Chapter 5
Optimal Hedging Under Robust-Cost Constraints

5.1 Introduction

In this chapter we analyze hedging of a short position in a European call option by an optimal strategy in the underlying asset under a *robust cost constraint* (RCC), that is, under the restriction that the worst-case costs do not exceed a certain a priori given upper bound. This relates to a Value-at-Risk (VaR) condition, which is usually defined for stochastic models as the maximum costs for a specified confidence level. As compared to VaR, an RCC denotes a level of worst-case costs that *cannot* be exceeded within a given interval model.

More specifically, the asset is modeled by an interval model $\mathbb{I}^{u,d}$ in N equal time steps from current time to expiry, cf. (3.9). Recall from Proposition 4.2 that a short position in the option, kept under a hedge strategy g, yields an outcome of costs in an interval I^g. For discontinuous strategies this interval I^g is not necessarily closed, and therefore *best-* and *worst-case costs* are defined as the infimum and supremum of costs:

$$\mathrm{BC}^g := \inf I^g = \inf_{S\in\mathbb{I}^{u,d}} Q^g(S).$$

$$\mathrm{WC}^g := \sup I^g = \sup_{S\in\mathbb{I}^{u,d}} Q^g(S).$$

We refer to $-\mathrm{BC}^g$ also as the *maximum profit* under g.

The RCC condition simply limits the worst-case costs WC^g. In this section we analyze the impact of such a restriction for the set of admissible hedging strategies and provide an algorithm to solve this constrained optimization problem. To that end we first introduce some notation.

The set of all strategies with price paths S in some interval model is denoted by $\mathbb{G}$. Thus

$$\mathbb{G} := \{g = (g_0,\dots,g_{N-1}) \mid g_j : (S_0,\dots,S_j) \to \gamma_j \in \mathbb{R}\}.$$

P. Bernhard et al., *The Interval Market Model in Mathematical Finance*, Static & Dynamic Game Theory: Foundations & Applications, DOI 10.1007/978-0-8176-8388-7_5,

Let V denote the RCC limit; then the set of all admissible strategies under this RCC limit is defined by

$$\mathbb{G}^V := \{g \in \mathbb{G} | \, \mathrm{WC}^g \leq V\}.$$

Furthermore, by Δ_j we will denote the delta-hedging strategy [see (3.4), (3.5)]

$$\Delta_j(S_j) = \lambda \Delta_{j+1}(uS_j) + (1-\lambda)\Delta_{j+1}(dS_j), \text{with}$$
$$\Delta_{N-1}(S_{N-1}) = \frac{[uS_{N-1} - X]^+ - [dS_{N-1} - X]^+}{(u-d)S_{N-1}},$$

where $\lambda = \frac{u(1-d)}{u-d}$, and by $f_j(S_j)$ we will denote the corresponding Cox–Ross–Rubinstein option premium [see (3.2), (3.3)]

$$\begin{aligned} f_N(S_N) &= [S_N - X]^+, \\ f_j(S_j) &= q f_{j+1}(uS_j) + (1-q) f_{j+1}(d_j S_j), \end{aligned} \tag{5.1}$$

where $q := \frac{1-d}{u-d}$.

5.2 Effect of Cost Constraints on Admissible Strategies

Since delta hedging yields the lowest upper bound of costs among all strategies in $\mathbb{G}$ (Theorem 4.5), we have the next result.

Proposition 5.1. *If $V < f_0(S_0)$, then $\mathbb{G}^V$ is empty. If $V \geq f_0(S_0)$, then the delta-hedging strategy belongs to $\mathbb{G}^V$.* □

So the arbitrage-free Cox–Ross–Rubinstein price of the option C in the binomial tree model $\mathbb{B}^{u,d}$ is the smallest RCC limit that is achievable for a hedged short position in the call option C with underlying asset $S \in \mathbb{I}^{u,d}$.

As may be expected, for RCC beyond this minimal level, the space of admissible strategies is centered around the delta-hedging strategy. To formulate the precise result, we introduce the following concepts. For a given strategy, H_j denotes the *realized hedge costs* at t_j:

$$H_j := -\Sigma_{k=0}^{j-1} \gamma_k (S_{k+1} - S_k). \tag{5.2}$$

In view of the previous result we also define the *current latitude*

$$\bar{V}_j(S_j, H_j) := V - H_j - f_j(S_j), \tag{5.3}$$

which is the excess of the RCC limit V over the past hedge costs H_j plus the minimal future worst-case costs (given by the Cox–Ross–Rubinstein price) $f_j(S_j)$, or, equivalently, the current wealth offset to the total minimal cost-to-go $f_j(S_j)$.

The next theorem shows that $\bar{V}$ indeed determines the extent to which admissible strategies may differ from delta hedging.

Theorem 5.2. *The set $\mathbb{G}^V$ of strategies admissible under the RCC level $V \geq f_0(S_0)$ for $S \in \mathbb{I}^{u,d}$ is given by*

$$\{g \in \mathbb{G} | \gamma_j^{\min}(S_j, H_j) \leq g_j(S_0, \ldots, S_j) \leq \gamma_j^{\max}(S_j, H_j)\}, \tag{5.4}$$

with

$$\gamma_j^{\min}(S_j, H_j) := \Delta_j(S_j) - \frac{\bar{V}_j(S_j, H_j)}{(u-1)S_j},$$

$$\gamma_j^{\max}(S_j, H_j) := \Delta_j(S_j) + \frac{\bar{V}_j(S_j, H_j)}{(1-d)S_j},$$

with $\bar{V}$ defined by (5.3) and H_j the realized hedge costs (5.2). □

See the appendix for a proof.

Summarizing, the consequence of including a restriction on the worst-case costs is that the set of admissible strategies is restricted to an interval around delta hedging, with fixed proportional centering determined by u and d and time-varying interval length determined by realized hedge costs.

5.3 Calculating Maximum Profit Under a Cost Constraint

In this section we present a numerical algorithm to maximize profits under a limit for worst-case costs in a given interval model for the asset. Or, stated differently, with an RCC limit V on worst-case costs, we look for a strategy g that provides the highest lower bound for best-case costs. We will denote this lower bound for best-case costs by

$$\mathrm{BC}^*(S_0, V) := \inf_{\{g \in \mathbb{G}^V, S \in \mathbb{I}^{u,d}\}} Q^g(S), \tag{5.5}$$

with $Q^g(S)$ as defined in (4.5).

Thus $\mathrm{BC}^*(S_0, V)$ is a lower bound for best-case costs (and hence an upper bound on maximum profit) under the RCC level V and with asset prices in $\mathbb{I}^{u,d}$ with initial price S_0. Notice that by choosing g equal to the stop-loss strategy, we in fact obtain just the opposite of what we want, i.e., a maximum profit that is outperformed by any other admissible strategy under optimal conditions.

To determine solutions, we first analyze the recursive structure of this minimization (5.5). It amounts to the dynamic programming problem

$$\begin{aligned}
\text{Minimize } J &:= \Sigma_{j=0}^{N-1} F(j,x_j,u_j,v_j) + G(x_N) \\
\text{with } x_{j+1} &= h(j,x_j,u_j,v_j) \\
\text{for } (u_j,v_j) &\in D(j,x_j),
\end{aligned}$$

with the following definitions of the variables:

$$\begin{aligned}
x_j &:= \begin{bmatrix} S_j \\ H_j \end{bmatrix}, \\
u_j &:= \gamma_j, \\
v_j &:= S_{j+1}/S_j,
\end{aligned}$$

with the domain $D(j,x_j)$ the rectangle specified by the conditions

$$\begin{aligned}
&u_j \in \Gamma_j(S_j,H_j) := [\gamma_j^{\min}(S_j,H_j), \gamma_j^{\max}(S_j,H_j)], \\
&v_j \in [d,u],
\end{aligned}$$

with $\gamma_j^{\min}$ and $\gamma_j^{\max}$ defined as in Theorem 5.2, and with the state recursion and cost function given by

$$\begin{bmatrix} S_{j+1} \\ H_{j+1} \end{bmatrix} = \begin{bmatrix} v_j S_j \\ H_j - u_j(v_j-1)S_j \end{bmatrix}; \begin{bmatrix} S_0 \\ H_0 \end{bmatrix} = \begin{bmatrix} S_0 \\ 0 \end{bmatrix};$$
$$J = -\Sigma_{j=0}^{N-1} u_j(v_j-1)S_j + [S_N - X]^+.$$

Notice the path dependence of the criterion, which becomes apparent in the occurrence of the realized hedge costs in the state. Equivalently, the criterion is path dependent through "current wealth" $V - H_j$, where the RCC limit V is interpreted as the initial wealth.

The corresponding *value function* consists of the best-case costs conditioned on the current asset price and past hedge costs:

$$\begin{aligned}
\mathrm{BC}_N(S_N,H_N) &:= [S_N - X]^+ + H_N, \\
\mathrm{BC}_j(S_j,H_j) &:= \min_{\{\gamma_j\in\Gamma_j, S_{j+1}\in[dS_j,uS_j]\}} \left(\mathrm{BC}_{j+1}(S_{j+1}, H_j - \gamma_j(S_{j+1}-S_j)\right), \quad (5.6)
\end{aligned}$$

with the domain for BC_j taken as

$$\{(S_j,H_j)|S_j > 0, H_j \geq V - f_j(S_j)\} \tag{5.7}$$

to avoid minimization over an empty domain. Not coincidentally, this definition of domain is consistent with the recursion in (5.6) because Γ_j is determined just to guarantee that $\bar{V}_j = V - H_j - f_j(S_j)$ remains nonnegative.

Thus BC_j denotes the best-case costs, given that at t_j the asset price is S_j and that past hedge costs accumulated to H_j or, equivalently, that $x_j = (S_j, H_j)$.

Before we go into computations, we show that the optimization problem has a solution. A proof of the next result can be found again in the appendix.

Proposition 5.3. *There exists an optimal strategy g^* and a best-case price path S^* such that $Q^{g^*}(S^*) = \mathrm{BC}^*(S_0, V)$.* □

As it seems too complicated to obtain closed-form solutions, we develop a numerical procedure that exploits some specific features of the dynamic programming problem, enabling a relatively simple *forward* recursion for a "frontier function" in one variable, with known initial conditions. Note that this approach differs from the standard numerical solution of the dynamic programming problem, which would amount to a backward recursion for a function in two variables, conditioned on *unknown* final values of asset prices S_N and realized hedge costs H_N. The following method hence avoids the use of a rather large grid matrix of sample points.

First a frontier function of minimal realized hedge costs is determined as a function of asset prices, then the best-case asset price path is determined by backward recursion, and finally the optimal strategy is reconstructed.

Algorithm 5.4 (Maximum profit under RCC).

Data: Initial asset price S_0, an interval model for assets $\mathbb{I}^{u,d}$, excercise price X and time T of a European call option, and an RCC limit on worst-case total costs at expiry, $\mathrm{WC}^g \le V$.

Step 1: Determine the "frontier function" of minimal realized hedge costs $H^*(j, S_j)$ by

$$H^*(0, S_0) := 0, \tag{5.8}$$

$$H^*(1, S_1) := -\gamma_0^\sharp (S_1 - S_0) \text{ for } S_1 \in [dS_0, uS_0], \tag{5.9}$$

$$H^*(j+1, S_{j+1}) := \min_{S_j \in I_j} [H^*(j, S_j) - (S_{j+1} - S_j)\gamma_j^\sharp (S_j, H^*(j, S_j))], \tag{5.10}$$

with $S_j \in [d^j S_0, u^j S_0]$, $I_j := [d^j S_0, u^j S_0] \cap [\frac{S_{j+1}}{u}, \frac{S_{j+1}}{d}]$ and $\gamma_j^\sharp$ defined by

$$\gamma_j^\sharp := \begin{cases} \gamma_j^{\max}(S_j, H^*(j, S_j)) & \text{for } S_{j+1} > S_j, \\ \gamma_j^{\min}(S_j, H^*(j, S_j)) & \text{for } S_{j+1} \le S_j. \end{cases} \tag{5.11}$$

Step 2: Determine the optimal price path S^* recursively by

$$S_N^* := \operatorname{argmin}_{S_N \in [d^N S_0, u^N S_0]} [H^*(N, S_N) + [S_N - X]^+], \tag{5.12}$$

$$S_j^* := \operatorname{argmin}_{S_j \in I_j} [H^*(j, S_j^*) - \gamma_j^\sharp (S_{j+1}^* - S_j)], \tag{5.13}$$

with $\gamma_j^\sharp$ defined as in (5.11) with S_{j+1} replaced by S_{j+1}^*.

Step 3: Determine the optimal strategy g^* by

$$g_j^*(S_0,\ldots,S_j) = \begin{cases} \gamma_j^{\max}(S_j,H_j) & \text{if } S_{j+1}^* > S_j^*, \\ \gamma_j^{\min}(S_j,H_j) & \text{if } S_{j+1}^* \leq S_j^*. \end{cases} \tag{5.14}$$

Result: The strategy g^* yields the maximum profit $-\mathrm{BC}^g$ under the restriction that worst-case costs WC^g are at most V and asset prices are in accordance with the interval model $\mathbb{I}^{u,d}$. These best-case costs are achieved for the price path S^* under strategy g^*. □

Remark 5.5. From (5.14) we see that the optimal hedge depends on the realized hedge costs in the past and that at each time step all gained reserves beyond the RCC limit are put at risk. This is a typical feature of the modeling we have used so far. If one is unhappy with these kinds of strategies because they are too risky, one should take this into account explicitly in the modeling. At this moment there is no incentive in the modeling to avoid this kind of behavior. We will return to this issue later on. □

That Algorithm 5.4 indeed achieves the advertised result is shown in the appendix. Before illustrating the algorithm with an example, we give a brief explanation. Initial hedge costs are set to zero in (5.8), and (5.9) simply denotes the realized hedge costs under optimal hedging $\gamma_0^\sharp$ as a function of the current price S_1 after the first step. For the second time step, $H^*(2,S_2)$ denotes optimal realized hedge costs, which now not only involve optimization over hedge position γ_1 but also over all paths in $\mathbb{I}^{u,d}$ starting at S_0 and ending at a fixed value S_2. The interval I_1 specifies all possible values for the asset price at t_1 for such paths. It is important to note that the algorithm postpones optimization over current prices, so hedge costs $H^*(j,S_j)$ are *conditionally* optimal, assuming an arbitrary fixed price level S_j at time t_j. Optimal price paths are then determined by a backward recursion (5.13) starting at an easy-to-evaluate final condition (5.12).

Example 5.6. We consider an at-the-money European call option with exercise price $X = 1$. We assume that the underlying asset follows a price path in the interval model $\mathbb{I}^{u,d}$, with $u = 5/4$ and $d = 4/5$, $N = 4$ time steps, and initial asset price $S_0 = 1$. Sampling of asset prices is done with a logarithmically regular grid, with 51 points on $[d,u]$. A further decrease of this mesh hardly affects the outcome of the algorithm.

The unique Cox–Ross–Rubinstein arbitrage-free option price in the corresponding binomial tree $\mathbb{B}^{u,d}$ is given by $f_0(S_0) = 0.1660$, cf. (5.1). With the aid of the algorithm we computed that maximum profit, under strategies that guarantee this limit, are given by $1/9$ and are achieved for the price path

$$S^* = (S_0^*, S_1^*, S_2^*, S_3^*, S_4^*) = (1,1,1,5/4,25/16). \tag{5.15}$$

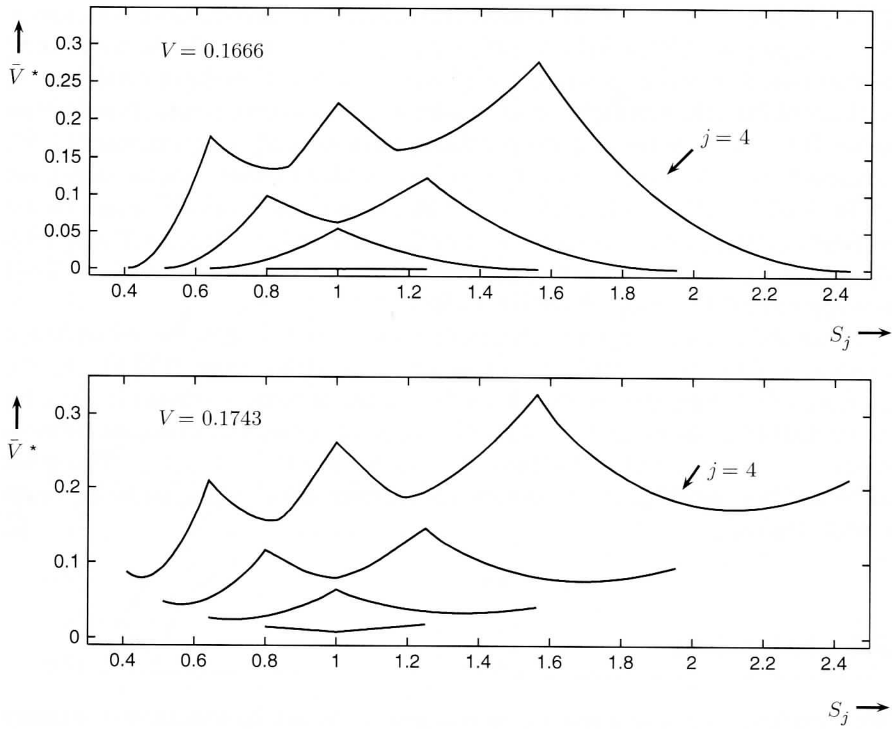

Fig. 5.1 Optimal costs under RCC restrained strategies. Both plots contain the graphs of the optimal current latitude $\bar{V}^*(j,s) := V - H^*(j,s) - f_j(s)$ with $s \in [d^j, u^j]$, for $j = 1,\ldots,4$. In the *upper plot* the RCC limit is chosen equal to the lowest achievable cost limit, i.e., $V = f_0(S_0) = 0.1666$, while in the *lower plot* this is increased by 5% to $V = 0.1743$. Thus $\bar{V}^*(j,s)$ denotes the maximum current latitude compatible with a price $S_j = s$ under strategies that are admissible under these RCC limits. In the *upper plot* $\bar{V}^*$ is zero at the boundary points d^j and u^j because these are only achievable by a sequence of extreme jumps in prices, which keeps the latitude at the zero level, by definition of delta hedging. The fact that $\bar{V}^*(1,s) = 0$ for all $s \in [d,u]$ is somewhat coincidental because this would not be the case for exercise prices unequal to 1

The same analysis is repeated with a slighly higher RCC limit $1.05 f_0(S_0) = 0.1743$. Maximum profit turns out to be 0.1531 and is achieved for the price path

$$S^* = (S_0^*, S_1^*, S_2^*, S_3^*, S_4^*) = (1, 0.9564, 1, 5/4, 25/16). \tag{5.16}$$

To give an impression of the outcome of the optimal realized hedge cost functions $H^*(j,s)$ for $j = 1,2,3,4$ [cf. (5.9) and (5.10)], we have plotted the corresponding optimal current latitude $\bar{V}^*(j,s) = V - H^*(j,s) - f(j,s)$, both for the tight RCC limit $V = f_0(S_0)$ and for $V = 1.05 f_0(S_0)$, in Fig. 5.1. This has the following interpretation. If at time t_j the asset price is given by $S_j = s$, then the realized hedge costs H_j, defined by (5.2), are under optimal circumstances equal to $H^*(j,s)$, i.e., under optimal admissible hedging and for the best price path in $\mathbb{I}^{u,d}$ from S_0 to

$S_j = s$. Hence $\bar{V}^*(j,s)$ denotes the maximum current latitude compatible with a price $S_j = s$. In particular, $\bar{V}^*(4,S_4) = V - H^*(4,S_4) - f(4,S_4)$. By (5.12), the last price S_4^* of the optimal price path minimizes $H^*(4,S_4) + f(4,S_4)$. Thus the optimal price S_4^* is obtained from the plot as that price at which the $j = 4$ curve attains its maximum value. If $V = 0.1666$, then we see that this optimal price S_4^* is approximately 1.56, whereas for $V = 0.1743$ the price S_4^* is slightly smaller. Furthermore, the maximum profit, $-\mathrm{BC}^*(S_0,V) = -H^*(4,S_4^*) - f(4,S_4^*)$, is obtained from this curve as the difference between the maximum value of the curve and V. Thus, for $V = 0.1666$ the maximum profit is approximately $0.28 - 0.1666 \approx 0.1104$ and for $V = 0.1743$ it is approximately $0.325 - 0.1743 \approx 0.1507$.

If the RCC limit is fixed at its smallest value, $f_0(S_0)$, then the initial hedge position is fixed because it must be equal to the delta hedge 0.5830. For the increased RCC limit the interval of admissible initial hedge positions is given by $[0.5369, 0.6863]$. In Fig. 5.2 we show the range of admissible strategies for time instants t_1, t_2, and t_3 under RCC limits $V = f_0(S_0)$ and $V = 1.05 f_0(S_0)$. This gives an idea of how far hedge positions may deviate from delta hedging under best-case circumstances. □

5.4 Extensions

We have described how a worst-case cost restriction can be translated to strategy limits and shown how to determine maximum profits under such a constraint. In this section we discuss extensions of this result with respect to the choice of the RCC limit and variants of the cost criterion.

First we pursue the pure interval calculus a little further in Sect. 5.4.1. It is shown how cost limits can be chosen on the basis of the maximum loss/profit ratio and how an option premium can be based on this criterion.

These criteria, which are based solely on interval limits for asset prices, have some degenerate features as a performance measure for investments, especially if the number of time steps is large. Under the RCC restriction, the downside risk is limited, by construction, but (as was already mentioned in Remark 5.5) in each step hedge volumes are driven to the maximum amount, and consequently all the gained reserves beyond the RCC limit are put at risk at each step. In particular, for a long sequence of time steps this seems odd, and it may be more desirable to secure profit, at least partially.

Therefore, in Sect. 5.4.2 we also analyze how to minimize expected costs under additional stochastic assumptions within interval models. This relates to a fairly general result that depends only on the expected growth factor, $E(S_{j+1}/S_j)$. However, despite the different nature of the criterion, we will see that the optimal hedge volumes turn out to be maximal again at each step.

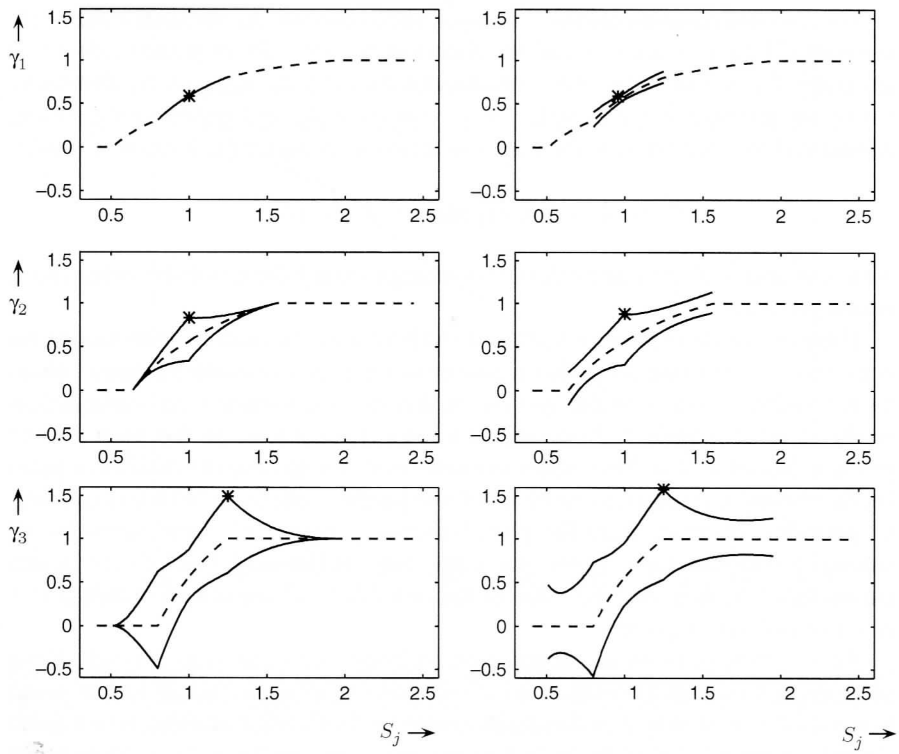

Fig. 5.2 RCC-admissible strategies. The plots on the *left-hand side* correspond to the tight RCC limit $V = f_0(S_0) = 0.1666$, on the *right-hand side* to $V = 1.05 f_0(S_0) = 0.1743$. The *dashed lines* indicate hedge positions according to delta hedging, as a function of prices S_j at time t_j, with $j = 1$ in the upper plots, $j = 2$ in the center, and $j = 3$ in the lower plots. The *solid lines* denote the graphs of $\gamma_j^{\max}$ and $\gamma_j^{\min}$, as a function of S_j, and with realized hedge costs $H^*(j, S_j)$, which are the optimal hedge costs compatible with asset price S_j at t_j, cf. (5.11). Whenever the corresponding current latitude is zero (Fig. 5.1), the strategy must coincide with delta hedging. The prices S_1^*, S_2^*, and S_3^* in the best-case price paths (5.15) (*left plots*) and (5.16) (*right plots*) are indicated by an asterisk and hence mark the actual outcome $\gamma_j^\sharp$ of the strategy for this path

5.4.1 Loss/Profit Ratio

In Sect. 5.3 we described an algorithm for solving the minimization problem (5.5). The solution consists of an optimal strategy and the construction of a corresponding price path at which profits are maximized. It is not hard to verify that worst-case costs under this strategy reach the prescribed RCC limit V, so that in fact the entire cost interval of the strategy (4.6) is known:

$$I^{g^*} = [\mathrm{BC}^*(S_0, V), V] \tag{5.17}$$

for g^* a solution of (5.5).

The question arises as to how to compare these intervals for different values of V if the RCC limit is a design variable rather than an externally imposed value. More generally, the question is how to evaluate cost intervals of strategies. By definition, the option premium is not included in the cost intervals, and incorporating it as an additional factor we arrive at the question of how to compare the interval of results

$$[a-f, b-f] \text{ and } [c-f, d-f]$$

with $[a,b]$ and $[c,d]$ cost intervals of two strategies and f the (yet-to-be-determined) option premium.

There is an infinite number of ways to evaluate uncertain costs. In this section we confine ourselves to the somewhat academic assumption that *nothing is known* about costs besides the limits of the interval. In this way we illustrate the consequences of the interval models without any additional assumptions on the asset pricing process. Notice that in this context the strategies that achieve the maximum profit for fixed worst-case costs, as constructed previously, relate to an "effective frontier" of portfolios because, from the pure "interval perspective," these dominate all strategies with the same worst-case costs and smaller profits. The only design parameters left, then, are the value of the cost limit and the option premium if it is not considered as given.

As a consequence of the absence of arbitrage, we must consider only those intervals that contain zero because entirely positive or negative cost ranges relate to the existence of strategies that yield certain profit. Recall from Theorem 4.5 that the precise bounds for arbitrage-free option premiums are the intrinsic value of the option, $[S_0 - X]^+$ (lower bound), and the Cox–Ross–Rubinstein price $f_0(S_0)$ for maximum volatility (upper bound). The *loss/profit ratio* (LPR) of such an interval $[a-f, b-f]$ is defined as $\frac{b-f}{f-a}$. There is a simple argument for considering this ratio as the main criterion. Suppose there are strategies that lead to cost intervals $[a,b]$ and $[c,d]$, so that the net results with option premium f are respectively $[a-f, b-f]$ and $[c-f, d-f]$. If we allow for portfolio rescaling, we may scale the second one by a factor $\frac{a-f}{c-f}$, which gives $[a-f, \frac{(a-f)(d-f)}{c-f}]$. Comparing this with $[a-f, b-f]$ is now simply a matter of comparing the right bounds, and their ordering is precisely determined by the LPR of the intervals. Observe that the cost interval with the smallest ratio is preferable in the absence of additional information.

From (5.17) it now follows that for a given (arbitrage-free) option premium f and an achievable RCC limit V [hence not below the Cox–Ross–Rubinstein price $f_0(S_0)$], the *optimal LPR criterion* is given by

$$\mathrm{LPR}^*(f, V) := \frac{V - f}{f - \mathrm{BC}^*(S_0, V)}.$$

Here $\mathrm{BC}^*(S_0, V)$ is given by (5.14), and it is interesting to see how this ratio depends on V and f. In particular, one might address the question of how to minimize this ratio over V for a fixed f. Notice that for the maximum arbitrage-free premium,

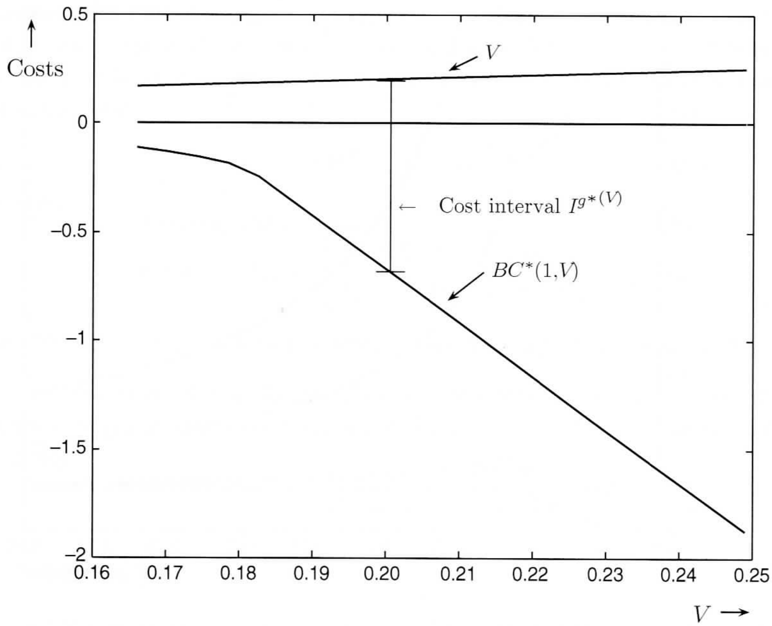

Fig. 5.3 Best-case costs as a function of RCC limit. The *lower curve* shows best-case costs $\mathrm{BC}^*(S_0,V)$ as a function of RCC limit V, with $S_0 = 1$; cf. (5.5). The *upper line* in the plot equals V. The minimally achievable RCC limit is 0.1666, which equals the Cox–Ross–Rubinstein option premium

$f = f_0(S_0)$, the RCC that minimizes this $\mathrm{LPR}^*(V)$ ratio is $V = f_0(S_0)$ because then the ratio is zero. Moreover, it is easily verified that for a fixed RCC limit V, $\mathrm{LPR}^*(f)$ is a decreasing function. So its minimum value is attained at $f = f_0(S_0)$.

In the following example we consider this dependency of the optimal LPR on its RCC limit V and option prices f in more detail.

Example 5.7. We proceed with Example 5.6. Figure 5.3 illustrates the dependence of best-case costs $\mathrm{BC}^*(S_0,V)$ on cost limit V by applying the algorithm to several values of V. It seems that $\mathrm{BC}^*(S_0,V)$ depends piecewise linearly on this RCC limit V. There seems to be a change in the slope of the best-case costs around $V = 0.18$.

In Fig. 5.4 the LPRs are shown for a range of arbitrage-free option premiums. The fair price interval is in this case given by $[0, 0.1666]$. We plotted the optimal LPR for different choices of the option price from this fair price interval. It turns out that there are two ranges of option premiums for which $\mathrm{LPR}^*(V)$ has a different minimum location. For option prices sufficiently close to the minimal RCC bound $f_0(S_0)$, the minimal $\mathrm{LPR}^*(V)$ is attained at $V = f_0(S_0)$. For option prices below a certain threshold, $\mathrm{LPR}^*(V)$ has an infimum at $V = \infty$. Thus, by relaxing in those cases the RCC constraint, we improve the LPR^*. □

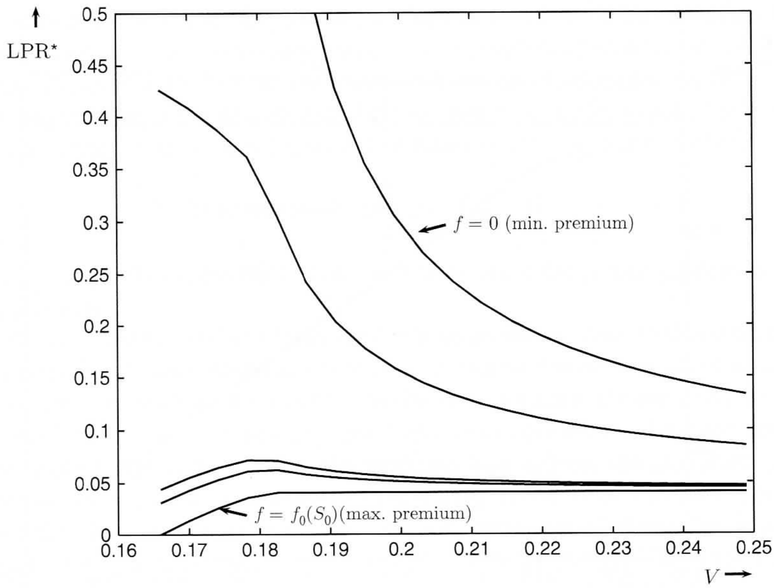

Fig. 5.4 Optimal loss/profit ratio for several option prices. The *curves* denote the LPR* for several option prices. The lowest one corresponds to the Cox–Ross–Rubinstein price $f_0(S_0) = 0.1666$; hence it is zero at $V = f_0(S_0)$. The other *curves* (*bottom* to *top*) correspond to option premiums of, respectively, 0.95, 0.93, 0.5, and 0 times $f_0(S_0)$. Prices outside this range are not arbitrage free. For factors from 0.93 to 1, the minimum is at the left, so the minimum LPR* is achieved for the lowest achievable cost bound $V = f_0(S_0)$, while for lower premiums the optimum switches to infinity

5.4.2 *Maximum Expected Profit Under a Cost Constraint*

In this section we take a step from the purely indeterministic features, as represented by interval models, to the probabilistic properties of prices. We assume that, in addition to the *limits* that an interval model induces for the growth factor of prices, its *expected value* is also given. Thus we assume that in each step

$$E_j[S_{j+1}/S_j] = 1 + e(j, S_j), \tag{5.18}$$

where $E_j[\cdot]$ denotes the expectations conditioned on past prices up to and including t_j, and e is a real-valued function denoting the expected growth rate at t_j for current price level S_j. Notice that under a risk-neutral probability measure and prices relative to a tradable asset, there are theoretical arguments for setting $e = 0$, making the measure a martingale. With e expressing "real" expectations, the value of e is typically positive and depends on the market price of risk.

It turns out that minimizing the expected costs under RCC still corresponds to hedging at the strategy limits imposed by the cost constraint.

Proposition 5.8 (Minimizing expected cost under RCC). *Assume that asset prices follow paths in $\mathbb{I}^{u,d}$ and have an expected growth rate of $1+e(j,S_j)$ for price S_j at time t_j. The strategy that minimizes the expected value of costs (4.5) under a cost constraint V,*

$$g^* := \operatorname{argmin}_{g\in\mathbb{G}^V} E[Q^g(S)],$$

is given by

$$g_j^*(S_0,\ldots,S_j) = \begin{cases} \gamma_j^{\max}(S_j,H_j) & \text{if } e(j,S_j) > 0, \\ \gamma_j^{\min}(S_j,H_j) & \text{if } e(j,S_j) < 0, \end{cases}$$

where $\gamma_j^{\min}$ and $\gamma_j^{\max}$ are the RCC strategy bounds as defined in Theorem 5.2. □

The proof of this result can be found in the appendix. This result shows that the expected-cost criterion does not restrain us from strategies that are at the cost limit at each step.

5.5 Summary

In retrospect, we showed that in interval models the price of an option is in general not uniquely determined. A fair price for an option with a convex payoff may be any price in a compact interval. The lower bound of this interval is determined by the smallest price that can occur in the interval model using, for example, a stop-loss strategy, and the upper bound is determined by the largest price that can occur in the interval model using a delta-hedging strategy. In a simulation study, we showed that in a discrete-time setting with uncertain volatility the use of binomial tree models may severely underestimate the involved cost of hedging. This applies particularly when the hedging strategy underestimates the volatility of prices. Finally, we considered the question of how to find a hedging strategy for a call option that maximizes potential profits under the restriction that costs must not exceed an a priori given bound. We derived a numerical algorithm to calculate such a strategy under the assumption that this cost bound is not too strict. This strategy has the property that gained reserves beyond this cost bound are put at risk at every step. This outcome is on the one hand quite rational given our modeling framework. On the other hand, the strategy does not seem to be in line with the basic idea behind hedging, which is to reduce risk. Clearly, the strategy satisfies the strict requirement of not crossing the cost bound. Profit maximization, however, might not be the correct specification of the hedger's objective. To find hedging strategies that are perhaps more in line with the idea of risk reduction, one could look for different objective specifications or restrict the set of admissible strategies to an a priori defined class. Together with the issue of seeing how this theory works in practice, these are challenging subjects for future research.