximal realized hedge costs over all paths in $\mathbb{I}^{u,d}$ that end in S_N. Once $S_N^{\min}$ and $S_N^{\max}$ are determined in (4.17) and its maximum analog, the correctness of (4.19) is obvious. The backward recursions for $S^{\min}$ and $S^{\max}$ simply reconstruct the paths corresponding to the interval bounds. □
To prove Theorem 5.2, we need the next two lemmas.

Lemma 6.3. *A strategy is compatible with the restriction on worst-case costs V if and only if along all paths the current latitude is always nonnegative, i.e.,*

$$g \in \mathbb{G}^V \text{ iff } \bar{V}_j(S_j,H_j) \geq 0 \,\forall\, S \in \mathbb{I}^{u,d},\ j=0,\ldots,N-1.$$

Proof. As soon as $\bar{V}_j$ drops below zero for some j and some price path, there is a worst-case path in the tree $\mathbb{B}^{u,d}$ that brings total costs above level V. Conversely, the condition is sufficient for g to be in $\mathbb{G}^V$, as then for all price paths $S \in \mathbb{I}^{u,d}$, $\bar{V}_N = V - H_N - \bar{V}_N(S_N,H_N) = V - Q^g(S) \geq 0$. □

Lemma 6.4. *For all $S_j \in \mathbb{R}^+$, for all $0 \leq j \leq N$,*

1. *$f_j(S_j)$ is convex in S_j;*
2. *On $S_{j+1} \in [dS_j,uS_j]$, $f_{j+1}(S_{j+1}) - \Delta_j(S_{j+1}-S_j)$ has equal boundary maxima $f_j(S_j)$.* □

Proof of Theorem 5.2.

First notice that, as $V \geq f_0(S_0)$, $\mathbb{G}^V$ is nonempty and contains delta hedging.

Now suppose we apply a strategy $g_j(S_j,H_j)$ at t_j, i.e., we choose the portfolio $C - \gamma_j S$ at t_j, with γ_j the outcome of g_j, given the past price path (which determines, in turn, the realized hedge costs H_j). Then at t_{j+1}, $H_{j+1} = H_j - \gamma_j(S_{j+1}-S_j)$, with $S_{j+1} \in [dS_j,uS_j]$, and $\bar{V}_{j+1} = V - f_{j+1}(S_{j+1}) - H_{j+1}$. From Lemma 6.3 it follows that g_j is admissible if and only if $\bar{V}_{j+1} > 0$ for all $S_{j+1} \in [dS_j,uS_j]$. So the strategy position γ_j is admissible if and only if for all $S_{j+1} \in [dS_j,uS_j]$, $V - H_j - f_{j+1}(S_{j+1}) + \gamma_j(S_{j+1}-S_j) \geq 0$. Substituting $\gamma_j =: \Delta_j + \bar{\gamma}_j$ this gives $V - H_j - f_{j+1}(S_{j+1}) + (S_{j+1}-S_j)\Delta_j - (S_{j+1}-S_j)\bar{\gamma} \geq 0$. With Lemma 6.4 we obtain that the left-hand side of this inequality is a concave function in S_{j+1}, with boundary values $V - f_j(S_j) - H_j - \bar{\gamma}(d-1)S_j$ on the left (for $S_{j+1} = dS_j$) and $V - f_j(S_j) - H_j - \bar{\gamma}(u-1)S_j$ on the right (for $S_{j+1} = uS_j$). As $d-1 < 0$ and $u-1 > 0$, this induces an upper and lower bound for $\bar{\gamma}$, from which the strategy bounds follow. □

Proof of Proposition 5.3.

The existence of g^* and S^* is equivalent to the existence of subsequent solutions of the minimizations in the definition of BC_j. First we write out BC_{N-1}:

$$\mathrm{BC}_{N-1}(S_{N-1}, H_{N-1})$$

$$:= \min_{\{\gamma_{N-1}\in\Gamma_{N-1}, S_N\in[dS_{N-1}, uS_{N-1}]\}} [S_N - X]^+ - H_{N-1} - \gamma_{N-1}(S_N - S_{N-1}),$$

with $\Gamma_{N-1} = [\gamma_{N-1}^{\min}, \gamma_{N-1}^{\max}]$

$$= \left[\Delta_{N-1}(S_{N-1}) - \frac{V - f_{N-1}(S_{N-1}) - H_{N-1}}{(u-1)S_{N-1}}, \Delta_{N-1}(S_{N-1}) + \frac{V - f_{N-1}(S_{N-1}) - H_{N-1}}{(1-d)S_{N-1}}\right].$$

For each S_{N-1}, H_{N-1} this involves minimization of a continuous function over a compact domain, so BC_{N-1} is well defined. Further, this domain of optimization is itself a continuous function of S_j and H_j, so BC_{N-1} is continuous itself on the entire domain given in (5.7).

The existence of solutions for $j = N-2, \ldots, 0$ now follows from an obvious inductive argument. □

To prove the correctness of the statements made in the results of Algorithm 5.4 we need the next lemma.

Lemma 6.5.
For each price path $S \in \mathbb{I}^{u,d}$,

$$\min_{g\in\mathbb{G}^V} Q^g(S) = Q^{g^\sharp}(S) \tag{6.2}$$

with the (noncausal "strategy") $g^\sharp$ defined by $\gamma^\sharp$,

$$g_j^\sharp(S_0, \ldots, S_j, S_{j+1}) := \gamma_j^\sharp.$$

Proof. We have

$$\mathrm{BC}_j(S_j, H_j) = \min_{S_{j+1}\in[dS_j, uS_j]} \left(\min_{\gamma_j\in\Gamma_j} \mathrm{BC}_{j+1}(S_{j+1}, H_j - \gamma_j(S_{j+1} - S_j)) \right).$$

The inner minimization has solutions at the boundary of Γ_j [see (6.4) below for a motivation]. More precisely, the minimum is at the left bound if $S_{j+1} < S_j$ and at the right bound if $S_{j+1} > S_j$. Finally, notice that if $S_{j+1} = S_j$, the value of γ_j affects neither the hedge costs over the jth time step nor future cost constraint implications, so its value may be chosen arbitrarily in Γ_j. In particular, then, $\gamma^\sharp$ can be taken, arbitrarily, at the left boundary of Γ_j, without affecting optimality.

This implies that the right-hand side in (6.2) is a lower bound for the left-hand side: price jumps cannot be amplified by larger factors with the right sign under the RCC restriction V.

Equality does not follow immediately, as $g^\sharp$ is *not* a strategy, due to the fact that it anticipates whether an increase or decrease will follow the current asset price. To derive equality, observe that for each fixed price path S' there is a (by definition *causal*) strategy $g \in \mathbb{G}^V$ that coincides with $g^\sharp$ for that particular price path, namely,

$$g_j(S_0,\ldots,S_{j+1}) = \gamma_j = \begin{cases} \gamma_j^{\max} & \text{if } S'_{j+1} > S'_j \\ \gamma_j^{\min} & \text{if } S'_{j+1} \le S'_j \end{cases}. \tag{6.3}$$

Then $g(S') - g^\sharp(S')$, and hence the outcome of costs are the same: $Q^g(S') = Q^{g^\sharp}(S')$. By taking S' a best-case price path (which exists according to Proposition 5.3), a causal strategy is obtained with the same best-case costs as $g^\sharp$. □

Proof of Correctness of Algorithm 5.4.

First we note a specific feature of the dynamic programming problem that underlies the first step:

$$\mathrm{BC}_j(S_j,H_j) \le \mathrm{BC}_j(S_j,H_j+h) \;\; \forall\, h \ge 0.$$

In fact, it even holds that the difference in realized hedge costs h can be maintained until the final time because any strategy that is admissible under the RCC with initial state $(S_j, H_j + h)$ at t_j is also admissible from a state with lower accrued hedge costs, so

$$\mathrm{BC}_j(S_j,H_j+h) \ge \mathrm{BC}_j(S_j,H_j) + h \;\; \forall\, h \ge 0. \tag{6.4}$$

From Lemma 6.5 we have now that due to this monotonicity in hedge costs we can reduce the double optimization over paths and strategies to a single one over price paths. This eliminates optimization over strategies in the best-case criterion (5.5).

A further reduction in computational complexity is achieved by selecting optimal paths among all those that recombine in the same price. In view of the previous results, this is simply a matter of comparing the realized "hedge" costs under $g^\sharp$ in each step.

Let $\mathbb{I}^{u,d}(j,s)$ denote the price paths in $\mathbb{I}^{u,d}$ with price s at time t_j, and let $H^*(j,s)$ denote the minimally achievable realized hedge costs for those paths under limit V

on worst-case costs. Then the optimal realized hedge costs at t_j for given asset price S_j are given by

$$\mathbb{I}^{u,d}(j,s) := \{S \in \mathbb{I}^{u,d} | S_j = s\}, \tag{6.5}$$

$$H^*(j,s) := \min_{\{g \in \mathbb{G}^V, S \in \mathbb{I}^{u,d}(j,s)\}} -\Sigma_{k=0}^{j-1} \gamma_k (S_{k+1} - S_k), \tag{6.6}$$

with $\gamma_k = g_k(S_0, \ldots, S_k)$ the outcome of the strategy for a given price path S.

Now (5.8) is trivial because hedge costs are zero before hedging starts. The formula for $H^*(1, S_1)$ follows from

$$H^*(1,S_1) = \min_{\gamma^{\min} \le \gamma_0 \le \gamma^{\max}} -\gamma_0(S_1 - S_0)$$

because $\mathbb{I}^{u,d}(1,s)$ consists of at most one path. This minimum is achieved for $\gamma_0 = \gamma_0^\sharp$ [see (5.11)], from which (5.9) follows.

To prove (5.10), observe that (6.6) can be rewritten as

$$H^*(j+1, S_{j+1}) = \min_{\{\gamma_j \in \Gamma_j, s \in I^j, S \in \mathbb{I}^{u,d}(j,s)\}} -\gamma_j(S_{j+1} - s) + H^*(j,s)). \tag{6.7}$$

For each fixed value s for S_j it is optimal to take $\gamma_j = \gamma_j^\sharp$, according to Theorem 5.2, and then (5.10) follows from the fact that the domain for s is indeed given by I^j.

Hence $H^*(N, S_N)$ denotes the minimal hedge costs that are compatible with final price S_N for strategies that are admissible by the RCC restriction. Thus S_N^* does indeed occur in a best-case price path. Further, $S_0^*, \ldots, S_{N-1}^*$ is the price path to S_N^* that realizes the minimal hedge costs $H^*(N, S_N)$ if g^* is applied. Thus indeed $\mathrm{BC}^*(S_0, V) = Q^{g^*}(S^*)$. □

Proof of Proposition 5.8.

Define the value function J_j for $j = 0, \ldots, N$ by

$$J_N := [S_N - X]^+ + H_N,$$

$$J_{j-1}(S_{j-1}, H_{j-1}) := \min_{\gamma_{j-1} \in \Gamma_{j-1}} E_j[J_j(S_j, H_{j-1} - \gamma_{j-1}(S_j - S_{j-1}))].$$

Then J_j denotes the expected costs at t_j under an optimal strategy as a function of the current asset price S_j and realized hedge costs H_j. So J is indeed a value function, and in particular, J_0 denotes the expected costs under optimal hedging.

To show that g^* is indeed a solution, we first derive that

$$J_j(S_j, H_j) = \beta_j H_j + h_j(S_j), \tag{6.8}$$

with $\beta_j \in \mathbb{R}^+$, and h_j a function of S_j that is independent of H_j. This is obviously true for $j = N$, with $\beta_N = 1$ and $h_N(S_N) = [S_N - X]^+ =: f_N(S_N)$. Now take (6.8) as induction hypothesis; then for $j-1$ we have, with the assumption $e = e(j-1, S_{j-1}) > 0$ and omitting the function arguments S_{j-1} and S_j in order to avoid confusion with multiplication,

$$\begin{aligned}
J_{j-1}(S_{j-1}, H_{j-1}) :&= \min_{\gamma_{j-1}\in\Gamma_{j-1}} E_j[J_j(S_j, H_{j-1} - \gamma_{j-1}(S_j - S_{j-1}))] \\
&= \min_{\gamma_{j-1}\in\Gamma_{j-1}} E_j[\beta_j(H_{j-1} - \gamma_{j-1}(S_j - S_{j-1})) + h_j] \\
&= \beta_j H_{j-1} - \max_{\gamma_{j-1}\in\Gamma_{j-1}} \gamma_{j-1}\beta_j e S_{j-1} + E_j[h_j] \\
&= \beta_j H_{j-1} - \gamma_{j-1}^{\max}\beta_j e S_{j-1} + E_j[h_j] \\
&= \beta_j H_{j-1} - \beta_j\left(\Delta_{j-1} + \frac{V - f_{j-1} - H_{j-1}}{(1-d)S_{j-1}} e S_{j-1}\right) + E_j[h_j] \\
&= \beta_j H_{j-1} - \beta_j \frac{e}{1-d}(\Delta_{j-1}(1-d)S_{j-1} + V - f_{j-1} - H_{j-1}) + E_j[h_j].
\end{aligned}$$

Now, with function arguments included, delta hedging has the property

$$(1-d)S_{j-1}\Delta_{j-1}(S_{j-1}) - f_{j-1}(S_{j-1}) = -f_j(dS_{j-1}).$$

Substituting the left-hand side in the last formula for J_{j-1}, we obtain

$$J_{j-1}(S_{j-1}, H_{j-1}) = \beta_j\left(1 + \frac{e}{1-d}\right)H_{j-1} + E_j[h_j(S_j)] + \beta_j\frac{e}{1-d}(f_j(dS_{j-1}) - V).$$

Now take $\beta_{j-1} := \beta_j(1 + \frac{e}{1-d})$ and $h_{j-1}(S_{j-1}) := E_j[h_j(S_j)] + \beta_j\frac{e}{1-d}(f_j(dS_{j-1}) - V)$; then $J_{j-1}(S_{j-1}, H_{j-1}) = \beta_{j-1}H_{j-1} + h_{j-1}(S_{j-1})$ with h_{j-1} being indeed independent of H_{j-1}. Hence, by induction, (6.8) must be valid for all j.

For negative $e(j-1, S_{j-1})$ the computations are analogous, with $\gamma^{\max}$ replaced by $\gamma^{\min}$. The derivation of the formula immediately reveals that g^* is indeed optimal. □