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Brief paper

Time-scaling symmetry and Zeno solutions ☆

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ABSTRACT

Symmetries are often helpful in reducing the complexity of the analysis of dynamical systems. Here we discuss a symmetry that combines a transformation in space with a scaling of time. Examples are given of a number of nonsmooth dynamical systems in which a symmetry of this type occurs. The time-scaling symmetry can be combined with return mapping analysis to achieve a dimension reduction in addition to the one already obtained from considering the return map. The method is applied in a study of Zeno behavior in linear systems with relay feedback.

1. Introduction

The notion of symmetry has been used for the study of dynamical systems in physics as well as in control theory (cf. for instance Fagnani and Willems (1993), Marsden and Ratiu (1999) and Nijmeijer and van der Schaft (1985)). Standard examples include translational and rotational symmetry. These symmetries refer to groups of space transformations which map the set of solutions of a given system into itself. The type of symmetry considered in this paper is based on a combination of space transformations and scaling of time. We call this type of symmetry a *time-scaling symmetry*.

Time scaling has been employed extensively in control theory (cf. for instance Kokotovic, O'Reilly, and Khalil (1986)), but usually not in conjunction with space scaling. Space–time scaling invariance has been used in geophysics, as a means of carrying over information from one scale to the other; see for instance (Venugopal, Foufoula-Georgiou, & Sapozhnikov, 1999). The basic idea in this application is that smaller features evolve similarly to larger ones, but on a faster time scale. Typically the space and time scales are not linearly related. Brownian motion constitutes another example of a dynamic process that exhibits a space/time scaling invariance in which the space and time scales are not linearly related.

This paper shows that there are close connections between time-scaling symmetry and the *Zeno phenomenon*. The latter term is used in the context of nonsmooth dynamical systems to describe a situation in which an infinite number of mode switches takes place in a finite interval of time. We study the Zeno phenomenon in this paper by making use of Poincaré mappings (also called return mappings) which are a well-known tool for the study of nonsmooth systems (cf. for instance Brogliato (1999, Ch. 7)) and for nonlinear systems in general. The technique makes it possible to study an n-dimensional continuous-time system through an $(n-1)$-dimensional discrete-time system. It is shown below that, in the presence of a time-scaling symmetry, a further dimension reduction may be possible which leads to the study of an $(n-2)$-dimensional system defined by the *orbit return map* that is introduced below. Under certain conditions, periodic solutions of this system give rise to Zeno solutions of the original n-dimensional continuous-time system.

Sufficient conditions for the *absence* of Zeno solutions have been developed in many papers; cf. for instance Çamlıbel and Schumacher (2001), Heemels, Çamlıbel, and Schumacher (2002), Pogromsky, Heemels, and Nijmeijer (2003), Shen and Pang (2005) and Zhang, Johansson, Lygeros, and Sastry (2001). On the other hand, as noted in Lamperski and Ames (2007), not many sufficient conditions are known for the *presence* of such solutions. Here, as in Lamperski and Ames (2007), we focus on conditions of the latter type. Lamperski and Ames (2007) consider hybrid systems with resets and find sufficient conditions for Zeno behavior in situations that generalize the well known bouncing ball example;

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here we study systems with continuous trajectories, covering Zeno examples of Filippov and Fuller.

2. Definitions and first examples

To define the notion of a time-scaling symmetry, we first of all need scaling of time. Scaling is performed by the multiplicative group of positive real numbers, denoted by $\mathbb{R}_+$. We also need a family $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathbb{R}_+\}$ of mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that F_1 is the identity mapping and $F_{\alpha_1}F_{\alpha_2} = F_{\alpha_1\alpha_2}$ for all $\alpha_1, \alpha_2 \in \mathbb{R}_+$. Such a family is called a *spatial transformation group parametrized by* $\mathbb{R}_+$. In this paper we consider only spatial transformation groups parametrized by the multiplicative group of the positive reals and so for brevity we shall frequently omit the qualification. Of course one can convert from a parametrization by the additive group of the reals to the multiplicative parametrization by taking exponentials.

If x is a function from a (finite or infinite) interval (t_-, t_+) of $\mathbb{R}$ to $\mathbb{R}^n$, then its *time-scaling transform* with parameter $\alpha \in \mathbb{R}_+$ relative to the transformation group $\mathcal{F}$ is the function $T_\alpha x$ from $(\alpha^{-1}t_-, \alpha^{-1}t_+)$ to $\mathbb{R}^n$ defined by

$$(T_\alpha x)(t) = F_\alpha(x(\alpha t)), \quad t \in (\alpha^{-1}t_-, \alpha^{-1}t_+). \tag{1}$$

The collection of mappings $\{T_\alpha \mid \alpha \in \mathbb{R}_+\}$ is said to form a *time-scaling transformation group*.

We consider systems described by equations of the form

$$\dot{x}(t) = f(x(t)) \tag{2}$$

on an open subset $\mathcal{X}$ of $\mathbb{R}^n$, where f is a function from $\mathcal{X}$ to $\mathbb{R}^n$. In most applications below, the function f is not continuous. The following solution concept will suffice for the purposes of the present paper. An absolutely continuous function $x : (t_-, t_+) \to \mathcal{X}$ defined on a finite or infinite open interval (t_-, t_+) of $\mathbb{R}$ is said to be a *solution* of the system above if it satisfies the Eq. (2) almost everywhere on (t_-, t_+). A solution x with domain (t_-, t_+) is said to be *maximal* if there does not exist another solution $\tilde{x}$ with domain $(\tilde{t}_-, \tilde{t}_+) \supset (t_-, t_+)$ such that $\tilde{x}(t) = x(t)$ for $t_- < t < t_+$, and at least one of the inequalities $\tilde{t}_- \leq t_-$ and $\tilde{t}_+ \geq t_+$ is strict. As a matter of terminology, we shall use the term "trajectory" both to refer to triples (t_-, t_+, x) and to refer to time functions $x : (t_-, t_+) \to \mathcal{X} \subset \mathbb{R}^n$.

The notion of time-scaling symmetry can now be defined as follows.

Definition 1. The dynamical system (2) is said to exhibit *time-scaling symmetry* with respect to a given spatial transformation group $\mathcal{F}$ if, for any solution $x : (t_-, t_+) \to \mathcal{X}$ of (2) and any $\alpha > 0$, the function $T_\alpha x : (\alpha^{-1}t_-, \alpha^{-1}t_+) \to \mathcal{X}$ defined in (1) is also a solution of (2).

It is straightforward to reformulate the notion of time-scaling symmetry in a "behavioral" (representation-free) way. The time-scaling symmetry is then a symmetry in the sense of Fagnani and Willems (1993).

The following are classical examples of systems that exhibit Zeno behavior. In both cases a time-scaling symmetry is present.

Example 2. In 1960, Fuller (1961) considered the problem of optimizing the quadratic cost function $\int_0^\infty x_1^2(t)\,dt$ for the linear system $\dot{x}_1 = x_2, \dot{x}_2 = u$ subject to the input constraint $|u(t)| \leq 1$ for all t. He found the closed-loop dynamics under the optimal control to be of the form

$$\dot{x}_1(t) = x_2(t) \tag{3a}$$

$$\dot{x}_2(t) = -\mathrm{sgn}(x_1(t) + \frac{1}{2}mx_2^2(t)\,\mathrm{sgn}\,x_2(t)) \tag{3b}$$

where $0 < m < 1$. He also showed that the system reaches the origin in finite time, after infinitely many switches. It is easily verified that Fuller's system is subject to time-scaling symmetry induced by the spatial transformation group $F_\alpha = \mathrm{diag}(\alpha^{-2}, \alpha^{-1})$.

Example 3. Filippov (1988, p. 116) discusses the following example:

$$\dot{x}_1(t) = \mathrm{sgn}\,x_1(t) - 2\,\mathrm{sgn}\,x_2(t) \tag{4a}$$

$$\dot{x}_2(t) = 2\,\mathrm{sgn}\,x_1(t) + \mathrm{sgn}\,x_2(t). \tag{4b}$$

It is straightforward to verify that this system exhibits a time-scaling symmetry corresponding to the spatial transformation group $F_\alpha = \mathrm{diag}(\alpha^{-1}, \alpha^{-1})$.

The bouncing ball (see for instance van der Schaft and Schumacher (2000, Section 2.2.3)) provides another standard example of Zeno behavior. While this system has resets and so is not in the class of systems that we consider in this paper, it may be noted that a time-scaling symmetry is present in this example as well. We now continue with a discussion of time-scaling symmetry in linear systems with relay feedback.

3. Linear and piecewise affine systems

Suppose that the system defined by (2) is linear; we then write

$$\dot{x}(t) = Ax(t). \tag{5}$$

Suppose also that the mappings F_α are linear. It is immediately apparent from the properties of the matrix exponential that, for any given $n \times n$ matrix K, the collection of linear mappings defined by $\alpha^K := \exp(K \log \alpha)$ $(0 < \alpha < \infty)$ forms a spatial transformation group. Conversely, if F_α $(0 < \alpha < \infty)$ is a collection of linear mappings such that $F_0 = I$ and $F_{\alpha_1\alpha_2} = F_{\alpha_1}F_{\alpha_2}$ for all $\alpha_1, \alpha_2 \in (0, \infty)$, then it follows from semigroup theory (see for instance Curtain and Zwart (1995, Ch. 2)) that the limit $K := \lim_{\alpha\to 1}(F_\alpha - I)/(\alpha - 1)$ exists, and $F_\alpha = \alpha^K$ for all α.

Necessary and sufficient conditions for the spatial transformation group defined by K to provide a time-scaling symmetry for the system (5) are provided below.

Proposition 4. *The transformation group $F_\alpha = \alpha^K$ establishes a time-scaling symmetry for the linear dynamical system (5) if and only if*

$$AK - KA = A. \tag{6}$$

Proof. If $z(t) = \alpha^K x(\alpha t)$ where $x(t)$ is a solution of (5), then $\dot{z}(t) = \alpha^{K+I}Ax(\alpha t)$. Therefore the time-scaling symmetry holds if and only if $\alpha^{K+I}A = A\alpha^K$ for all $\alpha > 0$. To see that this relation implies (6), differentiate with respect to α and set $\alpha = 1$ in the result. For the converse, first consider the dynamical system $\dot{v}(t) = Kv(t)$ with initial condition v_0. Define $w(t) = Av(t)$. Then $\dot{w}(t) = A\dot{v}(t) = AKv(t) = (I + K)Av(t) = (I + K)w(t)$. It follows that, for all $t \in \mathbb{R}$, $e^{(K+I)t}Av_0 = Ae^{Kt}v_0$. Since this holds for all v_0, we obtain $e^{(K+I)t}A = Ae^{Kt}$ for all t. The desired conclusion follows by taking $t = \log \alpha$. □

The following proposition shows for which matrices A the Eq. (6) can be satisfied.

Proposition 5. *Let A be a square matrix. There exists a matrix K such that (6) holds if and only if the matrix A is nilpotent.*

Proof. The necessity follows from the properties of the solutions X of the matrix equation $AX - XB = 0$ Gantmacher (1959, Ch. VIII), when A is looked at as a solution of the equation

$XK = (I + K)X$. For a more direct proof, see also Prop. 2.2 in Burde (2005). For the sufficiency part, it is convenient to use, following Fuhrmann (1976), the linear space $X_D = \{f(z) \in \mathbb{R}^k[z] \mid D^{-1}(z)f(z)$ is strictly proper rational$\}$ where $D(z)$ is a polynomial matrix of the form $\mathrm{diag}(z^{n_1}, \ldots, z^{n_k})$. The mapping $A : f(z) \mapsto (f(z) - f(0))/z$ is a nilpotent operator on this space, and it is easily verified that any nilpotent operator on a finite-dimensional real vector space is isomorphic to a mapping of this form. Moreover, one verifies by straightforward calculation that the mapping $K : f(z) \mapsto zf'(z)$ takes X_D to itself and is such that $AK - KA = A$. □

If K solves (6), then so does $K + MA$ for any matrix M such that $AM = MA$. In particular, $K + \mu I$ is a solution if K is, for any $\mu \in \mathbb{R}$. Some properties of matrices K that satisfy (6) are given below.

Lemma 6. *Let a linear mapping A be given that acts on a real vector space $\mathcal{X}$. For $i = 0, 1, 2, \ldots$, define $\mathcal{X}_i = \{x \in \mathcal{X} \mid A^i x = 0\}$. If K is a linear mapping such that (6) holds, then all subspaces $\mathcal{X}_i$ are invariant under K.*

Proof. The proof is by induction. For $i = 0$ the statement is trivial. Suppose that $K\mathcal{X}_i \subset \mathcal{X}_i$ for given i, and take $x \in \mathcal{X}_{i+1}$. Then $AKx = KAx + Ax \in \mathcal{X}_i$ because $Ax \in \mathcal{X}_i$, so that $Kx \in \mathcal{X}_{i+1}$. □

Proposition 7. *Let A be a matrix of size n and suppose its minimal polynomial has degree n. If the relation (6) holds, then there exists $\mu \in \mathbb{R}$ such that the eigenvalues of K are equal to $\mu, \mu+1, \ldots, \mu+n-1$.*

Proof. We already know that A must be nilpotent. The assumption on the minimal polynomial of A implies that the subspaces $\mathcal{X}_i$ introduced in the lemma above satisfy $\dim \mathcal{X}_i = i$ for $i = 1, \ldots, n$. We can choose a basis $x_1, \ldots, x_n$ of the state space $\mathcal{X}$ such that, for all i, $\{x_1, \ldots, x_i\}$ is a basis of $\mathcal{X}_i$, and $Ax_i = x_{i-1}$ for $i = 1, \ldots, n$ with $x_0 = 0$. By the K-invariance of the subspaces $\mathcal{X}_i$ as shown in the lemma above, we can write $Kx_i = \sum_{j=1}^{i} k_{ij}x_j$. We have

$$AKx_i = \sum_{j=1}^{i} k_{ij}x_{j-1} = \sum_{j=1}^{i-1} k_{i,j+1}x_j$$

$$(I + K)Ax_i = (I + K)x_{i-1} = x_{i-1} + \sum_{j=1}^{i-1} k_{i-1,j}x_j.$$

By the independence of the vectors $x_1, \ldots, x_n$, the relation (6) implies that $k_{ii} = k_{i-1,i-1} + 1$ for $i = 2, \ldots, n$ (and also that $k_{i,j+1} = k_{i-1,j}$ for $j < i - 1$). Because of the upper triangular form of K, the eigenvalues of K are equal to the diagonal elements k_{ii} ($i = 1, \ldots, n$). The statement follows (take $\mu = k_{11}$). □

Consider now the class of piecewise affine systems defined by the equation

$$\dot{x}(t) = Ax(t) - b\,\mathrm{sgn}\,(c^{\mathrm{T}}x(t)) \tag{7}$$

and suppose that there exists a matrix K such that

$$AK - KA = A, \qquad Kb = -b, \qquad c^{\mathrm{T}}K = \lambda c^{\mathrm{T}} \tag{8}$$

for some $\lambda \in \mathbb{R}$. We can then verify that the system defined by (7) admits a time-scaling symmetry with respect to the spatial transformation group defined by $F_\alpha(x) = \alpha^K x$. Indeed, suppose that $x(t)$ is a solution of (7), and define $z(t)$ by $z(t) = \alpha^K x(\alpha t)$. From the properties (8), it follows that

$$\alpha^{K+I}A = A\alpha^K, \qquad \alpha^{K+I}b = b, \qquad c^{\mathrm{T}}\alpha^K = \alpha^\lambda c^{\mathrm{T}}. \tag{9}$$

Therefore we have

$$\begin{aligned}\dot{z}(t) &= \alpha^K \alpha \dot{x}(\alpha t)\\ &= \alpha^{K+I}(Ax(\alpha t) - b\,\mathrm{sgn}\,(c^{\mathrm{T}}x(\alpha t)))\\ &= A\alpha^K x(\alpha t) - b\,\mathrm{sgn}\,(\alpha^{-\lambda}c^{\mathrm{T}}\alpha^K x(\alpha t))\\ &= Az(t) - b\,\mathrm{sgn}\,(c^{\mathrm{T}}z(t)).\end{aligned}$$

For future use we note the following property of systems that satisfy (8).

Proposition 8. *If the system (A, b, c) satisfies (8), then $c^{\mathrm{T}}A^j b = 0$ for all $j = 0, 1, 2, \ldots$ such that $j + \lambda + 1 \neq 0$.*

Proof. From (6), one easily proves by induction that $A^jK - KA^j = jA^j$ for all $j = 1, 2, \ldots$. Moreover, this relation trivially also holds for $j = 0$. So for all $j = 0, 1, 2, \ldots$ we have $jc^{\mathrm{T}}A^jb = c^{\mathrm{T}}A^jKb - c^{\mathrm{T}}KA^jb = -c^{\mathrm{T}}A^jb - \lambda c^{\mathrm{T}}A^jb$, that is, $(j + \lambda + 1)c^{\mathrm{T}}Ab = 0$. □

4. The orbit return mapping

We revert to the general setting of Section 2. A subset $\mathcal{S}$ of the collection $\mathcal{AC}(\mathcal{X})$ of absolutely continuous functions of time with values in $\mathcal{X}$ will be said to be *parametrized by initial conditions* if for every $x_0 \in \mathcal{X}$ there is exactly one maximal solution $(t_-, t_+, x) \in \mathcal{S}$ such that $x(0) = x_0$, with $t_- < 0$ and $t_+ > 0$. We shall refer to trajectory sets that are parametrized by initial conditions as *state trajectory sets*. When a state trajectory set is given, the value at time t of the trajectory passing through a given vector x_0 is denoted by $x(t; x_0)$, and the domain of definition of the trajectory is denoted by $(t_-(x_0), t_+(x_0))$.

Lemma 9. *Let $\mathcal{S} \subset \mathcal{AC}(\mathcal{X})$ be a state trajectory set that satisfies a time-scaling symmetry relative to a spatial transformation group $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathbb{R}_+\}$. For every $\alpha \in \mathbb{R}_+$ and $x_0 \in \mathcal{X}$ we have $t_\pm(F_\alpha(x_0)) = \alpha^{-1}t_\pm(x_0)$, and for all $t \in (t_-(F_\alpha(x_0)), t_+(F_\alpha(x_0)))$ we have*

$$x(t; F_\alpha(x_0)) = F_\alpha(x(\alpha t; x_0)). \tag{10}$$

Proof. Due to the time-scaling symmetry, the function $t \mapsto F_\alpha(x(\alpha t; x_0))$ belongs to $\mathcal{S}$; moreover the value of this function at $t = 0$ is $F_\alpha(x_0)$. The claims now follow from the assumption that $\mathcal{S}$ is a state trajectory set. □

An important tool in the study of dynamical systems is the so called *Poincaré map* or *return map*. This map is defined on a suitably selected *Poincaré section* (a hypersurface of codimension 1 that is transversal to the motion defined by the system, cf. for instance Guckenheimer and Holmes (1983)). In the context of systems that show time-scaling symmetry, we are interested in particular in Poincaré sections that are invariant under the spatial transformation group. A subset S of $\mathcal{X}$ is said to be *invariant* under the group $\mathcal{F}$ if $F_\alpha S \subset S$ for all $\alpha \in \mathbb{R}_+$. In this case we must actually have $F_\alpha S = S$, because $x = F_\alpha F_{1/\alpha}x$ for all x. The *orbit* corresponding to a state $x_0 \in S$ is the set $\{F_\alpha x_0 \mid \alpha \in \mathbb{R}_+\}$. Any invariant set is a union of orbits, and vice versa.

Let a state trajectory set on $\mathcal{X} \subset \mathbb{R}^n$ be given, and let S be a subset of $\mathcal{X}$. The *return set* S_0 is the set of initial conditions x_0 for which there exists a time $\tau > 0$ such that $x(\tau; x_0) \in S$. If τ is moreover such that $x(t; x_0) \notin S$ for $0 < t < \tau$, then τ is said to be the *first return time* of x_0 and we write $\tau = \tau(x_0)$. If $x(t; x_0) \in S$ for all $t > 0$, then we set $\tau(x_0) = 0$. The *return map* (or *Poincaré map*) $R : S_0 \to S$ is defined by

$$R : x_0 \mapsto x(\tau(x_0); x_0). \tag{11}$$

Assume now that we work with a given state trajectory set on $\mathcal{X}$ that shows time-scaling symmetry with respect to the spatial transformation group $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathbb{R}_+\}$.

Lemma 10. *Let $S \subset \mathcal{X}$ be invariant. Then the return subset S_0 of S is invariant as well. Moreover, if τ is the first return time of $x_0 \in S_0$, then $\alpha^{-1}\tau$ is the first return time of $F_\alpha(x_0)$.*

Proof. The claim follows from

$$\begin{aligned}x(t; F_\alpha(x_0)) \in S &\Leftrightarrow F_\alpha(x(\alpha t; x_0)) \in S\\ &\Leftrightarrow x(\alpha t; x_0) \in S.\end{aligned}$$

The first of these equivalence relations follows from Lemma 9, the second from the invariance of S. □

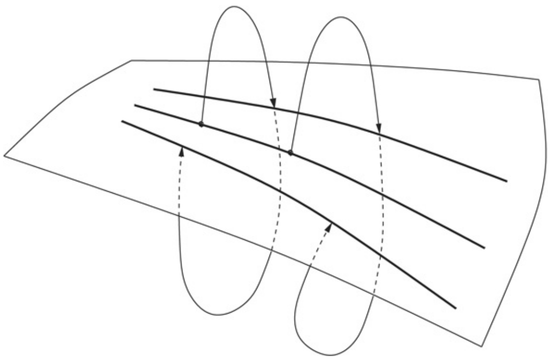

Fig. 1. Orbit return mapping. A Poincaré section is shown together with two dynamical system trajectories. Bold lines indicate orbits of the spatial transformation group.

Theorem 11. *If $S \subset \mathcal{X}$ is invariant, then the spatial transformation group restricted to S commutes with the return map. In other words, for all $x_0 \in S_0$ and all $\alpha \in \mathbb{R}_+$ we have $R(F_\alpha(x_0)) = F_\alpha(R(x_0))$.*

Proof. By the preceding lemmas, we have $R(F_\alpha(x_0)) = x(\alpha^{-1}\tau(x_0); F_\alpha(x_0)) = F_\alpha(x(\tau(x_0); x_0)) = F_\alpha(R(x_0))$. □

It follows from the theorem above that *the return map takes orbits to orbits*. In other words, points on the same orbit are mapped by the return mapping to points on the same orbit, as illustrated in Fig. 1. The quotient map on the orbit space will be called the *orbit return map*.

Concretely, the orbit return map may be constructed as follows. Assume we have an invariant Poincaré section S. Typically the orbits are of dimension 1 and the orbits on S can be parametrized by an $(n-2)$-dimensional coordinate vector which we shall call v. One may choose α as a parameter on the orbits, by selecting an arbitrary point on each orbit corresponding to $\alpha = 1$. As a more intrinsic parameter, one may take the return time; by Lemma 10, the return time scales inversely with α on a given orbit, so that different points on a one-dimensional orbit indeed have different return times associated to them. The Poincaré map takes the "triangular" form $R(v, \tau) = (v', \tau')$ with

$$v' = R_1(v) \tag{12a}$$

$$\tau' = R_2(v, \tau). \tag{12b}$$

The orbit return map is then given by R_1. It is a map on an $(n-2)$-dimensional space.

A case of particular interest arises when the sequence generated by the iteration (12a) is periodic. Consider in particular the case in which the orbit return map R_1 has a fixed point, i.e. in the Poincaré section there is an orbit of the transformation group which is mapped to itself by the return mapping. Take an initial condition x_0 on this orbit; we then have $R(x_0) = F_\alpha(x_0)$ for some $\alpha > 0$. By the commutativity of the return mapping and the transformation group, we have $R(F_{\alpha^j}x_0) = F_{\alpha^j}(R(x_0)) = F_{\alpha^{j+1}}(x_0)$. The return times corresponding to the sequence $x_0, F_\alpha(x_0), F_{\alpha^2}(x_0), \ldots$ are, by Lemma 10, $\tau(x_0), \alpha^{-1}\tau(x_0), \alpha^{-2}\tau(x_0), \ldots$. Thus the return times form a geometric sequence, and if $\alpha > 1$ the sum of the return times is finite. In the case in which the Poincaré section is in fact a switching surface, we have infinitely many events in a finite time interval; since convergence takes place "to the right" (i.e. there exists a time point t_Z such that every interval of the form $(t_Z - \varepsilon, t_Z)$ contains infinitely many events), we speak in this case of a right Zeno solution. In case $\alpha < 1$, the same reasoning applies backwards in time and we have a left Zeno solution. In the examples given above, Fuller's system has right Zeno solutions, whereas Filippov's example admits left Zeno trajectories.

In both examples discussed in Section 2, the switching surfaces are invariant and actually consist of only finitely many orbits of the spatial transformation group, as expected since the state space dimension is 2 in both cases. Because the orbit return mapping acts on a finite set, it has to be periodic.

5. Zeno solutions in relay systems

Consider the class of systems (7) under the conditions (8). The conditions imply that b is a right eigenvector of K with eigenvalue -1, and c^{T} is a left eigenvector with eigenvalue λ. We know from Proposition 8 that at most one of the Markov parameters $c^{\mathrm{T}}A^jb$ can be nonzero. The case in which all Markov parameters are zero is not of interest here, and so we require that λ is a negative integer, and that $c^{\mathrm{T}}A^{-\lambda-1}b \neq 0$. Actually the behavior of the system (7) is not affected if we multiply the vector c by a positive constant; therefore, reverting time if necessary, we can without loss of generality assume that $c^{\mathrm{T}}A^{-\lambda-1}b = 1$. This sign of the leading Markov parameter is the one that has been shown in Lootsma, van der Schaft, and Çamlıbel (1999) to ensure uniqueness of solutions in the "forward" sense (without considering left Zeno solutions). A state trajectory set as in the previous section can be constructed by considering a collection of initial conditions which is closed under the transformation group, together with one maximal solution trajectory for each initial condition.

Our purpose is to obtain a sufficient condition for the presence of Zeno solutions on the basis of the study of the orbit return mapping. We will concentrate on solutions of (7) that alternate between "positive" phases in which $c^{\mathrm{T}}x(t) > 0$ and "negative" phases where $c^{\mathrm{T}}x(t) < 0$. Since the dynamical system (7) has the property that $-x(t)$ is a solution whenever $x(t)$ is a solution, the trajectories that alternate between positive and negative phases can also be looked at as trajectories of the hybrid system with resets defined by $\dot{x}(t) = Ax(t) - b$, $c^{\mathrm{T}}x(t) \geq 0$ for all t, and $x(\tau^+) = -x(\tau^-)$ when $c^{\mathrm{T}}x(\tau) = 0$. If $c^{\mathrm{T}}x(0) = 0$, $c^{\mathrm{T}}x(t) > 0$ for $0 < t < \tau$, $c^{\mathrm{T}}x(\tau) = 0$, and $x(\tau^+) = -x(\tau^-)$ is on the same orbit of the spatial transformation group as $x(0)$, then, by the time-scaling symmetry, there exists a solution which starts at $x(\tau^+)$ and which returns to the surface $c^{\mathrm{T}}x = 0$ at a point which is again on the same orbit. In this way, a fixed point of the orbit return mapping of the original system (7) is obtained, so that Zeno solutions may emerge in the way described above. We will concentrate on the route to Zenoness which is based on fixed points of what might be called the half-return mapping. It should be noted, however, that all cycles of this mapping also give rise of fixed points of the orbit return mapping or an iterated version of it, and these provide alternative routes to Zenoness.

By Lemma 10, we are not essentially constrained in the search for fixed points if we assume that the return time τ is equal to 1 (replace if necessary the initial condition $x(0)$ by $F_\tau x(0)$). The solution to (7) on the interval $[0, 1]$ is given by

$$x(t) = \mathrm{e}^{At}x_0 - \int_0^t \mathrm{e}^{A(t-s)}b\,\mathrm{d}s \tag{13}$$

and in particular $x(1) = \mathrm{e}^A(x_0 - (\int_0^1 \mathrm{e}^{-As}\mathrm{d}s)b)$. Since the matrix A is nilpotent, we have in fact

$$\mathrm{e}^A = \sum_{j=0}^{m-1}\frac{A^j}{j!}, \qquad \int_0^1 \mathrm{e}^{-As}\mathrm{d}s = \sum_{j=0}^{m-1}\frac{(-1)^jA^j}{(j+1)!} \tag{14}$$

where m is the nilpotency index of A.

To find fixed points of the orbit return mapping, we therefore can look for initial conditions x_0 that satisfy $c^{\mathrm{T}}x_0 = 0$ and

$$-\alpha^K x_0 = \mathrm{e}^A\left(x_0 - \int_0^1 \mathrm{e}^{-As}\mathrm{d}s\, b\right) \tag{15}$$

for some $\alpha > 0$. A solution exists if the matrix $I + \mathrm{e}^{-A}\alpha^K$ is invertible and $h(\alpha) = 0$ for some $\alpha > 0$, where the function h is defined by

$$h(\alpha) = c^{\mathrm{T}}(I + \mathrm{e}^{-A}\alpha^K)^{-1} \int_0^1 \mathrm{e}^{-As}\mathrm{d}s b. \tag{16}$$

In this way we obtain a characteristic equation for fixed points of the orbit return mapping in the system (7) subject to the conditions (8). Right Zeno solutions are associated with solutions of the equation $h(\alpha) = 0$ that satisfy $\alpha > 1$, whereas left Zeno solutions relate to roots α in $(0, 1)$.

The characteristic equation needs to be supplemented by a viability condition, namely that $c^{\mathrm{T}}x(t) > 0$ must hold for $0 < t < 1$. In view of Proposition 8 we can write

$$c^{\mathrm{T}}x(t) = \sum_{j=1}^{m-1} \frac{1}{j!} c^{\mathrm{T}}A^j x_0 t^j - \frac{1}{(-\lambda)!} t^{-\lambda}. \tag{17}$$

The absence of roots in the interval $(0, 1)$ of this polynomial in t may be verified for instance by constructing its Sturm sequence (Jacobson, 1987); positivity may then be checked by verifying that the first nonzero element in the sequence $c^{\mathrm{T}}Ax_0, c^{\mathrm{T}}A^2x_0, \ldots$ is positive.

Some properties of the function h defined in (16) can be obtained by making use of Proposition 8 in combination with power series developments as in (14). In particular, if the eigenvalues of the matrix K are negative, then $\lim_{\alpha\to\infty} h(\alpha) = (-1)^{\lambda+1}/((-\lambda)!)$. The number $h(1)$ is equal to the coefficient of $x^{-\lambda}$ in the power series development of the function $(1-\mathrm{e}^{-x})/(1+\mathrm{e}^{-x}) = \tanh \frac{1}{2}x$. These coefficients are zero for even values of $-\lambda$ and alternate in sign for odd values. In relation to the situation at $\alpha = 0$, note that the function $h(\alpha)$ defined in (16) can also be written as

$$h(\alpha) = \alpha^{-\lambda} c^{\mathrm{T}}(\alpha^{-K} + \mathrm{e}^{-A})^{-1} \int_0^1 \mathrm{e}^{-As}\mathrm{d}s\, b.$$

It follows that, if the eigenvalues of the matrix K are negative, the function $h(\alpha)$ is of the form $h(\alpha) = \alpha^{-\lambda}/((-\lambda)!) +$ higher-order terms in α. In particular, the lowest-order coefficient is positive. The sign considerations demonstrate, for the case in which the eigenvalues of K are negative, that the function $h(\alpha)$ has roots both in $(0, 1)$ and in $(1, \infty)$ if $-\lambda + 1$ is a multiple of 4.

Under the assumption that the pair (c^{T}, A) is observable, the characteristic equation can be obtained alternatively by applying the linear mappings $c^{\mathrm{T}}A^j$ $(j = 0, \ldots, n-1)$ to both sides of the fixed-point Eq. (15) and making use of the properties in (9) and Proposition 8. We obtain an upper triangular system of linear equations for the variables $c^{\mathrm{T}}A^jx_0$ whose solution depends on α; the additional requirement $c^{\mathrm{T}}x_0 = 0$ then leads to the characteristic equation. For instance, in the case $\lambda = -m = -n$ (note that observability of (c^{T}, A) implies that the nilpotency index of A is equal to the state space dimension), the equations are

$$\left(1 + \alpha^{-n+j}\right) c^{\mathrm{T}}A^jx_0 + \sum_{\ell=1}^{n-j-1} \frac{1}{\ell!} c^{\mathrm{T}}A^{j+\ell}x_0 = \frac{1}{(n-j)!} \tag{18}$$

from which $c^{\mathrm{T}}x_0$ can be obtained in terms of α. The procedure also produces the coefficients needed in (17).

A formal statement of conditions for the presence of Zeno solutions is given in the following theorem.

Theorem 12. *Consider a system of the form (7). Suppose that there exist a matrix K and a negative integer λ such that the conditions in (8) are satisfied, and $c^{\mathrm{T}}A^{-\lambda-1}b = 1$. The system (7) admits a right Zeno solution if there exists $\alpha > 1$ such that*

(i) *the matrix $I + \mathrm{e}^{-A}\alpha^K$ is invertible,*
(ii) *$h(\alpha) = 0$, where h is the function defined in (16),*
(iii) *the polynomial appearing in (17) with x_0 defined by*

$$x_0 = (I + \mathrm{e}^{-A}\alpha^K)^{-1} \int_0^1 \mathrm{e}^{-As}\mathrm{d}s\, b \tag{19}$$

is positive on the interval $(0, 1)$.

If the same conditions hold for a number $\alpha \in (0, 1)$, then the system admits a left Zeno solution.

Proof. Assume that the conditions hold with $\alpha > 1$. Define x_0 as in (19). The function $x(t)$ defined by (13) on the interval $0 \le t \le 1$ is a solution of (7) and satisfies $x(0) = x_0$, $c^{\mathrm{T}}x(0) = 0$, $c^{\mathrm{T}}x(1) = 0$, and $c^{\mathrm{T}}x(t) > 0$ for $0 < t < 1$. Moreover, the condition $h(\alpha) = 0$ implies that $x(1) = -\alpha^K x(0)$. By the time-scaling invariance due to the conditions (8), the function $y(t) = \alpha^K x(\alpha t)$ $(0 \le t \le \frac{1}{\alpha})$ is a solution of (7) as well. On account of the sign symmetry and the time-homogeneity of (7), the same is true for the function $z(t) = -y(t-1)$ $(1 \le t \le 1 + \frac{1}{\alpha})$. The concatenation of $x(t)$ $(0 \le t \le 1)$ and $z(t)$ $(1 \le t \le 1 + \frac{1}{\alpha})$, which with some abuse of notation we denote again by $x(t)$, is a solution of (7) on the time interval $0 \le t \le 1 + \frac{1}{\alpha}$; in particular, note that $x(1) = z(1)$ so that continuity is ensured. The newly defined function $x(t)$ satisfies $x(1 + \frac{1}{\alpha}) = \alpha^{2K}x(0)$. Extend $x(t)$ once more by defining $x(t) = \alpha^{2K}x(\alpha^2(t - 1 - \frac{1}{\alpha}))$ for $1 + \alpha^{-1} \le t \le 1 + \alpha^{-1} + \alpha^{-2} + \alpha^{-3}$; the function constructed in this way is again a solution of (7). Going on in this way, we find a solution that undergoes switches at times $\sum_{i=0}^{k} \alpha^{-i}$ for $k = 0, 1, 2, \ldots$. Since $\alpha > 1$, the event times converge to a finite limit and we find a right Zeno solution. In the case $0 < \alpha < 1$ we can follow the same trajectory in the reverse time direction and in this way obtain a left Zeno solution. □

Example 13. The following system was studied by Pogromsky et al. (2003):

$$\dot{x}_1(t) = x_2(t) \tag{20a}$$
$$\dot{x}_2(t) = x_3(t) \tag{20b}$$
$$\dot{x}_3(t) = -\operatorname{sgn} x_1(t). \tag{20c}$$

The system satisfies (8) with $K = \operatorname{diag}(-3, -2, -1)$ and $\lambda = -3$. The sign considerations above already demonstrate the presence of roots both in $(0, 1)$ and in $(1, \infty)$. In this case, because the expression (17) is a third-degree polynomial with negative leading coefficient having zeros in 0 and 1, the condition $c^{\mathrm{T}}Ax_0 > 0$ is necessary and sufficient for positivity of $c^{\mathrm{T}}x(t)$ on the interval $(0, 1)$. From the Eq. (18) we find $(1 + \alpha^{-2})c^{\mathrm{T}}Ax_0 = \frac{1}{2} - \frac{\alpha}{\alpha+1}$ so that $c^{\mathrm{T}}Ax_0 > 0$ for $\alpha < 1$. This shows that at least one left Zeno solution is present, as was already established in Pogromsky et al. (2003) by a different method. Since $c^{\mathrm{T}}Ax_0 < 0$ for $\alpha > 1$, no right Zeno solutions arise in this way. The characteristic function is

$$h(\alpha) = \frac{\alpha^3(\alpha^2 - 3\alpha + 1)}{6(1+\alpha^2)(1+\alpha^3)}$$

which shows more specifically that there is exactly one root in the interval $(0, 1)$, namely $\alpha = \frac{3}{2} - \frac{1}{2}\sqrt{5}$.

6. Conclusion

Dimension reduction by means of the Poincaré mapping is a well known technique in the analysis of dynamical systems, in particular in the study of periodic solutions. This paper has shown that under certain conditions a second dimension reduction is possible. The technique has been applied to give sufficient conditions for the presence of Zeno solutions in a class of linear systems with relay feedback.

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