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On the Zeno behavior of linear complementarity systems

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Abstract

In this paper, the so-called Zeno phenomenon is addressed for linear complementarity systems which are interconnections of linear systems and complementarity conditions. We present some sufficient conditions for absence of Zeno behavior. It is also shown that the zero state, which is the most obvious candidate for being a Zeno state, cannot be a Zeno state in certain cases.

1 Introduction

The occurrence of an infinite number of mode transitions in hybrid systems is called Zeno behavior by referring to ancient Greek philosopher Zeno's paradoxes[3]. One may face Zeno phenomenon in modeling, controlling and simulating hybrid systems. For instance, it may occur in the optimal control involving chattering, variable structure systems and relay control systems (see [11] and references therein). It has particular importance in the (event-tracking) simulation of hybrid systems since existence of Zeno behavior causes a lot of problems.

In this paper, we study Zeno behavior of a class of hybrid systems, namely linear complementarity systems (LCS). For a treatment of this phenomenon in a more general framework, we refer to [11, 12]. An LCS consists of a linear system and the so-called complementarity conditions. The most typical examples of this class are linear electrical networks with ideal diodes. In this case, existence of Zeno behavior means that the diodes change their states (from conducting to blocking or vice versa) infinitely many times in finite time.

We will be particularly interested in LCS for which the underlying linear systems is passive. For such systems, it has been shown (see e.g. [9]) that there are no left accumulation points of event times. A generalization of passive systems, namely the systems that are passifiable by pole shifting, will play a key role in the subsequent development. These systems do not have left accumulation points either. A subclass of them, for which passifiability is invariant under time reversion, does not exhibit Zeno behavior.

In what follows, the notational conventions that will be in force throughout the paper are introduced. The symbol $\mathbb{R}$ denotes the real numbers, $\mathbb{C}$ complex numbers. All vector inequalities must be understood componentwise. Let $A \in \mathcal{X}^{n\times m}$ be a matrix of the elements of the set $\mathcal{X}$. We write A_{ij} for the (i,j)th element of A. The transpose of A is denoted by $A^\top$. For an integer n, $\overline{n} = \{1, 2, \ldots, n\}$. For $J \subseteq \overline{n}$, and $K \subseteq \overline{m}$, A_{JK} denotes the submatrix $\{A_{jk}\}_{j\in J, k\in K}$. If $J = \overline{n}$ ($K = \overline{m}$), we also write $A_{\bullet K}$ ($A_{J\bullet}$). For the vectors x and y, we write $x \perp y$ if $x^\top y = 0$. A matrix $A \in \mathbb{R}^{n\times n}$ (not necessarily symmetric) is said to be positive (nonnegative) definite if $x^\top Ax > 0$ ($x^\top Ax \geq 0$) for all $0 \neq x \in \mathbb{R}^n$. Given two matrices $A \in \mathcal{X}^{n_a\times m}$ and $B \in \mathcal{X}^{n_b\times m}$, the matrix obtained by stacking A over B is denoted by $\mathrm{col}(A, B)$. The set of all functions that are defined from the set $\mathcal{A}$ to $\mathcal{B}$ is denoted by $\mathcal{F}(\mathcal{A}, \mathcal{B})$. The symbol $\mathcal{L}_2(\Omega)$ stands for all Lebesgue measurable square-integrable functions that are defined on Ω.

2 Hybrid systems and Zeno behavior

Very roughly speaking, a hybrid system consists of a number of *modes* and *mode transition rules* (see e.g. [17]). Each mode has its own dynamics. The mode transition rules specify when mode changes (called *events*) may and/or must occur, which mode will be *active* after the mode change and *re-initialization* of the state of the system during the mode change. By referring to Zeno's paradox of Achilles and the turtle, the behavior of the system in the case of a presence of an accumulation point of the event times is called *Zeno behavior*. The most typical example is the bouncing ball. Consider a ball bouncing on a surface for which the bounces are instantaneous with a restitution coefficient $e \in (0, 1)$. Let the variable q denote the distance between the surface and the ball. There is only one mode whose dynamics can be given by $\ddot{q} = -1$. An event occurs at time τ if and only if $q(\tau) = 0$ and $\dot{q}(\tau) \leq 0$. The state of the system is re-initialized according to $\dot{q}(\tau) = -e\dot{q}(\tau^-)$. Elementary calculation shows that the event times have an accumulation point. For instance, if the initial state is

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[3]The most well-known of these four paradoxes dealing with counterintuitive aspects of space and time is Achilles and the turtle.

$(q(0), \dot{q}(0)) = (2, 0)$ then the event times can be computed as $2, 2+4e, 2+4e+4e^2, \cdots$. Hence, event times have an accumulation point at $2\frac{1+e}{1-e}$. In this paper, we will investigate Zeno behavior of a special class of hybrid systems, namely *linear complementarity systems*.

3 Linear complementarity systems

Consider a continuous-time, linear and time-invariant system given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$ and A, B, C, and D are matrices with appropriate sizes. We denote (1) by $\Sigma(A, B, C, D)$. A triple $(u, x, y) \in \mathcal{L}_2^{m+n+m}(t_0, t_1)$ is said to be an *$\mathcal{L}_2$-solution* on (t_0, t_1) of $\Sigma(A, B, C, D)$ with the initial state x_0 if it satisfies (1a) in the sense of Carathéodory, i.e., for almost all $t \in (t_0, t_1)$,

$$x(t) = x_0 + \int_{t_0}^{t} [Ax(s) + Bu(s)]ds \tag{2}$$

and (1b) holds almost everywhere. We get a linear complementarity system (LCS) by making the u and y variables subject to the *complementarity* conditions

$$0 \leq u(t) \perp y(t) \geq 0. \tag{3}$$

The overall system (i.e., (1) and (3) together) will be denoted by LCS(A, B, C, D) throughout the paper. The LCS has been introduced in [15] and further studied in [16]. Two of the main themes of study have been the well-posedness (in the sense of existence and uniqueness of solutions) and regularity of the solutions (see e.g. [4, 5, 9, 10]). Zeno behavior is one of the issues that makes the analysis of LCSs difficult. By means of the following example, we illustrate that a LCS may exhibit Zeno behavior.

Example 3.1 Consider the following example (its time-reversed version is due to Filippov [7, p. 116])

$$\dot{x}_1 = -\operatorname{sgn} x_1 + 2\operatorname{sgn} x_2$$
$$\dot{x}_2 = -2\operatorname{sgn} x_1 - \operatorname{sgn} x_2$$

where $\operatorname{sgn} x$ is the set-valued function given by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

As shown in [8, 14], this type of systems can be cast as LCS. Solutions of the system are spiraling towards the origin, which is an equilibrium. Since

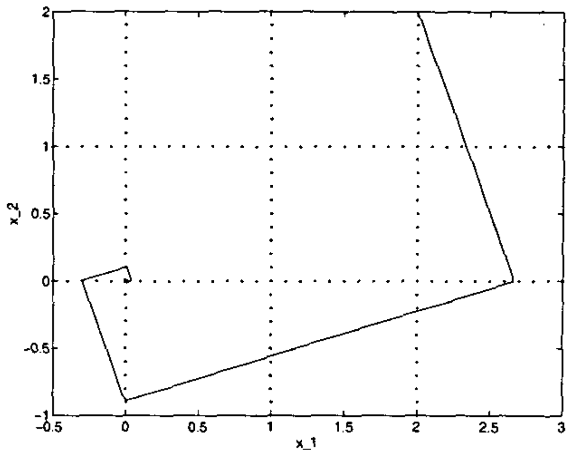

Figure 1: Trajectory with initial state $(2, 2)^\top$.

$\frac{d}{dt}(|x_1(t)| + |x_2(t)|) = -2$ when $x(t) \neq 0$ along trajectories x of the system, solutions reach the origin in finite time (see Figure 1 for a trajectory). Every crossing from one quadrant to another corresponds to an event. Therefore, on a finite time interval there are infinitely many events, i.e., the system exhibits Zeno behavior.

In the sequel, we show that for a class of LCS Zeno behavior can be ruled out. This will be done by employing the passivity notion and the existing results on linear passive complementarity systems (LCS when the underlying linear system is passive). We shall first recall the notion of passivity and summarize some of the previous results in order to be self-contained.

4 Passive systems

Ever since it was introduced in system theory by V. M. Popov, the notion of passivity has played an important role in various contexts such as stability issues, adaptive control, identification, etc. Particularly, the interest in stability issues led to the theory of dissipative systems [19] due to J. C. Willems. In this paper, we will utilize passivity in the analysis of a class of hybrid systems. First, let us recall what passivity means.

Definition 4.1 [19] The system $\Sigma(A, B, C, D)$ given by (1) is said to be *passive (dissipative with respect to the supply rate $u^\top y$)* if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ (called a *storage function*), such that

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t) y(t) dt \geq V(x(t_1)) \tag{4}$$

holds for all t_0 and t_1 with $t_1 \geq t_0$, and for all $\mathcal{L}_2$-solutions $(u, x, y) \in \mathcal{L}_2^{m+n+m}(t_0, t_1)$ of $\Sigma(A, B, C, D)$.

The inequality (4) is sometimes called the *dissipation inequality*. Next, we quote a very well-known characterization of passivity.

Theorem 4.2 [19] Assume that (A, B, C) is minimal. Let $G(s) := C(sI - A)^{-1}B + D$ be the transfer matrix of $\Sigma(A, B, C, D)$. Then the following statements are equivalent:

1. $\Sigma(A, B, C, D)$ is passive.

2. The matrix inequalities

$$K = K^\top > 0 \text{ and } \begin{pmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{pmatrix} \leq 0$$

   have a solution.

3. $G(s)$ is positive real, i.e., $G(\lambda) + G^\top(\bar{\lambda}) \geq 0$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$.

Moreover, $V(x) = \frac{1}{2}x^\top Kx$ defines a quadratic storage function if and only if K satisfies the above system of linear matrix inequalities.

The equivalence of the statements 2 and 3 is sometimes called the positive real lemma or the Kalman-Yakubovich-Popov lemma (see e.g. [13, p.406]). We will often employ the following generalization of the passivity notion in the sequel.

Definition 4.3 [3] A system $\Sigma(A, B, C, D)$ is said to be *passifiable by pole shifting* if there exists a $\rho \in \mathbb{R}$ such that $\Sigma(A + \rho I, B, C, D)$ is passive.

Before giving necessary and sufficient conditions for passifiability by pole shifting, we state the following assumption which will often be used later on.

Assumption 4.4 (A, B, C) is a minimal representation and $\mathrm{col}(B, D + D^\top)$ has full column rank.

Necessary and sufficient conditions for passifiability by pole shifting are in order.

Theorem 4.5 [2, Theorem 3.4.3] *Consider a matrix 4-tuple (A, B, C, D) satisfying Assumption 4.4. Let E be such that* $\ker E = \{0\}$ *and* $\mathrm{im}\, E = \ker\,(D + D^\top)$. *Then $\Sigma(A, B, C, D)$ is passifiable by pole shifting if and only if D is nonnegative definite and $E^\top CBE$ is symmetric positive definite.*

5 Solution concepts for LCS

In this section, we work with two different solution concepts for LCS. The first one represents a physical system modelling point of view.

Definition 5.1 A triple $(u, x, y) \in \mathcal{L}_2^{m+n+m}([0, T])$ is called an $\mathcal{L}_2$-*solution* on $[0, T]$ of $\mathrm{LCS}(A, B, C, D)$ with the initial state x_0 if it is an $\mathcal{L}_2$-solution of $\Sigma(A, B, C, D)$ and for almost all $t \in [0, T]$, $0 \leq u(t) \perp y(t) \geq 0$.

In what follows, we will state a rather trivial fact about the *time-reverse* system. The *time-reverse* operator $\mathrm{rev}_{[t', t'']} : \mathcal{L}_2([t', t'']) \to \mathcal{L}_2([t', t''])$ is defined by

$$(\mathrm{rev}_{[t', t'']}v)(t) = v(t' + t'' - t).$$

Fact 5.2 If $(u, x, y) \in \mathcal{L}_2^{m+n+m}([0, T])$ is an $\mathcal{L}_2$-solution on $[0, T]$ of $\Sigma(A, B, C, D)$ then $\mathrm{rev}_{[0,T]}(u, x, y)$ is an $\mathcal{L}_2$-solution on $[0, T]$ of $\Sigma(-A, -B, C, D)$.

Another rather obvious fact concerning pole shifting will be used later.

Fact 5.3 If the triple $t \mapsto (u(t), x(t), y(t))$ is an $\mathcal{L}_2$-solution on some finite interval of $\mathrm{LCS}(A, B, C, D)$ with some initial state then $t \mapsto e^{\rho t}(u(t), x(t), y(t))$ is an $\mathcal{L}_2$-solution on the same interval of $\mathrm{LCS}(A + \rho I, B, C, D)$ with the same initial state.

Our second solution concept represents a hybrid system point of view. We begin with the definition of the event times set.

Definition 5.4 A set $\mathcal{E} \subset \mathbb{R}_+$ is called an *admissible event times set* if it is closed and countable, and $0 \in \mathcal{E}$. To each admissible event times set $\mathcal{E}$, we associate a collection of intervals between events $\tau_{\mathcal{E}} = \{(t_1, t_2) \subset \mathbb{R}_+ \mid t_1, t_2 \in \mathcal{E} \cup \{\infty\} \text{ and } (t_1, t_2) \cap \mathcal{E} = \emptyset\}$.

Next, we define a *hybrid* solution concept that is enough to cover the behavior of linear passive complementarity systems.

Definition 5.5 A 5-tuple $(\mathcal{E}, \mathcal{S}, u, x, y)$ where $\mathcal{E}$ is an admissible event times set, $\mathcal{S} : \tau_{\mathcal{E}} \to 2^{\overline{m}}$, and $(u, x, y) \in \mathcal{F}(\mathbb{R}_+, \mathbb{R}^{m+n+m})$ is said to be a *hybrid solution* of $\mathrm{LCS}(A, B, C, D)$ with the initial state x_0 if the following conditions hold.

1. x is continuous, piecewise differentiable and $x(0) = x_0$.

2. For each $\tau \in \tau_{\mathcal{E}}$,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\\ u_{\mathcal{S}(\tau)}(t) &\geq 0\\ y_{\mathcal{S}(\tau)}(t) &= 0\\ u_{\overline{m}\setminus\mathcal{S}(\tau)}(t) &= 0\\ y_{\overline{m}\setminus\mathcal{S}(\tau)}(t) &\geq 0\end{aligned}$$

hold for all $t \in \tau$.

Moreover, we say that a hybrid solution $(\mathcal{E}, \cdot, u, x, y)$ is *redundant* if there exists $t \in \mathcal{E}$ and t', t'' with $t' < t < t''$ such that (u, x, y) is analytic on (t', t''). It is said to be *nonredundant* otherwise.

As stated in the following lemma, every hybrid solution is also an $\mathcal{L}_2$-solution.

Lemma 5.6 *Consider a matrix 4-tuple (A, B, C, D) satisfying Assumption 4.4. Suppose that $\Sigma(A, B, C, D)$ is passifiable by pole shifting. If $(\cdot, \cdot, u, x, y)$ is a hybrid solution of LCS(A, B, C, D) with some initial state then for any $T > 0$ (u, x, y) is an $\mathcal{L}_2$-solution on $[0, T]$ of LCS(A, B, C, D) with the same initial state.*

Proof: It follows from [2, Proposition 3.3.9 and Lemma 3.8.2 item 4].■

We summarize some existence and uniqueness results in the following theorem. To do this, we need to introduce some nomenclature. First, a language for the (possible) accumulation points will be set up.

Definition 5.7 An element t of an admissible set $\mathcal{E}$ is said to be a *left (right) accumulation point* if for all $t' > t$ ($t' < t$) $(t, t') \cap \mathcal{E}$ ($(t', t) \cap \mathcal{E}$) is not empty. An admissible event times set $\mathcal{E}$ is said to be *left (right) Zeno free* if it does not contain any left (right) accumulation points. A hybrid solution is said to be *left (right) Zeno* if the corresponding event times set contains at least one left (right) accumulation point and *non-Zeno* if the corresponding event times set contains no left or right accumulation points.

The concept of the dual cone needs to be introduced as well. For a given nonempty set $\mathcal{S}$, we say that the set $\{v \mid v^\top w \geq 0 \text{ for all } w \in \mathcal{S}\}$ is the *dual cone* of $\mathcal{S}$. It is denoted by $\mathcal{S}^*$. In particular, we will be interested in the dual cone of the set $\mathcal{Q}_D = \{v \in \mathbb{R}^m \mid 0 \leq v \perp Dv \geq 0\}$ for a given $D \in \mathbb{R}^{m \times m}$.

Theorem 5.8 *Consider a matrix 4-tuple (A, B, C, D) satisfying Assumption 4.4. Suppose that $\Sigma(A, B, C, D)$ is passifiable by pole shifting. The following statements hold.*

1. *There exists a nonredundant left Zeno free hybrid solution of LCS(A, B, C, D) if and only if $Cx_0 \in \mathcal{Q}_D^*$.*
2. *There exists at most one $\mathcal{L}_2$-solution for a given initial state.*
3. *There exists at most one nonredundant hybrid solution for a given initial state.*

Moreover, the state trajectory of a hybrid ($\mathcal{L}_2$-)solution of LCS(A, B, C, D) is uniformly continuous.

Proof: *1:* It follows from [2, Corollary 3.4.4 and Proposition 3.3.7].
2: It follows from [2, Corollary 3.4.4].
3: It follows from the definition of hybrid solutions, Lemma 5.6 and the previous item.
The uniform continuity of x-trajectories follows from the proof of [2, Theorem 3.3.3] and [2, Proposition 3.4.4].■

6 Zeno behavior of LCS

The lemma below rules out left accumulation points for systems that are passifiable by pole shifting.

Lemma 6.1 [2, Lemma 3.5.1] *Consider a matrix 4-tuple (A, B, C, D) satisfying Assumption 4.4. Assume that $\Sigma(A, B, C, D)$ is passifiable by pole shifting. Then, there is no nonredundant left Zeno solution of LCS(A, B, C, D).*

As an implication of the previous lemma, passifiable systems for which the passifiability is invariant under *time-reversion* enjoy the non-Zenoness property.

Lemma 6.2 *Consider a matrix 4-tuple (A, B, C, D). Suppose that (A, B, C) is minimal and D is positive definite. Then, there is no nonredundant Zeno solution of LCS(A, B, C, D).*

Proof: Since D is positive definite, col$(B, D + D^\top)$ is of full column rank. Hence, Assumption 4.4 is satisfied by the hypotheses. Furthermore, positive definiteness of D indicates that both $\Sigma(A, B, C, D)$ and $\Sigma(-A, -B, C, D)$ are passifiable by pole shifting due to Theorem 4.5. Note that $\mathcal{Q}_D = \{0\}$ and hence $\mathcal{Q}_D^* = \mathbb{R}^m$. Let $(\mathcal{E}^1, \mathcal{S}^1, u^1, x^1, y^1)$ and $(\mathcal{E}^2, \mathcal{S}^2, u^2, x^2, y^2)$ be hybrid solutions of LCS(A, B, C, D) and LCS$(-A, -B, C, D)$ with some initial state x_0 according to Theorem 5.8 item 1. Lemma 6.1 implies that $\mathcal{E}^1$ is left Zeno free. Suppose that it is not right Zeno free, i.e., there exists a right accumulation point of $\mathcal{E}^1$. Our aim is to construct a left Zeno hybrid solution for LCS$(-A,-B,C,D)$ by using $(\mathcal{E}^1,\mathcal{S}^1,u^1,x^1,y^1)$ and reversing the time. Clearly, this would give a contradiction due to Lemma 6.1 since $\Sigma(-A,-B,C,D)$ is passifiable by pole shifting. Let $t^* \in \mathcal{E}^1$ be a right accumulation point. Define $\mathcal{E} = (t^* - (\mathcal{E}^1 \cap [0,t^*])) \cup (t^* + \mathcal{E}^2)$ where $t+\mathcal{X}$ denotes the set $\{t+t' \mid t' \in \mathcal{X}\}$. Clearly, $\mathcal{E}$ is an admissible event times set and $\tau_{\mathcal{E}} = \tau_{t^*-(\mathcal{E}^1\cap[0,t^*])} \cup \tau_{t^*+\mathcal{E}^2}$. Define $\mathcal{S}$ as

$$\mathcal{S}(\tau) = \begin{cases} \mathcal{S}^1(t^*+\tau) & \text{if } \tau \in \tau_{t^*-(\mathcal{E}^1\cap[0,t^*])} \\ \mathcal{S}^2(t^*-\tau) & \text{if } \tau \in \tau_{t^*+\mathcal{E}^2} \end{cases}$$

and (u,x,y) by

$$(u,x,y)|_{[0,t^*]} = \mathrm{rev}_{[0,t^*]}(u^1,x^1,y^1) \tag{5}$$

$$(u,x,y)|_{(t^*,\infty)} = (u^2,x^2,y^2). \tag{6}$$

It can be verified that $(\mathcal{E},\mathcal{S},u,x,y)$ is a nonredundant hybrid solution of LCS$(-A,-B,C,D)$ with the initial state $x^1(t^*)$. Since $\Sigma(-A,-B,C,D)$ is passifiable by pole shifting, Lemma 6.1 reveals that $\mathcal{E}$ is left Zeno free. By construction t^* is a left accumulation point of $\mathcal{E}$ since it is a right accumulation point of $\mathcal{E}^1$. We reach a contradiction therefore. ∎

Remark 6.3 We can use input scaling to weaken the positive definiteness condition on the matrix D in Lemma 6.2. To do this, consider the LCS$(A,B\Lambda,C,D\Lambda)$ where Λ is a positive definite diagonal matrix. Clearly, if (u,x,y) is a solution of LCS(A,B,C,D) then $(\Lambda^{-1}u,x,y)$ is a solution of LCS$(A,B\Lambda,C,D\Lambda)$ and vice versa. So, Lemma 6.2 can be applied to systems with a *diagonally stable* [1] D matrix. One may also think of utilize output scaling to weaken the condition more. However, as shown in [6, Theorem 3.3.9], it does not yield anything extra.

Note that *Zeno states* (i.e., the states at the accumulation points) are well-defined due to the fact that x is uniformly continuous (see Theorem 5.8) for a hybrid solution $(\cdot,\cdot,\cdot,x,\cdot)$. Intuitively, the most natural candidates for Zeno states are equilibrium states, in particular the zero state, of the system. The following theorem indicates that the zero state cannot be a Zeno state for a class of passifiable complementarity systems. The relation between the equilibrium states and Zeno states can be found in [18] for a fairly large class of hybrid systems.

Theorem 6.4 *Consider a matrix 4-tuple (A,B,C,D) satisfying Assumption 4.4. Assume that $\Sigma(A,B,C,D)$ is passifiable by pole shifting and there exists an index set $J \subseteq \overline{m}$ such that D_{JJ} is positive definite, $D_{J,\overline{m}\setminus J} = 0$, $D_{\overline{m}\setminus J,J} = 0$ and $D_{\overline{m}\setminus J,\overline{m}\setminus J}$ is skew-symmetric. Let $(\mathcal{E},\cdot,\cdot,x,\cdot)$ be a nonredundant hybrid solution of LCS(A,B,C,D) with some initial state. If t^* is a right accumulation point of $\mathcal{E}$ then $x(t^*) \neq 0$.*

[1] A matrix A is said to be *diagonally stable matrix* if there exists a positive definite diagonal matrix X such that AX is positive definite. See [1] for a detailed discussion of this class of matrices.

Proof: Let $(\mathcal{E},\mathcal{S},u,x,y)$ be a nonredundant hybrid solution of LCS(A,B,C,D) with some initial state such that t^* is a right accumulation point of $\mathcal{E}$ and $x(t^*)=0$. According to Lemma 5.6, (u,x,y) is an $\mathcal{L}_2$-solution on $[0,t^*]$ of LCS(A,B,C,D) with the same initial state and hence

$$u_i(t)y_i(t) = 0 \tag{7}$$

for almost all t and for all $i \in \overline{m}$. Define $(\bar{u},\bar{x},\bar{y}) = \mathrm{rev}_{[0,t^*]}(u,x,y)$. From Fact 5.2, we know that $(\bar{u},\bar{x},\bar{y})$ is an $\mathcal{L}_2$-solution on $[0,t^*]$ of $\Sigma(-A,-B,C,D)$ with the initial state $\bar{x}(0) = x(t^*) = 0$. Define $z_J = \bar{y}_J$ and $z_{J^c} = -\bar{y}_{J^c}$ where $J^c = \overline{m}\setminus J$. Clearly, the triple

$$(u^*,x^*,y^*) := \left(\begin{pmatrix}\bar{u}_J\\ \bar{u}_{J^c}\end{pmatrix}, \bar{x}, \begin{pmatrix}z_J\\ z_{J^c}\end{pmatrix}\right)$$

is an $\mathcal{L}_2$-solution on $[0,t^*]$ of the system Σ_0^* where

$$\Sigma_\rho^* = \Sigma(-A+\rho I, \begin{pmatrix}-B_{\bullet J} & -B_{\bullet J^c}\end{pmatrix}, \begin{pmatrix}C_{\bullet J}\\ -C_{\bullet J^c}\end{pmatrix}, \begin{pmatrix}D_{JJ} & 0\\ 0 & -D_{J^cJ^c}\end{pmatrix}).$$

On the other hand, the passifiability of $\Sigma(A,B,C,D)$ immediately implies that the system (obtained by renumbering of inputs and outputs)

$$\Sigma' = \Sigma(A, \begin{pmatrix}B_{\bullet J} & B_{\bullet J^c}\end{pmatrix}, \begin{pmatrix}C_{\bullet J}\\ C_{\bullet J^c}\end{pmatrix}, \begin{pmatrix}D_{JJ} & 0\\ 0 & D_{J^cJ^c}\end{pmatrix})$$

is passifiable. This means that the system Σ_0^* is also passifiable since $D_{J^cJ^c}$ is skew-symmetric. Let ρ be such that Σ_ρ^* is passive. It follows from Fact 5.3 that $e^{\rho\cdot}(u^*,x^*,y^*)$ is an $\mathcal{L}_2$-solution of Σ_ρ^* with zero initial state. Then, the dissipation inequality shows that for all $t \in [0,t^*]$

$$\int_0^t e^{2\rho s}(u^*(s))^\top y^*(s)\,ds \geq e^{2\rho t}(x^*(t))^\top K x^*(t) \tag{8}$$

where $x \mapsto x^\top Kx$ is a storage function for the system Σ_ρ^*. It follows from (7) and the definition of y^* that

$$\begin{aligned}\int_0^t e^{2\rho s}(u^*(s))^\top y^*(s)\,ds &= \sum_{i=1}^m \int_0^t e^{2\rho s}u_i^*(s)y_i^*(s)\,ds\\ &= \sum_{i\in J}\int_0^t e^{2\rho s}u_i(s)y_i(s)\,ds - \sum_{i\in J^c}\int_0^t e^{2\rho s}u_i(s)y_i(s)\,ds\\ &= 0.\end{aligned} \tag{9}$$

Therefore, $x^*(t) = 0$ for all $t \in [0,t^*]$ due to (8) and the positive definiteness of K. However, this means that $x(0) = x^*(t^*) = 0$ and hence $(u,x,y) \equiv 0$ since

(u, x, y) is the unique $\mathcal{L}_2$-solution of LCS(A, B, C, D) with the zero initial state. Consequently, $\mathcal{E} = \{0\}$ since the hybrid solution is nonredundant. Hence, we reach a contradiction. ■

Although the hypothesis of the above theorem on the special structure of the D matrix is rather restrictive, it is satisfied for instance when D is skew-symmetric. In particular, lossless systems (see [19]) are covered.

7 Conclusions

Two solution concepts for linear complementarity systems are discussed briefly. After establishing existence and uniqueness of solutions for a class of LCS for which the underlying linear system enjoys the property of passifiability by pole shifting, we have shown that Zeno behavior cannot arise for a subclass of systems that are passifiable by pole shifting and for which passifiability is invariant under time reversion. For a larger subclass, it has been shown that the zero state, which is the most natural candidate, cannot be a Zeno state. The problem of showing absence of Zeno behavior of linear complementarity systems in general is currently under consideration.

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