

# Multi-Period Risk Sharing under Financial Fairness

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## Abstract

We work with a multi-period system where a finite number of agents need to share multiple monetary risks. We look for the solutions that are both Pareto efficient utility-wise and financially fair value-wise. A buffer enables the intertemporal capital transfer. Expected utility is used to evaluate the utility, and a risk-neutral measure is essential for determining the risk sharing rules. It can be shown that in the model setting there always exists a unique risk sharing rule that is both Pareto efficient and financially fair. An iterative algorithm is introduced to calculate this rule numerically.

*Keywords:* Intertemporal risk sharing, Pareto efficiency, financial fairness, contract design

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## 1. Introduction

This paper explores the intertemporal risk sharing in a multi-period setting under the notion of *Pareto efficiency and financial fairness* (PEFF). Pareto efficiency means that the utility of nobody can be improved without hurting the utility of some others, while financial fairness indicates that the market values of the risk positions before and after risk sharing should be equal. A risk-sharing system with respect to monetary uncertainties – the stochastic returns from the financial market, for instance – can be viewed as a financial contract. On the one hand, Pareto efficiency is fundamental in risk-sharing systems, while on the other hand financial fairness is important in the design of financial contracts.

The model is motivated and abstracted from systems that allow for intertemporal risk sharing. One example is the collective defined-contribution pension systems which can be viewed as a multilateral financial contract among both current and future cohorts. The possibility of intertemporal risk sharing with respect to investment risk is due to the incompleteness of the market, i.e. the inability of generations to be exposed to risks outside their own (mature) lifespan. A risk-sharing system tries to partly fix this problem by allowing later generations to take risks before they become participants. Risk sharing shall result in welfare gains to the generations; meanwhile, the pension contract should also be fair from a valuation perspective. Another example is the reinsurance market, in which insurance companies reallocate the risks by way of reinsurance contracts among themselves. A multi-period contract is appropriate for dealing with long-term risks, or simply when companies agree to make multi-period arrangements. A similar example is the design of structured derivatives, for instance, the practice of tranching. In these examples, Pareto efficiency is pertinent for designing the optimal allocation of risks, while financial fairness guarantees that the contract is fairly priced.

The characterization of Pareto efficient solutions in a single-period setting is well studied in a lot of papers, which date back to the 1960s with the focus mainly on the field of insurance. For instance, Borch [1] gives a characterization

of the Pareto efficient solutions under the situation where expected utility is used to describe the agents' risk preferences, and later DuMouchel [2] gives proof to these results. Similar work also includes Raviv [3] which takes into consideration the existence of market frictions. The fairness criterion is first considered alongside the Pareto efficiency by, amongst others, Gale [4], Bühlmann and Jewell [5] and Balasko [6] in different settings. In these literature, the risk sharing is built over both a utility basis and a valuation basis.

The risk sharing problem in a multi-period setting is investigated by Barrieu and Scandolo [7] in a general setting; they talk about risk exchanges between two agents over more than one period without taking into consideration any fairness conditions. Other work has been mainly focused on the design of pension systems and the space of intergenerational risk sharing, where risk redistribution can be organized among both the existing and future cohorts. Pareto-efficient risk sharing can be achieved by maximizing the aggregate expected utility of all generations in the situation where a social planner is present (e.g. Gordon and Varian [8], Gollier [9], Bovenberg and Mehlkopf [10]) or by looking for an equilibrium (see Ball and Mankiw [11], Krueger and Kubler [12]). Financial fairness has been considered by Cui et al. [13]; however, the valuation approach is only used to check afterwards whether the distribution rule is fair for the participants. Kleinow and Schumacher [14] analyze the pension system with conditional indexation from the perspective of market value; they investigate whether the pension contract is financially fair for existing and incoming cohorts as well as the sponsor. Risk-neutral valuation becomes essential in Bovenberg and Mehlkopf [10] to determine a unique risk sharing solution by setting the ex-ante market values of the intergenerational transfers to zero.

This paper explores the Pareto efficient and financially fair risk sharing in a multi-period environment. Expected utility is adopted to evaluate the welfare, and a risk-neutral measure works for the valuation purpose. We shall show the existence and uniqueness of the PEFF solution, and give a numerical algorithm to find it. This can be seen as a direct generalization of the research by Pazdera et al. [15], which explores the Pareto efficient and financially fair risk-sharing

rule in a single-period case. Compared to Barrieu and Scandolo [7], we restrict ourselves to the case of expected utility as the preference functional, and risk-neutral valuation is built into the system to determine the uniqueness of the solution. Different from Bovenberg and Mehlkopf [10], no parameterization on the risk-sharing rules is needed here; the rules are determined totally under the notion of PEFF. Mathematically, our results resemble the famous consumption-savings model for intertemporal substitution to some extent. The intertemporal balance equation, as we call it, has a close relationship with the Euler equation in the intertemporal substitution theory; see Hall [16]. The main difference is that the model here introduces no subjective discount factor for impatience. The characterization of Pareto efficiency leads to a weighted optimization problem where the weights are unknowns to be determined uniquely by the financial fairness constraints, making use of a risk-neutral measure.

The rest of the paper is structured as follows. The model setting is set up in Section 2 and we formulate the problem of finding PEFF solutions mathematically. Next we establish the existence and uniqueness of the solution in Section 5. Explicit solution exists when we assume exponential utility functions to all the agents and deterministic asset returns; other than that, there appears to be no hope for an explicit solution in general. We then develop an iterative algorithm to numerically find the solution. The case of the explicit solution is dealt with in Section 7; besides, we also give a simple example where the numerical algorithm is implemented. Some remarks will conclude the paper in the end.

## 2. Model Framework

We assume a finite discrete-time system in which a finite number of agents gather to share their risks. As a result of the risk sharing, the agents expect to receive contingent payments from the system. Each agent is assumed to get one single contingent payment. The term “contingent payment” is general and can have various interpretations in different circumstances. For instance, it can refer to the risk exposure of a insurance company after risk sharing in the case of a

reinsurance contract, or the investment risk in the case of a collective pension fund. Alongside there is also a long-lived *buffer* which makes the intertemporal money transfer possible.

The system starts at time  $t_0$ . Assume that altogether there are  $N$  contingent  
 95 payments happening at time  $t_1 \leq t_2 \leq \dots \leq t_N$ , where  $N$  is some positive integer.  $C_n$  will stand for the contingent payment paid out from the system at time  $t_n$ . Let  $F_n$  be the buffer size at time  $t_n$ .  $X_n$  denotes the financial risk coming into the system from the agents from time  $t_{n-1}$  to  $t_n$ , that is, it is the sum of all the stochastic cash inflows from the agents from time  $t_{n-1}$  to  $t_n$ . The  
 100 risk stream  $X = (X_1, \dots, X_N)$  is defined in a financial market in which prices are given exogenously. The buffer is invested in a risky asset  $R$  which produces stochastic per-dollar gross return  $R_n$  from time  $t_{n-1}$  to  $t_n$ . Here the  $C_n$ 's and  $F_n$ 's are decision variables, and the  $X_n$ 's and  $R_n$ 's are the risks to be shared.

The  $X_n$ 's and  $R_n$ 's are random variables defined on a finite probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  where  $\mathbb{P}$  is the objective measure.  $\mathbb{F}$  is the filtration generated by the  $X$ 's and  $R$ 's:

$$\mathbb{F} = \{\mathcal{F}_n | n = 1, \dots, N\}, \quad \mathcal{F}_n = \sigma\{(X_1, R_1), \dots, (X_n, R_n)\}.$$

There is also a risk-neutral measure  $\mathbb{Q}$  defined on the probability space besides the objective measure  $\mathbb{P}$ . There is no need to assume the completeness of the market; any given risk-neutral measure  $\mathbb{Q}$  will suffice. The only assumption is that the agents have agreed to adopt some probability measures  $\mathbb{P}$  and  $\mathbb{Q}$ , or the measures are simply specified in a situation where a social planner is present. Let

$$\mathbb{E}_n[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_n].$$

It is assumed that the process  $\{(X_n, R_n)\}$  is sequentially independent under  $\mathbb{P}$  and  $\mathbb{Q}$ , that is,  $(X_t, R_t)$  and  $(X_s, R_s)$  are independent for  $t \neq s$  under  $\mathbb{P}$  and  $\mathbb{Q}$ . For  $n = 1, \dots, N$ , the random variables  $X_n$  and  $R_n$  need not be independent, and their joint distribution is known. As we are working on a finite probability space, the total number of outcomes of  $(X_n, R_n)$  is finite for all  $n$ . Illustrated

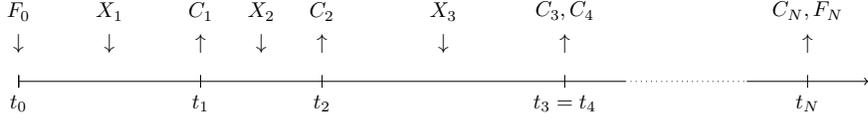


Figure 1: The risk sharing system

by Figure 2, the risks can be seen as a multinomial tree and every pair  $(X_n, R_n)$  can then be totally characterized by

$$\left\{ ((X_n^{j_n}, R_n^{j_n}), \mathbb{P}(j_n), \mathbb{Q}(j_n)) \mid j_n = 1, \dots, m_n \right\}$$

where  $(X_n^{j_n}, R_n^{j_n})$  represents all the possible and distinct values of  $(X_n, R_n)$  and  $\mathbb{P}(j_n), \mathbb{Q}(j_n)$  are the corresponding  $\mathbb{P}$ - and  $\mathbb{Q}$ -probabilities. A technical requirement is that for any  $n = 1, \dots, N$

$$\mathbb{Q}(\{\omega \in \Omega \mid X_n(\omega) = \max X_n, R_n(\omega) = \max R_n\}) > 0, \quad (2.1)$$

which means that  $X_n$  and  $R_n$  can attain their maximum under  $\mathbb{Q}$  simultaneously.

105 Furthermore we assume that  $R_n > 0$  for all  $n$  as the  $R$ 's have the interpretation as the gross return of the asset  $R$ .

Write  $J_n = j_1 j_2 \dots j_n$  as the trajectory  $((X_1^{j_1}, R_1^{j_1}), \dots, (X_n^{j_n}, R_n^{j_n}))$ . Let  $\mathcal{J}_n$  be the set of all the possible trajectories of  $(X, R)$  up to time  $t_n$ .  $J_n j_{n+1}$  will denote any trajectory whose up-to-time- $t_n$  part is  $J_n$ . In such a situation  
110 we write  $j_{n+1} \in \mathcal{J}_n^{n+1}$  where  $\mathcal{J}_n^{n+1}$  denotes the set of all the possible cases of  $(X_{n+1}, R_{n+1})$ .

The risk-neutral measure  $\mathbb{Q}$  is used to price the risks  $X$  as well as the investment returns  $R$ . In this generic setting, write

$$x_n := \mathbb{E}^{\mathbb{Q}} X_n, \quad 1 + r_n := \mathbb{E}^{\mathbb{Q}} R_n, \quad n = 1, \dots, N.$$

The  $x_n$ 's are the market prices of the risks  $X$  and the  $r_n$ 's are the risk-free returns implied by the pricing measure  $\mathbb{Q}$ . Please note that now and later we directly work with future values for convenience.

115 Note that the time points  $\{t_0, t_1, \dots, t_N\}$  need not be equidistant. As shown in Figure 1, two or more time points can be equal if there are more than one

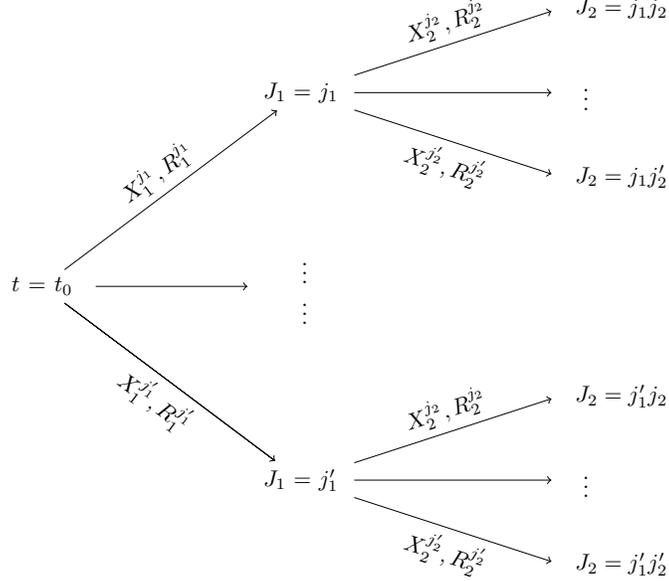


Figure 2: The first two periods of the multinomial tree for the risks

contingent payment paid out at the same time. In that case, say  $t_{n-1} = t_n$  for some  $n$ , we shall have  $X_n \equiv 0$  and  $R_n \equiv 1$ , because there will be no risks coming in and the buffer will not evolve with respect to asset return.

120 The utilities of the agents depend solely on the contingent payments they receive. Utility function  $u_n(\cdot)$  will be used to evaluate contingent payment  $C_n$ . The function  $u_n(x)$  is defined on  $x \in (b_n, +\infty)$ , where  $b_n$  is a constant, either a finite real number (e.g. shifted power utility) or  $-\infty$  (e.g. exponential utility). These utility functions are *stereotype utility functions* defined as follows:

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1. it is continuous and differentiable;
  2. it is strictly concave;
  3. the marginal utility satisfies the *Inada conditions*

$$\lim_{x \downarrow b_n} u'_n(x) = +\infty, \quad \lim_{x \rightarrow \infty} u'_n(x) = 0.$$

For any agent, define  $I_n = (u'_n)^{-1}$ , which is the inverse function of the marginal utility function. Since  $u'_n$  satisfies the Inada conditions, we know that

$I_n$  is a strictly decreasing function mapping  $(0, +\infty)$  into  $(b_n, +\infty)$  and is a  
 130 bijection.

The budget constraints of the system are then straightforward: at each time point, the invested capital will be distributed between the buffer and the current contingent payment, i.e.

$$F_n + C_n = X_n + F_{n-1}R_n \quad n = 1, \dots, N. \quad (2.2)$$

The key problem is to determine the decision variables  $C_n$ 's and  $F_n$ 's along each trajectory. The interpretation is to divide the total amount of available asset in the system between the current payment and the buffer for future payments.

It is assumed without loss of generality that

$$F_0 = 0.$$

The budget constraint is

$$C_1 + F_1 = X_1 + F_0R_1 := \tilde{X}_1,$$

which suggests that the situation when  $F_0$  is nonzero or even a random variable  
 135 can always be dealt with by regarding  $X_1 + F_0R_1$  as a new random variable  $\tilde{X}_1$ .

The buffer size by the end of the system,  $F_N$ , will be referred to as the *end buffer*. Depending on whether the end buffer also takes the risks or not, we may have the following two cases:

- *Closed end buffer (CEB)* case:  $F_N$  will be a constant, that is, the buffer will only make the intertemporal transfer possible, but it does not take any risks by the end. Without loss of generality we assume

$$F_N = 0.$$

In the situations where  $F_N$  is supposed to be a nonzero constant, we can then redefine a new random variable  $\tilde{X}_N$  such that

$$C_N = (X_N - F_N) + F_{N-1}R_N := \tilde{X}_N + F_{N-1}R_N.$$

140 • *Open end buffer (OEB)* case:  $F_N$  will be a decision variable just as the  $C$ 's. This means that the buffer provider will also participate in the risk sharing besides acting as a vehicle for intertemporal transfer. In this case, a stereotype utility function  $u_p$  will be employed to evaluate the utility of  $F_N$ . The function  $u_p(x)$  is defined on  $x \in (b_p, +\infty)$ , and  $b_p$  can be either a finite constant or equal to  $-\infty$ .

145 It is worth mentioning that there is no explicit constraint on the interim status of the buffer  $F_n$ ,  $n = 1, \dots, N - 1$ , thus in general they can be positive or negative.

It can be argued as follows that any OEB case can always be converted into a CEB case. For any OEB case  $(C_1, \dots, C_N, F_N)$  with utility functions  $(u_1, \dots, u_N, u_p)$ , we define a new time point  $t_{N+1} := t_N$  with  $X_{N+1} := 0$  and  $R_{N+1} := 1$ . The OEB setting is thus formulated into a CEB one with an extra contingent payment  $C_{N+1}$  with utility  $u_p$

$$C_{N+1} = X_{N+1} + F_N R_{N+1} = F_N.$$

On the other hand, any CEB setting can be turned into an OEB setting in the sense of Pareto efficiency as we shall see later. In this paper we will proceed 150 mainly with the OEB setting. The utility of the end buffer  $F_N$  will be evaluated according to the utility function  $u_p$ .

We will try to determine the  $C$ 's and the  $F$ 's. For any  $n = 1, \dots, N$ , both  $F_n$  and  $C_n$  are by nature  $\mathcal{F}_n$ -measurable random variables. We then have the following important definition.

155 DEFINITION 2.1 (*Risk-sharing rule.*) A vector of random variables  $(C_1, C_2, \dots, C_N, F_N)$  is called a *risk-sharing rule* if it satisfies

- the measurability condition:  $C_n \in \mathcal{F}_n$  for  $n = 1, \dots, N$  and  $F_N \in \mathcal{F}_N$ ,
  - the budget constraints (2.2), and
  - the domain requirements of the utility functions, i.e.  $C_n > b_n$  for all  $n$  and  $F_N > b_p$  along any trajectory.
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One last thing to mention in this section is that the budget constraints (2.2) imply a single *global budget constraint* by eliminating the  $F$ 's:

$$\sum_{n=1}^{N-1} \left[ C_n \left( \prod_{i=n+1}^N R_i \right) \right] + C_N + F_N = \sum_{n=1}^{N-1} \left[ X_n \left( \prod_{i=n+1}^N R_i \right) \right] + X_N. \quad (2.3)$$

This implies that in order to make the problem well-posed, one needs to have that

$$\sum_{n=1}^{N-1} \left[ b_n \left( \prod_{i=n+1}^N R_i \right) \right] + b_N + b_p < \sum_{n=1}^{N-1} \left[ X_n \left( \prod_{i=n+1}^N R_i \right) \right] + X_N.$$

This should hold for any realizations of  $X$  and  $R$  as we now have a finite probability space. Otherwise there will be no possible risk-sharing rules as the domain requirements of the utility functions can never be satisfied.

EXAMPLE 2.2 (*The autarky.*) A trivial solution to the risk sharing problem is the *autarky* where there is no risk-sharing effect: all agents will be on their own and the buffer will be left unused.

EXAMPLE 2.3 (*Possible variations of the model.*) The budget constraint (2.2) shows that the model is very general and can handle different risk sharing systems. Examples are

- if we let

$$\begin{aligned} t_1 &= t_2 = \dots = t_N \\ X_2 &= \dots = X_N \equiv 0 \\ R_2 &= \dots = R_N \equiv 1 \end{aligned}$$

then the system degenerates to a single-period problem as in Pazdera et al. [15] and the budget constraint becomes

$$\sum_{n=1}^N C_n + F_N = X_1$$

where  $X_1$  represents the aggregate risk to be shared.

- If we only let

$$X_2 = \dots = X_N \equiv 0$$

then this represents a *decumulation system* where the only cash inflow  $X_1$  will be distributed into several contingent payments in the future.

- A *defined-contribution pension fund* in the form of a non-overlapping generations model can be modeled by modifying the budget constraint to

$$F_n + C_n = (Y_{n-1} + F_{n-1})R_n \quad n = 1, \dots, N,$$

where the  $Y$ 's are the contributions paid into the system by the beginning of each period, the  $C$ 's are the benefits paid out from the system by the end of each period and the  $R$ 's now represent the returns from a fixed asset mix where the fund invest its capital.

### 3. Pareto Efficiency in the Multi-Period Setting

This section deals with the concept of Pareto efficiency in this multi-period setting, which is the first step to look for a PEFF risk-sharing rule. We shall characterize parametrically all the PE solutions, among which we look for the one that is also financially fair in the following sections.

It may be convenient to introduce first some notations. Let  $\mathbb{R}_+^{N+1}$  be the nonnegative cone in  $\mathbb{R}^{N+1}$ , i.e.  $\mathbb{R}_+^{N+1} := \{\theta \in \mathbb{R}^{N+1} | \theta_i \geq 0\}$ , and define  $\mathbb{R}_{++}^{N+1} := \{\theta \in \mathbb{R}^{N+1} | \theta_i > 0\}$  as the strictly positive cone. For two vectors  $a = (a_1, \dots, a_N)$ ,  $b = (b_1, \dots, b_N)$ ,  $a, b \in \mathbb{R}^N$ , we write  $a \succeq b$  if  $a_n \geq b_n$  for all  $n = 1, \dots, N$  and there exists some  $m = 1, \dots, N$  such that  $a_m > b_m$ .

For simplicity we write  $X := (X_1, \dots, X_N)$  and  $R := (R_2, \dots, R_N)$  which are vectors of random variables on  $\Omega$ . Write  $\rho := (C_1, C_2, \dots, C_N, F_N)$  as the generic notation for a risk-sharing rule and the set of all the possible  $\rho$ 's is denoted by  $\mathcal{RS}$ . We will be particularly interested in the subset  $\mathcal{P} \subset \mathcal{RS}$  which is the set of all Pareto-efficient risk-sharing rules. First we need the following definition.

DEFINITION 3.1 (*Multi-period Pareto efficiency.*) A risk-sharing rule  $(C_1, C_2, \dots, C_N, F_N)$  is called *Pareto efficient*, or *Pareto optimal*, if there does not exist another risk-sharing rule  $(\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_N, \tilde{F}_N)$  such that

$$\left( \mathbb{E}^{\mathbb{P}} u_1(\tilde{C}_1), \dots, \mathbb{E}^{\mathbb{P}} u_N(\tilde{C}_N), \mathbb{E}^{\mathbb{P}} u_p(\tilde{F}_N) \right) \succeq \left( \mathbb{E}^{\mathbb{P}} u_1(C_1), \dots, \mathbb{E}^{\mathbb{P}} u_N(C_N), \mathbb{E}^{\mathbb{P}} u_p(F_N) \right).$$

We then have the following important theorem in this discrete probability space, which can be seen as a generalization of the Borch-type characterization of the Pareto efficiency: every Pareto-efficient risk-sharing rule can be totally characterized by optimizing a weighted time-additive aggregate utility.

THEOREM 3.2 (*Characterization of Pareto efficiency.*) For a risk-sharing rule  $(C_1, C_2, \dots, C_N, F_N)$ , the following statements are equivalent.

1. The risk-sharing rule is Pareto efficient.
2. The risk-sharing rule maximizes

$$\mathbb{E}^{\mathbb{P}} \left[ \sum_{n=1}^N \theta_n u_n(C_n) + \theta_p u_p(F_N) \right] \quad (3.1)$$

for some positive constants  $\theta_1, \dots, \theta_N, \theta_p$ .

3. The risk-sharing rule satisfies the following which are hereafter called the *intertemporal balance equations (IBEs)* for some positive constants  $\theta_1, \dots, \theta_N, \theta_p$ :

$$\begin{aligned} \theta_n u'_n(C_n) &= \theta_{n+1} \mathbb{E}_n^{\mathbb{P}} [u'_{n+1}(C_{n+1}) R_{n+1}] \quad \text{for } n = 1, \dots, N-1, \\ \theta_N u'_N(C_N) &= \theta_p u'_p(F_N). \end{aligned}$$

PROOF See appendix. □

REMARK 3.3 (*Link to Borch [1].*) Consider the case when  $t_n = t_{n+1}$  for some  $n$ . We must have that  $X_{n+1} \equiv 0$  and  $R_{n+1} \equiv 1$ . Thus  $\mathcal{F}_n = \mathcal{F}_{n+1}$  and the IBE becomes

$$\theta_n u'_n(C_n) = \theta_{n+1} \mathbb{E}_n^{\mathbb{P}} [u'_{n+1}(C_{n+1}) R_{n+1}] = \theta_{n+1} u'_{n+1}(C_{n+1}).$$

This means that in a single period setting, the IBEs will coincide with the characterization of PE risk-sharing rules by Borch [1].

REMARK 3.4 (*Comparison to the Euler equation.*) The IBEs are very similar  
 205 to the famous Euler equation derived amongst others by Hall [16] for solving  
 the consumption-savings model. In fact, the model setting in this paper can be  
 used as a life-cycle model. If we let the time points  $\{t_n\}$  be equispaced and set  
 $R_n = 1 + r$  and  $u_n = u$  for all  $n$ , then the model setting is also similar to Hall's:  
 every period there is a stochastic earning and a consumption, which correspond  
 210 to the incoming "risk" and the "contingent payment" in this setting.

The optimization targets are different regarding weighing intertemporally  
 the utilities: Hall assumed a single rate of subjective time preference  $\delta$  while  
 the IBEs are parameterized by weight vector  $\theta := (\theta_1, \dots, \theta_N, \theta_p)$ .

Formula-wise, Hall gave

$$\mathbb{E}_n u'(C_{n+1}) = \left( \frac{1 + \delta}{1 + r} \right) u'(C_n),$$

while the IBE gives

$$\mathbb{E}_n u'(C_{n+1}) = \left( \frac{\theta_n}{\theta_{n+1}} \right) u'(C_n).$$

It is obvious that Hall adopts a specific set of weights in the scope of Theorem  
 215 3.2. As we shall see later, the weights  $\theta$  can be seen as unknowns within the  
 framework here and will be determined endogenously by the financial fairness  
 constraint. The interpretation is that, regarding the intertemporal substitution,  
 Hall adopts a single subjective discount factor while in the PEFF framework  
 the discount curve is determined by the market values of the consumption.

The theorem shows that it is equivalent to solve the optimization problem  
 (3.1) subject to the budget constraints when one wants to find the corresponding  
 PE risk-sharing rule given any  $\theta \in \mathbb{R}_{++}^{N+1}$ . We can then construct a mapping to  
 compute the PE solution given any  $\theta \in \mathbb{R}_{++}^{N+1}$ , which we will call  $\Phi : \mathbb{R}_{++}^{N+1} \rightarrow$   
 $\mathcal{P}$ . This can be done by solving the corresponding parameterized optimization

problem of time-additive utility functions:

$$\begin{aligned} & \max_{C_1, \dots, C_N} \mathbb{E}^{\mathbb{P}} \left[ \sum_{n=1}^N \theta_n u_n(C_n) + \theta_p u_p(F_N) \right] \\ \text{such that } & F_n + C_n = X_n + F_{n-1} R_n \quad n = 1, \dots, N, \\ & F_0 = 0. \end{aligned}$$

This optimization problem can be solved by dynamic programming. Add in a new time point  $t_{N+1} = t_N$ , and

$$X_{N+1} \equiv 0, \quad R_{N+1} \equiv 1.$$

Define

$$A_n := X_n + F_{n-1} R_n \quad n = 1, \dots, N+1,$$

which has the interpretation as the total available asset at time  $t_n$  to be divided into the current payment and the buffer for later use. Note that by definition  $A_{N+1} = F_N$ . The  $A$ 's are the state variables, the  $C$ 's are the decision variables and the  $X$ 's and  $R$ 's are the risks. Then we shall have the optimization problem formulated as

$$\begin{aligned} & \max_{C_1, \dots, C_N} \mathbb{E}^{\mathbb{P}} \left[ \sum_{n=1}^N \theta_n u_n(C_n) + \theta_p u_p(A_{N+1}) \right] \\ \text{such that } & A_{n+1} = X_{n+1} + (A_n - C_n) R_{n+1}, \quad n = 1, \dots, N, \\ & A_1 = X_1. \end{aligned}$$

Proposition 1.3.1 in [17] shows that in order to solve the problem one needs to define the value functions (indirect utility): first for the last period

$$V_{N+1}(A_{N+1}) = \theta_p u_p(A_{N+1}),$$

and then define backwards, for  $n = 1, \dots, N$

$$V_n(A_n) = \max_{C_n} \mathbb{E}_n^{\mathbb{P}} [\theta_n u_n(C_n) + V_{n+1}(X_{n+1} + (A_n - C_n) R_{n+1})]. \quad (3.2)$$

220 The final result is presented below. This mapping  $\Phi$  gives an explicit expression of the risk-sharing rule  $\rho$  as a function of the weights  $\theta$ , which makes it possible to express the financial fairness condition in terms of the weights later in the paper.

THEOREM 3.5 (*The construction of  $\Phi$ .*) For any given  $\theta = (\theta_1, \dots, \theta_N, \theta_p) \in \mathbb{R}_{++}^{N+1}$ , the corresponding PE solution  $\rho = (C_1, \dots, C_N, F_N)$  is given by

$$A_n = X_n + F_{n-1}R_n \quad n = 1, \dots, N, \quad (3.3)$$

$$C_n = I_n \left( \frac{g_n(A_n)}{\theta_n} \right) \quad n = 1, \dots, N, \quad (3.4)$$

$$F_n = H_n \left( \frac{g_n(A_n)}{\theta_{n+1}} \right) \quad n = 1, \dots, N-1, \quad (3.5)$$

$$F_N = I_p \left( \frac{g_N(A_N)}{\theta_p} \right), \quad (3.6)$$

where the functions are defined recursively by

$$\begin{aligned} G_N(x) &:= I_N \left( \frac{x}{\theta_N} \right) + I_p \left( \frac{x}{\theta_p} \right), \\ g_N(x) &:= G_N^{-1}(x), \end{aligned}$$

and for  $n = 1, \dots, N-1$

$$h_n(x) = \mathbb{E}_n^{\mathbb{P}} \left[ \frac{1}{\theta_{n+1}} g_{n+1}(X_{n+1} + xR_{n+1})R_{n+1} \right] \quad (3.7)$$

$$= \mathbb{E}^{\mathbb{P}} \left[ \frac{1}{\theta_{n+1}} g_{n+1}(X_{n+1} + xR_{n+1})R_{n+1} \right], \quad (3.8)$$

$$H_n = h_n^{-1},$$

$$G_n(x) := I_n \left( \frac{x}{\theta_n} \right) + H_n \left( \frac{x}{\theta_{n+1}} \right),$$

$$g_n(x) := G_n^{-1}.$$

The mapping (3.3) - (3.6) is denoted as  $\Phi : \theta \mapsto \rho, \mathbb{R}_{++}^{N+1} \rightarrow \mathcal{P}$ .

225 PROOF See appendix. Please note that from expression (3.7) to (3.8) we utilized the assumption that the processes  $X$  and  $R$  are sequentially independent.  $\square$

The functions above have the following interpretation. While  $u'_n$  is the marginal utility function of the contingent payment  $C_n$ , the function  $h_n$  is the implied marginal utility of the buffer  $F_n$  and  $g_n$  the implied marginal utility of the total available asset  $A_n$ . The capital-letter functions  $I, H, G$  are the corresponding inverse functions. The following relationships hold:

$$\begin{aligned} g_n(A_n) &= \theta_n u'_n(C_n) = \theta_{n+1} h_n(F_n), \quad n = 1, \dots, N-1, \\ g_N(A_N) &= \theta_N u'_N(C_N) = \theta_p u'_p(F_N). \end{aligned}$$

The function  $g$ 's are also the derivatives of the value functions. The proof in the appendix shows that for any  $n$

$$V'_n(A_n) = g_n(A_n).$$

Write

$$L_n := g_n(A_n),$$

which is interpreted as the weighted marginal utility of the contingent payments. Furthermore, the IBE will be translated into

$$L_n = \mathbb{E}_n^{\mathbb{P}}[L_{n+1} R_{n+1}].$$

The idea of dynamic programming indicates that in each period, the system has to ponder how to distribute the risks between the current contingent payment and all the future contingent payments: for any  $n < N$ , it compares the marginal utilities of paying out the money now (i.e.  $C_n$ ) or saving it for the future (i.e.  $F_n$ ):

$$\theta_n u'_n(C_n) \quad \text{v.s.} \quad \theta_{n+1} h_n(F_n).$$

The  $h_n$  function is calculated by “summarizing” the expectations over the future. This property allows us to convert an  $n$ -period problem into an *induced*  $(n-1)$ -period one, by regarding the time  $t_{n-1}$  as the new end of the system and  $F_{n-1}$  as the new end buffer with utility  $h_{n-1}$ .

230

This perspective is essential for the proofs later. As a first application, it can help us link the settings of CEB and OEB to each other. First, as we have

discussed, any OEB problem can be converted into a CEB problem by regarding  $F_N$  as an extra contingent payment  $C_{N+1}$  at  $t_{N+1} = t_N$ . The following result  
 235 shows that in the sense of Pareto efficiency, the OEB and CEB are equivalent, thus we can work with the two environments interchangeably.

PROPOSITION 3.6 (*Equivalence between CEB and OEB problems.*) *The CEB and the OEB are equivalent in the sense that they can always be converted into the form of the other which can produce the identical PE risk-sharing rule.*

PROOF We only need to consider the direction from CEB to OEB. Given a CEB case with PE risk-sharing rule  $(C_1, \dots, C_N)$ , utility functions  $(u_1, \dots, u_N)$  and weights  $(\theta_1, \dots, \theta_N)$ , we can create a corresponding OEB problem that replicates the original setting for  $n = 1, \dots, N - 1$  and truncate the system at time  $t_{N-1}$  by defining

$$h(x) := \mathbb{E}_{N-1}^{\mathbb{P}} [u'_N(X_N + xR_N)R_N]$$

as the marginal utility function for the new end buffer  $F_{N-1}$  together with weight  $\theta_N$ . Then according to the IBE for the CEB problem we have

$$\begin{aligned} \theta_{N-1}u'_{N-1}(C_{N-1}) &= \theta_N \mathbb{E}_{N-1}^{\mathbb{P}} [u'_N(C_N)R_N] \\ &= \theta_N \mathbb{E}_{N-1}^{\mathbb{P}} [u'_N(X_N + F_{N-1}R_N)R_N] \\ &= \theta_N h(F_{N-1}) \end{aligned}$$

240 which matches the final-period IBE in Theorem 3.2. Thus according to the theorem the two settings should produce the same PE risk-sharing rules. The only thing left is to verify that the function  $h(x)$  defined in this way is indeed a (stereotype) marginal utility function; this has been done in the proof of Theorem 3.5.  $\square$

245 There is one degree of freedom extra in determining  $\theta$ , as for any  $c \in \mathbb{R}_{++}$ ,  $\theta$  and  $c \cdot \theta$  will produce essentially the same optimization target. But if we choose a way of normalizing the  $\theta$ 's, e.g. restrict the  $\theta$ 's to the open unit simplex in  $\mathbb{R}_{++}^{N+1}$ , then we will have the following theorem which indicates that every PE

250 risk-sharing rule  $\rho \in \mathcal{P}$  can be uniquely characterized by the weights  $\theta$ , and the function  $\Phi$  is a meaningful bijection between all the PE risk-sharing rules  $\rho$ 's and the weights  $\theta$ 's.

THEOREM 3.7  $\Phi$  is a one-to-one mapping between the set of all the Pareto efficient risk-sharing rules  $\mathcal{P}$  and the open unit simplex in  $\mathbb{R}_{++}^{N+1}$ , i.e. the set  $\mathcal{U} := \{c \in \mathbb{R}_{++}^{N+1} | c_1 + \dots + c_{N+1} = 1\}$ .

255 PROOF This can be seen as a corollary of Theorem 3.2. We discuss the two directions.

1.  $\mathcal{U} \rightarrow \mathcal{P}$ : the mapping  $\Phi$  maps any  $\theta \in \mathbb{R}_{++}^{N+1}$  into  $\mathcal{P}$ . This mapping is not injective. Consider some  $\theta$  and  $\theta'$  such that  $\Phi(\theta) = \Phi(\theta')$ . Then we show that there will exist some  $c \in \mathbb{R}_{++}$  such that  $\theta = c\theta'$  thus  $\Phi$  is injective if restricted on  $\mathcal{U}$ .  
260

By the IBEs we know that

$$\frac{\theta_n}{\theta_{n+1}} = \frac{\mathbb{E}_n u'_{n+1}(C_{n+1}) R_{n+1}}{u'_n(C_n)} \quad \text{for } n = 1, \dots, N-1$$

and

$$\frac{\theta_N}{\theta_p} = \frac{u'_p(F_N)}{u'_N(C_N)}.$$

This indicates

$$\frac{\theta_n}{\theta_{n+1}} = \frac{\theta'_n}{\theta'_{n+1}} \quad \text{for } n = 1, \dots, N-1$$

and

$$\frac{\theta_N}{\theta_p} = \frac{\theta'_N}{\theta'_p}$$

We then have

$$\theta = \frac{\theta_1}{\theta'_1} \theta'.$$

$\Phi$  will be an injective mapping if restricted on  $\mathcal{U}$ .

2.  $\mathcal{P} \rightarrow \mathcal{U}$ : Theorem 3.2 shows that for any element  $\rho \in \mathcal{P}$ , there exists some  $\theta \in \mathbb{R}_{++}^{N+1}$  such that  $\Phi(\theta) = \rho$ .

We conclude from the above discussion that  $\Phi$  is both injective and surjective.

265 It must be bijective. □

We conclude this section by some useful properties of the PE risk sharing system. First, we give the following result which seems quite intuitive: every agent will be better off when the realization of the risks is (strictly) better. We call this the monotonicity property of the system with respect to the risks.

270 LEMMA 3.8 (*Monotonicity property of the system with respect to the risks.*)  
For any  $\theta \in \mathbb{R}_{++}^{N+1}$ , consider two trajectories  $J, J^* \in \mathcal{J}_N$  such that  $(X^J, R^J) \succeq (X^{J^*}, R^{J^*})$ . Then we have  $\rho^J \succeq \rho^{J^*}$ .

PROOF See appendix. □

275 The following result illustrates the impact of the weight  $\theta$  on the contingent payments: if some weight increases while the others stay the same, then along any trajectory, the corresponding contingent payment will increase while the other contingent payments will decrease.

LEMMA 3.9 (*Monotonicity property of the system with respect to the weights.*)  
Consider two weights  $\theta = (\theta_1, \dots, \theta_N, \theta_p), \theta' = (\theta'_1, \dots, \theta'_N, \theta'_p) \in \mathbb{R}_{++}^{N+1}$  such that there exists some  $n = 1, \dots, N, p$  that

$$\theta_n > \theta'_n, \quad \theta_i = \theta'_i \quad \forall i \neq n.$$

Then we have that for any trajectory  $J \in \mathcal{J}_N$ , the corresponding PE risk-sharing rules satisfy

$$C_n^J > C_n^{J'}, \quad C_i^J < C_i^{J'} \quad \forall i \neq n.$$

Here for convenience we let  $C_p = F_N$ .

PROOF See appendix. □

280 **4. Financial Fairness**

As we have discussed, the PE risk-sharing rules can be totally characterized by the points on the open unit simplex in  $\mathbb{R}_{++}^{N+1}$  and thus there will be infinitely many such PE rules. We will see in the following that the concept of financial fairness will help us narrow down our scope – finally we will arrive at a unique risk-sharing rule that is both PE and FF.

The concept of financial fairness means that when the system starts, for each agent involved, the market value of the risks he contributes into the system should be equal to that of the contingent payments he gets after risk sharing. FF will work via the concept of *value profile*, which is the vector of the values of the contingent payments under the risk-neutral measure  $\mathbb{Q}$ , that is, for any  $\rho = (C_1, \dots, C_N, F_N) \in \mathcal{RS}$

$$v = (v_1, v_2, \dots, v_N, v_p) := \mathbb{E}^{\mathbb{Q}}\rho = (\mathbb{E}^{\mathbb{Q}}C_1, \mathbb{E}^{\mathbb{Q}}C_2, \dots, \mathbb{E}^{\mathbb{Q}}C_N, \mathbb{E}^{\mathbb{Q}}F_N) \in \mathbb{R}^{N+1}. \quad (4.1)$$

As before we consider no discounting and we simply use the  $\mathbb{Q}$ -expectation as market values.

The set of all the possible value profiles  $\mathcal{V}$  can only be a restricted subset of  $\mathbb{R}^{N+1}$ . First note it is trivial that

$$v_n > b_n \quad \text{for } n = 1, \dots, N; \quad v_p > b_p$$

according to the domain requirements of the utility functions. Next, according to the global budget constraint (2.3) we shall have, by taking the expectation under  $\mathbb{Q}$  to both sides

$$\sum_{n=1}^{N-1} \left[ v_n \left( \prod_{i=n+1}^N (1+r_i) \right) \right] + v_N + v_p = \sum_{n=1}^{N-1} \left[ x_n \left( \prod_{i=n+1}^N (1+r_i) \right) \right] + x_N. \quad (4.2)$$

We can then write

$$\mathcal{V} = \left\{ v \in \mathbb{R}^N \mid \text{Eq (4.2) holds; } v_n > b_n \text{ for } n = 1, \dots, N; \quad v_{N+1} > b_p \right\} \quad (4.3)$$

as the set of all possible value profiles. Note that the set  $\mathcal{V}$  is totally determined by the market values of risks and the utility functions.

REMARK 4.1 The global budget constraint suggests that for any given value profile vector  $v := (v_1, \dots, v_N, v_p)$ , we only have to consider any  $N$  coefficients. For instance, if the following hold

$$\mathbb{E}^{\mathbb{Q}} C_n = v_n \quad n = 1, \dots, N$$

then

$$\mathbb{E}^{\mathbb{Q}} F_N = v_p$$

290 will automatically be satisfied.

## 5. Existence and Uniqueness of the PEFF Risk Sharing Rule

The theorems in this section will show that the solution exists and is actually unique if we combine the Pareto efficiency with financial fairness. We continue to work with the general situation when there are  $N$  contingent payments alongside the buffer,  $N \geq 1$ . For any given value profile  $v := (v_1, \dots, v_N, v_p) \in \mathcal{V}$ , the corresponding PEFF risk-sharing rule is the solution to the following equation system:

1. budget constraints (BCs):

$$F_n + C_n = X_n + F_{n-1}R_n \quad n = 1, \dots, N; \quad (5.1)$$

2. intertemporal balance equations (IBEs):

$$\begin{aligned} \theta_n u'_n(C_n) &= \theta_{n+1} \mathbb{E}_n^{\mathbb{P}} [u'_{n+1}(C_{n+1})R_{n+1}] \quad n = 1, \dots, N-1, \\ \theta_N u'_N(C_N) &= \theta_p u'_p(F_N); \end{aligned} \quad (5.2)$$

3. financial fairness constraints (FFs):

$$\mathbb{E}^{\mathbb{Q}} C_n = v_n \quad n = 1, \dots, N. \quad (5.3)$$

Please note that the BC and IBE equations above are actually equations between functions. The equations should hold for all possible trajectories.

300 The following theorem is one of the key results of this paper. It indicates that for the equation system above, the solution always exists and is unique, thus it establishes the existence and uniqueness of the PEFF risk-sharing rule.

THEOREM 5.1 (*The existence and uniqueness of the PEFF risk-sharing rule.*)

305 *For any given value profile vector  $v \in \mathcal{V}$ , the PEFF risk-sharing rule exists and is unique. The corresponding  $\theta$  is unique up to normalization.*

PROOF See appendix. □

According to Theorem 3.2, the function sets BC and IBE characterize all the possible PE risk-sharing rules by way of weights  $\theta \in \mathbb{R}_{++}^{N+1}$ . The theorem above then shows that the value profile determines a unique  $\theta$ .

310 Recall that in Theorem 3.5  $\Phi$  defines a bijective mapping from  $\mathcal{U}$  to the set of all PE risk-sharing rules  $\mathcal{P}$ . The mapping  $\Phi$  then induces a natural mapping  $\Psi$  from  $\mathcal{U}$  to  $\mathcal{V}$ :  $\Psi(\theta) = \mathbb{E}^{\mathbb{Q}}\Phi(\theta)$ . This  $\Psi$  links the set of all the possible weights  $\theta$  and the set of all the possible value profiles.

THEOREM 5.2  *$\Psi$  is a one-to-one mapping between the set of all possible value*  
 315 *profiles  $\mathcal{V}$  and the open unit simplex  $\mathcal{U}$  in  $\mathbb{R}_{++}^{N+1}$ .*

PROOF Theorem 5.1 shows that  $\Psi$  is surjective: for any given  $v \in \mathcal{V}$  there exists a  $\theta \in \mathbb{R}_{++}^{N+1}$  such that  $\Psi(\theta) = \mathbb{E}^{\mathbb{Q}}\Phi(\theta) = v$ .

This  $\Psi$  is also injective restricted on the open unit simplex  $\mathcal{U}$  because of the uniqueness of  $\theta$  up to normalization. Suppose there are  $\theta_1, \theta_2 \in \mathcal{U}$  such  
 320 that  $\Psi(\theta_1) = \Psi(\theta_2)$ . Theorem 5.1 indicates that  $\Phi(\theta_1) = \Phi(\theta_2)$ , as for each value profile, there will exist exactly one PE risk-sharing rule such that the FF condition is satisfied. According to Theorem 3.7, it must be that  $\theta_1 = \theta_2$  as they both belong to the open unit simplex  $\mathcal{U}$ . □

We can then say that the  $\theta$  uniquely determines the value profile of any PE  
 325 risk-sharing rule, and also *vice versa*. Instead of talking about the weights  $\theta$  we

can now talk about the value profiles which seem more tangible. However, we cannot say more of the mapping  $\Psi$ ; the structure of it can be very complicated depending on the utility functions one uses.

## 6. A General Algorithm for Finding PEFF Solution

330 Looking for the PEFF risk-sharing rule will come down to solving a system of both linear and non-linear equations. In most cases there's no hope for explicit solutions; fortunately, we have a numerical algorithm that helps to find the PEFF solution.

Recall that

$$L_n = \theta_n u'_n(C_n) \quad n = 1, \dots, N$$

are the weighted marginal utilities of the contingent payments as determined by the risk-sharing rule at time  $t_n$ . According to the IBEs

$$L_n = \mathbb{E}_n^{\mathbb{P}}[L_{n+1}R_{n+1}] \quad n = 1, \dots, N-1,$$

thus the whole sequence  $\{L_n\}$  is known once  $L_N$  is known.

In Theorem 3.5 we constructed a mapping  $\Phi : \mathbb{R}_{++}^{N+1} \rightarrow \mathcal{P}$  from the sets of functions BC and IBE. Given the mapping  $\Phi$ , we can deduce another mapping  $\varphi_1$  by

$$\varphi_1(\theta) = L_N = \theta_N u'_N(C_N) = \theta_N u'_N(\Phi_{(N)}(\theta)),$$

where  $\Phi_{(N)}(\cdot)$  stands for the  $N$ -th coordinate of this vector-valued function.  $\varphi_1$  maps any  $\theta \in \mathbb{R}_{++}^{N+1}$  into some  $L_N$ . For any  $L_N$ , another mapping  $\varphi_2 : L_N \mapsto \theta$  can be constructed based on the FF constraints: note that according to the mapping  $\Phi$  we have

$$C_n = I_n \left( \frac{L_n}{\theta_n} \right) \quad n = 1, \dots, N;$$

$$F_N = I_p \left( \frac{L_N}{\theta_p} \right),$$

and

$$L_n = \mathbb{E}_n[L_{n+1}R_{n+1}].$$

This allows us to find a  $\theta'$  such that the FF conditions are satisfied for the given  $L_N$ :

$$\mathbb{E}^{\mathbb{Q}}C_n = \mathbb{E}^{\mathbb{Q}}I_n \left( \frac{L_n}{\theta'_n} \right) = v_n \quad \text{for } n = 1, \dots, N; \quad (6.1)$$

$$\mathbb{E}^{\mathbb{Q}}F_N = \mathbb{E}^{\mathbb{Q}}I_p \left( \frac{L_N}{\theta'_p} \right) = v_p. \quad (6.2)$$

The function  $\varphi_2$  is well defined since

$$\mathbb{E}^{\mathbb{Q}}C_n = \mathbb{E}^{\mathbb{Q}}I_n \left( \frac{L_n}{\theta_n} \right) = \sum_{J \in \mathcal{J}_n} \mathbb{Q}(J)I_n \left( \frac{L_n^J}{\theta_n} \right)$$

is a strictly increasing and continuous function in  $\theta_n$  with  $\theta_n \in \mathbb{R}_{++}$ . Thus

$$\varphi_{2(n)}(L_N) := \left[ \mathbb{E}^{\mathbb{Q}}I_n \left( \frac{L_n}{\cdot} \right) \right]^{-1} (v_n)$$

335 is well defined. This holds for all  $n = 1, \dots, N$  and also for  $F_N$ , thus  $\varphi_2$  is well-defined. Please note that one and only one coordinate of the weight vector  $\theta$  is solved in every single equation (6.1) and (6.2).

Consider the composition of the two functions  $\varphi = \varphi_2 \circ \varphi_1$ : it is a mapping from  $\mathbb{R}_{++}^{N+1}$  into itself. Theorem 5.1 indicates that there always exists a unique  
340 fixed point of this mapping  $\varphi$ , which corresponds to the PEFF risk-sharing rule. The next theorem shows that  $\varphi$  suggests an iterative algorithm for finding the PEFF solution.

**THEOREM 6.1** (*Feasibility of an iterative algorithm by  $\varphi$ .*) *For any given starting point  $\theta \in \mathbb{R}_{++}^{N+1}$  with any proper normalization, the sequence of iterates*  
345  *$\{\varphi^{(n)}(\theta) | n \in \mathbb{N}_+\}$  will converge to the fixed point of  $\varphi$ .*

**PROOF** See Appendix. □

Theorem 6.1 suggests that starting with any given  $\theta$ , one first finds the corresponding  $L_N$  by  $\varphi_1$  and then updates the value of  $\theta$  by  $\varphi_2$ . It is more convenient, in fact, to use function  $\Phi$  instead of  $\varphi_1$ , i.e. we map  $\theta$  to  $\rho$  directly  
350 and in the second step we update the  $\theta$  accordingly. In the first step, we need to calculate numerically the functions  $g$ 's and  $h$ 's backwards in time, and once

all the functions are ready, we then go forwards in time and calculate all the  $C$ 's and  $F$ 's from the starting distribution  $X_1$ .

ALGORITHM 1 (*Numerical algorithm for finding the PEFF solution.*) The following gives a description of the numerical algorithm for finding the PEFF solution.

1. Start with some initial  $\theta^{(0)} \in \mathbb{R}_{++}^{N+1}$ .
2. For any given  $\theta^{(m)}$  with  $m \in \mathbb{N}$ , calculate backwards in time that

$$G_N^{(m)}(x) := I_N \left( \frac{x}{\theta_N^{(m)}} \right) + I_p \left( \frac{x}{\theta_p^{(m)}} \right),$$

$$g_N^{(m)}(x) := \left( G_N^{(m)} \right)^{-1}(x),$$

and for  $n = 1, \dots, N - 1$

$$h_n^{(m)}(x) = \mathbb{E}^{\mathbb{P}} \left[ \frac{1}{\theta_{n+1}^{(m)}} g_{n+1}^{(m)}(X_{n+1} + xR_{n+1})R_{n+1} \right],$$

$$H_n^{(m)} = \left( h_n^{(m)} \right)^{-1},$$

$$G_n^{(m)}(x) := I_n \left( \frac{x}{\theta_n^{(m)}} \right) + H_n^{(m)} \left( \frac{x}{\theta_{n+1}^{(m)}} \right),$$

$$g_n^{(m)}(x) := \left( G_n^{(m)} \right)^{-1}.$$

3. Calculate the decision variables forwards in time by

$$A_n^{(m)} = X_n + F_{n-1}^{(m)}R_n \quad n = 1, \dots, N,$$

$$C_n^{(m)} = I_n \left( \frac{g_n^{(m)}(A_n^{(m)})}{\theta_n^{(m)}} \right) \quad n = 1, \dots, N,$$

$$F_n^{(m)} = H_n^{(m)} \left( \frac{g_n^{(m)}(A_n^{(m)})}{\theta_{n+1}^{(m)}} \right) \quad n = 1, \dots, N - 1,$$

$$F_N^{(m)} = I_p \left( \frac{g_N^{(m)}(A_N^{(m)})}{\theta_p^{(m)}} \right).$$

4. Update the  $\theta$  from  $\theta^{(m)}$  to  $\theta^{(m+1)}$  by solving

$$\mathbb{E}^{\mathbb{Q}} C_n^{(m)} = \mathbb{E}^{\mathbb{Q}} I_n \left( \frac{g_n^{(m)}(A_n^{(m)})}{\theta_n^{(m+1)}} \right) = v_n \quad n = 1, \dots, N;$$

$$\mathbb{E}^{\mathbb{Q}} F_N^{(m)} = \mathbb{E}^{\mathbb{Q}} I_p \left( \frac{g_N^{(m)}(A_N^{(m)})}{\theta_p^{(m+1)}} \right) = v_p.$$

5. Normalize  $\theta^{(m+1)}$ .

6. If, for some pre-specified error tolerance  $\varepsilon$

$$|\mathbb{E}^{\mathbb{Q}} C_n^{(m)} - v_n| < \varepsilon \quad n = 1, \dots, N,$$

$$|\mathbb{E}^{\mathbb{Q}} F_N^{(m)} - v_p| < \varepsilon,$$

we conclude that  $\rho^{(m)} = (C_1^{(m)}, \dots, C_N^{(m)}, F_N^{(m)})$  is the PEFF risk-sharing rule we are looking for. Otherwise, go to step 2 with  $\theta^{(m+1)}$ .

360

REMARK 6.2 (*Comparison to the algorithm proposed by Pazdera et al. [15].*)

As has been mentioned, the framework introduced here can also deal with the single-period situation, which has been investigated by Pazdera et al. [15].

There is a significant difference between the two numerical algorithms, though.

365

The algorithm here makes use of the induction technique that the number of contingent payments is reduced by one recursively, thus in each iteration the algorithm always calculate the functions backwards and then the distributions of the decision variables forwards. In contrast, the algorithm in [15] need not use such an induction technique; functions and decision variables can be calculated

370

simultaneously in each iteration. The algorithm in [15] offers more efficiency for the single-period problem, while the algorithm here is more versatile and can deal with multi-period problems.

## 7. Examples

In this section we give two examples of the PEFF risk sharing. In the first example we implement Algorithm 1 in a simple case where each of the financial risks has only two possible outcomes. The second example deals with exponential utility functions where we may have explicit PEFF solution.

375

7.1. Implementing the Algorithm: the Case of Two-Valued Random Variables

We start with a 3-period setting where three agents gather to share their risks. As shown in Figure 3, there are four time points  $t = 0, 1, 2, 3$ . For  $n = 1, 2, 3$ , agent  $n$  exists between time points  $n - 1$  and  $n$ . He receives a stochastic income  $X_n$  as the risk and he gets  $C_n$  as the contingent payment. The distributions of  $X_n$  are given by

$$\begin{aligned}\mathbb{P}(X_n = 1.2) &= 0.6, & \mathbb{P}(X_n = 0.8) &= 0.4; \\ \mathbb{Q}(X_n = 1.2) &= 0.5, & \mathbb{Q}(X_n = 0.8) &= 0.5.\end{aligned}$$

In the *autarky* case where the agents are all on their own, agent  $n$  will get  $C_n = X_n$  and there is no risk sharing among the agents.

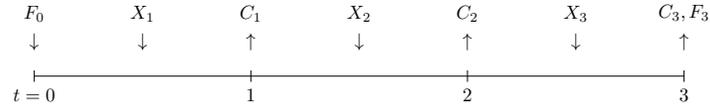


Figure 3: Timeline for Section 7.1

We consider the situation when the three agents gather to share their stochastic incomes in a PEFF way. The capital in the buffer is always invested in asset  $R$  with  $R_n \equiv 1$  for  $n = 1, 2, 3$  for simplicity. The budget constraints are then

$$C_n + F_n = X_n + F_{n-1} \quad n = 1, 2, 3.$$

The FF constraints are

$$\mathbb{E}^{\mathbb{Q}} C_n = 1 \quad n = 1, 2, 3$$

and

$$\mathbb{E}^{\mathbb{Q}} F_3 = F_0.$$

We assume that a buffer is available for the agents with initial capital  $F_0 = 1$ . The reason for starting with a positive buffer size is that we will later use power utility for  $F_3$  and  $F_3$  is required to be strictly positive, and so is  $\mathbb{E}^{\mathbb{Q}} F_3$ .

Power utility functions are used to evaluate the utility. We assume that the agents use

$$u_n(x) = \frac{x^{1-\gamma}}{1-\gamma} \text{ with } \gamma = 3, \quad n = 1, 2, 3.$$

We

385 We will consider both the OEB case where the end buffer is also a decision variable, and the CEB case where the end buffer will be a constant. In the OEB case, power utility with  $\gamma = 3$  is also used to evaluate the end buffer.

The IBEs in the OEB case are

$$\begin{aligned} \theta_1 u'_1(C_1) &= \theta_2 \mathbb{E}_1^{\mathbb{P}} [u'_2(C_2)], \\ \theta_2 u'_2(C_2) &= \theta_3 \mathbb{E}_2^{\mathbb{P}} [u'_3(C_3)], \\ \theta_3 u'_3(C_3) &= \theta_p u'_p(F_3). \end{aligned}$$

In the CEB case we only have the first two sets of IBEs since the end buffer size is a constant. In any case, the IBEs have to hold along all the trajectories.

390 Figures 4 and 5 show the distributions of the risks, under both the PEFF case and the autarky case. The details of the distributions are shown in the appendix.

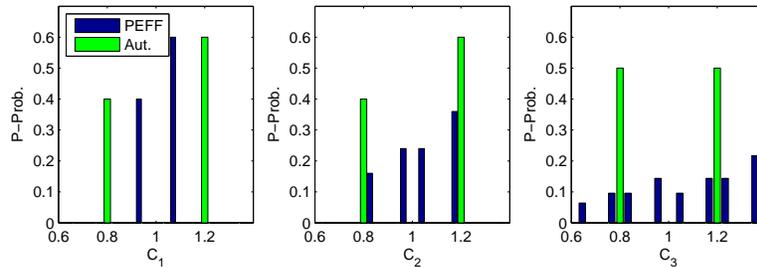


Figure 4: Payments to agents under  $\mathbb{P}$ , CEB

**The interpretation of the results.** In both the CEB case and the OEB case, agent 1 and 2 have effectively shifted some of the volatilities to the last agent, which can be seen from the fact that  $C_1$  and  $C_2$  from the PEFF solution 395 are less dispersed than the autarky situation. Agent 3 can be better off in the

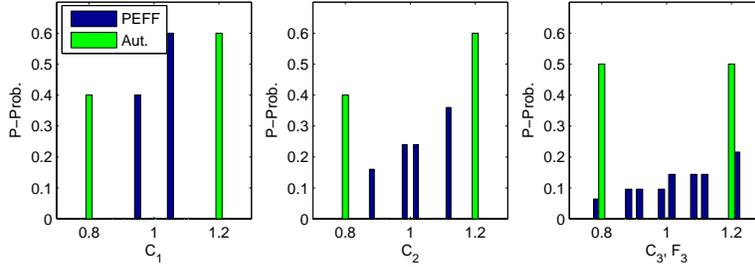


Figure 5: Payments to agents under  $\mathbb{P}$ , OEB

best-outcome scenario and worse off in the worst-outcome scenario compared to the autarky case. As a compensation for higher volatility, he benefits from a higher expected return. See Table 1. An important feature of the PEFF solution is that by design, the PEFF solution satisfies the FF constraints.

	Expected return			Standard deviation		
	PEFF, CEB	PEFF, OEB	Autarky	PEFF, CEB	PEFF, OEB	Autarky
Agent 1	1.0141	1.0101	1.0400	0.0689	0.0497	0.1960
Agent 2	1.0349	1.0236	1.0400	0.1235	0.0826	0.1960
Agent 3	1.0711	1.0431	1.0400	0.2247	0.1271	0.1960

Table 1: Comparison of PEFF and autarky solutions: statistics

The difference between the CEB and the OEB case is that in the OEB case,  $F_3$  can also absorb some risks. The results are lower expected returns under  $\mathbb{P}$  and smaller standard deviations for the payments  $C_n$  compared to the CEB case. Note that  $C_3$  and  $F_3$  have identical distributions. This is because they are assigned the same utility functions and the same ex-ante market values.

The results also shed some light on the issue of *individual rationality*, which says that if the agents are rational, they are willing to take part in the risk sharing system only when the risk sharing gives welfare improvements. This

is a different concept from Pareto efficiency, and in general the PEFF solution  
 410 does not necessarily result in larger expected utility for *every* agent. In this  
 example, it is possible to compare the expected utility for each agent. Table  
 2 shows the comparison of expected utilities in terms of certainty equivalents.  
 It is clear that in both the CEB and the OEB cases, the agents all experience  
 welfare improvements and should be willing to participate in the risk sharing  
 415 system.

	PEFF, CEB	PEFF, OEB	Autarky
Agent 1	0.9905	1.0064	0.9798
Agent 2	1.0113	1.0132	0.9798
Agent 3	1.0068	1.0183	0.9798

Table 2: Certainty equivalents

**Illustration of Algorithm 1.** Algorithm 1 indicates that to find the PEFF  
 solution, one starts with some initial weights  $\theta$ , calculates the functions  $g_n$  and  
 $h_n$ , gets the distributions of the decision variables, and then updates the weights  
 until they converge to some  $\theta^*$ . Setting the error tolerance  $\varepsilon = 10^{-6}$ , the weight  
 420 usually converges in less than ten iterations and is not sensitive to the starting  
 values.

Figure 6 shows the functions  $h_n$  and  $g_n$ . Recall that for  $n = 1, 2$ ,  $h_n$  can  
 be seen as the implied marginal utility function for the buffer size  $F_n$ , and for  
 $n = 1, 2, 3$ ,  $g_n$  can be seen as the implied indirect marginal utility function for  
 425 the total asset  $A_n$ .

### 7.2. Explicit PEFF Solution: the Case of Exponential Utility Function

This section discusses a special case when we assume the  $R_n$ 's are all con-  
 stants (thus only the risks  $X$  are stochastic) and exponential utility functions  
 (the constant-absolute-risk-aversion (CARA) utility) are used for all the contin-  
 gent payments

$$u_n(x) = 1 - e^{-\alpha_n x}, \quad \text{for } n = 1, \dots, N,$$

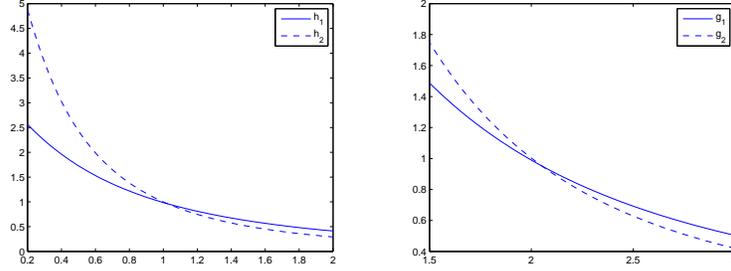


Figure 6: Functions  $h_n$  and  $g_n$

and also for the end buffer

$$u_p(x) = 1 - e^{-\alpha_p x}.$$

Then we will have an explicit PEFF solution: the contingent payments are actually linear functions of the risks.

**THEOREM 7.1** (*PEFF solution under CARA utilities and deterministic asset returns.*) *The PEFF solution to an  $N$ -period problem with exponential utility functions and deterministic asset returns  $\{R_n\}$  is of the form*

$$C_n = a_n [(X_n + F_{n-1}R_n) - w_n] + v_n = a_n(A_n - w_n) + v_n,$$

$$F_n = A_n - C_n = (1 - a_n)A_n - (v_n - a_n w_n),$$

where

$$w_n := \mathbb{E}^Q A_n$$

which can be calculated recursively from the budget constraints and the  $a_n$ 's are defined recursively by

$$a_N = \frac{\alpha_p}{\alpha_p + \alpha_N}, \quad (7.1)$$

$$a_n = \frac{a_{n+1}\alpha_{n+1}R_{n+1}}{\alpha_n + a_{n+1}\alpha_{n+1}R_{n+1}} \quad n = 1, \dots, N-1. \quad (7.2)$$

PROOF See appendix. □

430 Theorem 7.1 shows that under CARA utility, each contingent payment only takes a proportion  $a_n$  of  $A_n - w_n$  which is the excess return from total available asset, thus only takes part of the risk. The remaining part  $(1 - a_n)$  is shifted into the future. Under the CARA utility assumption, the risk-sharing rules don't depend on the distribution of the random variables.

REMARK 7.2 Suppose  $R_n \equiv R = 1 + r$  for  $n = 1, \dots, N$ . Also, let  $\alpha_n \equiv \alpha$  for  $n = 1, \dots, N$ , that is, we assume the same risk aversion level for all the agents except the buffer. The equations (7.2) become

$$a_n = \frac{a_{n+1}R}{1 + a_{n+1}R}.$$

If we let  $N \rightarrow \infty$ , then we shall have

$$a_n \rightarrow \frac{R - 1}{R} \approx r,$$

435 that is, given a sufficiently long horizon, the proportion that each agent takes from the total excess return is approximately equal to the risk-free rate.

## 8. Concluding Remarks

In this paper we have explored solving a multi-period risk sharing problem under the concept of Pareto efficiency and financial fairness. The important  
440 results are:

1. Theorem 3.2 characterizes the Pareto efficient risk-sharing rules: every PE risk-sharing rule can be associated uniquely to an optimization problem with the objective function being the weighted aggregate expected utility of the contingent payments, which can be further translated into the intertemporal balance equations. Theorem 3.5 shows how to compute the  
445 risk-sharing rule given the weights.
2. Theorem 5.1 establishes the existence and uniqueness of a PEFF risk-sharing rule. Furthermore, Theorem 5.2 indicates that the value profile will uniquely determine the weights.

450 3. Theorem 6.1 guarantees the possibility to find unique the PEFF rule numerically by a universal algorithm.

We conclude this paper with some comments on further possibilities. First, we have assumed that each agent can have only one contingent payment as a way of simplification. As a result, the optimization target (3.1) is time-additive and the value profile is straightforward to determine. If we make the generalization  
455 that each agent can have multiple contingent payments in different periods, two issues need to be resolved. Utility-wise, one needs to choose a preference functional for evaluating the welfare; value-wise, the fairness constraint in such a setting needs to be formulated. Some cases are essentially different from the setting in this paper, and the existence and uniqueness of the PEFF solution  
460 may have to be re-established.

In this paper the financial fairness is defined in an *ex ante* sense, i.e. the market values of the contingent payments will match the given value profile only at the time when the system starts. The FF will generally not hold *ex interim*,  
465 as the contingent payments are by nature contingent claims and their market values will change after the system starts. For a collective pension system which may include already the unborn cohorts at start, this issue may result in the so-called *discontinuity problem*: some future cohort may find themselves in a very disadvantageous position when they have to face a large deficit in the  
470 buffer left by the previous generations because of some preceding bad financial performance. The later cohort may argue that they didn't have a say when the system was initiated, thus they may choose not to step into the system. Strict ex-interim FF is meaningful, but leaves little space for intergenerational risk sharing. One may then adopt some fairness condition that lies between the two extremes as a compromise. These possibilities are beyond the scope of this  
475 paper and may be future topics of interest.

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### Appendix A. Proofs for Section 3

For any risk sharing rule  $\rho = (C_1, \dots, C_N, F_N) \in \mathcal{RS}$ , let

$$u(\rho) := (u_1(C_1), \dots, u_N(C_N), u_p(F_N))$$

and

$$\phi := \mathbb{E}^\mathbb{P} u(\rho) = (\mathbb{E}^\mathbb{P} u_1(C_1), \dots, \mathbb{E}^\mathbb{P} u_N(C_N), \mathbb{E}^\mathbb{P} u_p(F_N)) \in \mathbb{R}^{N+1}.$$

First note that  $\phi$  is a strictly concave and increasing function of  $\rho$  with co-domain  $\mathbb{R}^{N+1}$ . The PE optimization target then becomes

$$\mathbb{E}^\mathbb{P} \left[ \sum_{n=1}^N \theta_n u_n(C_n) + \theta_p u_p(F_N) \right] = \langle \theta, \phi \rangle$$

where  $\theta = (\theta_1, \dots, \theta_N, \theta_p) \in \mathbb{R}_{++}^{N+1}$ .

We need the following definitions and results in preparation for the proof of Theorem 3.2.

525 LEMMA APPENDIX A.1 *Consider  $n$  concave functions  $\{f_i | i = 1, \dots, n\}$  from a common domain  $K$  to  $\mathbb{R} \cup \{-\infty\}$ . Then  $F(K) - \mathbb{R}_+^n := \{x - y | \forall x \in F(K), y \in \mathbb{R}_+^n\}$  is convex where  $F := (f_1, f_2, \dots, f_n)$ .*

PROOF See the proof of Proposition 2.6 from Aubin [18]. □

530 We will use a separation theorem in the proof of Theorem 3.2. We then need to introduce the following definitions. <sup>1</sup>

DEFINITION APPENDIX A.2 (*Affine sets in  $\mathbb{R}^n$ .*) A subset  $M \in \mathbb{R}^n$  is called an affine set if  $(1 - \lambda)x + \lambda y \in M$  for any  $x, y \in M$  and  $\lambda \in \mathbb{R}$ .

DEFINITION APPENDIX A.3 (*Affine hull.*) The affine hull of any subset  $M \in \mathbb{R}^n$ , which is denoted as  $\text{aff}(M)$ , is the smallest affine set that contains  $M$ .

---

<sup>1</sup>Interested readers are referred to Rockafeller [19] for more details.

535 DEFINITION APPENDIXA.4 (*Relative interior and boundary.*) The relative interior of a convex set  $C \subset \mathbb{R}$ , which is denoted as  $ri(C)$ , is defined as the interior of  $C$  when it is regarded as a subset of  $aff(C)$ . The relative boundary of  $C$  is the difference of the closure of  $C$  and the relative interior of  $C$ .

The following lemma is crucial in proving Theorem 3.2.

540 LEMMA APPENDIXA.5 *Let  $C$  be a convex set. A point  $x \in C$  is a relative boundary point of  $C$  if and only if there exists a linear function not constant on  $C$  such that it achieves its maximum over  $C$  at  $x$ .*

PROOF See Corollary 11.6.2 by Rockafellar [19]. □

PROOF OF THEOREM 3.2.

1  $\Rightarrow$  2 : Let  $\rho = (C_1, C_2, \dots, C_N, F_N)$  be PE. Then we have that  $\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}$  is convex by Lemma AppendixA.1. Note that an element  $\rho^*$  is PE if and only if

$$\{\phi(\rho^*)\} \cap (\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}) = \{\phi(\rho^*)\}$$

and

$$\{\phi(\rho^*)\} \cap (\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1})^\circ = \emptyset.$$

545 Otherwise, if  $\{\phi(\rho^*)\} \in (\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1})^\circ$ , then there exist  $\rho' \in \mathcal{RS}$  and  $c \in \mathbb{R}_+^{N+1}$  with  $c \neq 0$  such that  $\phi(\rho^*) = \phi(\rho') - c$ , which means  $\rho'$  results in a Pareto improvement. This is in contradiction with the assumption that  $\rho^*$  is PE.

$\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}$  is a full-dimensional set thus its relative interior is the same as its interior. Write  $\phi^* = \phi(\rho^*)$ . Then  $\phi^*$  is a relative boundary point of  $\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}$ , as it belongs to  $\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}$ , thus to its closure, but not its relative interior. According to Lemma AppendixA.5, for this  $\phi^*$ , there exists a  $\theta^* \neq 0$  such that

$$\sup_{\phi \in \phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}} \langle \theta^*, \phi \rangle \leq \langle \theta^*, \phi^* \rangle.$$

First note that any coordinates of  $\theta^*$  cannot be negative as then

$$\sup_{\phi \in \phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}} \langle \theta^*, \phi \rangle = +\infty.$$

No coordinates of  $\theta^*$  can be zero. If this would be the case, suppose  $\theta_1^* = 0$  while  $\theta_2^* > 0$  without loss of generality. Then any  $\rho = (C_1, C_2, \dots, C_N, F_N)$  cannot be optimal since for any small  $\epsilon > 0$  such that  $C_1^{j_1} - \epsilon > b_1$  for all  $j_1 \in \mathcal{J}_1$ ,  $\rho_\epsilon = (C_1 - \epsilon, C_2 + R_2\epsilon, \dots, C_N, F_N)$  will result in a larger optimization target because  $u_2$  is strictly increasing.

2  $\Rightarrow$  1 : consider a risk-sharing rule  $\rho$  that maximizes  $\langle \theta, \phi \rangle$  for some  $\theta \in \mathbb{R}_+^{N+1}$ . If  $\rho$  is not PE, then there exists another  $\tilde{\rho}$  such that  $\tilde{\phi} \succeq \phi$  and hence

$$\langle \theta, \tilde{\phi} \rangle > \langle \theta, \phi \rangle$$

which results in a contradiction.

2  $\Leftrightarrow$  3 : as we are working with a finite probability space, we may use the Lagrangian multiplier method to solve the maximization problem.

For  $n = 1, \dots, N$ , reorganize the budget constraint and we have

$$F_n^{J_{n-1}j_n} + C_n^{J_{n-1}j_n} - X_n^{j_n} - F_{n-1}^{J_{n-1}} R_n^{j_n} = 0.$$

Define

$$F_n^{J_{n-1}j_n} + C_n^{J_{n-1}j_n} - X_n^{j_n} - F_{n-1}^{J_{n-1}} R_n^{j_n}$$

as  $BC^{J_{n-1}j_n}$  or  $BC^{J_n}$ .

We then maximize

$$\sum_{n=1}^N \left\{ \theta_n \sum_{J_n \in \mathcal{J}_n} \mathbb{P}(J_n) u_n(C_n^{J_n}) + \sum_{J_n \in \mathcal{J}_n} \lambda^{J_n} BC^{J_n} \right\} + \theta_p \sum_{J_N \in \mathcal{J}_N} \mathbb{P}(J_N) u_p(F_N^{J_N})$$

where the  $\lambda$ 's are the Lagrangian multipliers.

For any  $n < N$ , setting the first-order partial derivative with respect to  $C_n^{J_n}$  to zero will help us find a stationary point of the optimization problem. It gives

$$\mathbb{P}(J_n) \theta_n u'_n(C_n^{J_n}) + \lambda^{J_n} = 0 \quad \forall J_n \in \mathcal{J}_n.$$

For  $n + 1$  similarly we have, along the trajectory  $J_n$

$$\mathbb{P}(J_n j_{n+1}) \theta_{n+1} u'_{n+1}(C_{n+1}^{J_n j_{n+1}}) + \lambda^{J_n j_{n+1}} = 0 \quad \forall j_{n+1} \in \mathcal{J}_n^{n+1}.$$

Now take the partial derivative with respect to  $F_n$  and set to zero

$$\lambda^{J_n} = \sum_{j_{n+1} \in \mathcal{J}_n^{n+1}} \lambda^{J_n j_{n+1}} R_{n+1}^{j_{n+1}}.$$

This will lead to

$$\theta_n u'_n(C_n^{J_n}) = \theta_n \sum_{j_{n+1} \in \mathcal{J}_n^{n+1}} u'_{n+1}(C_{n+1}^{J_n j_{n+1}}) R_{n+1}^{j_{n+1}} \frac{\mathbb{P}(J_n j_{n+1})}{\mathbb{P}(J_n)}.$$

By the assumption of sequential independence we have

$$\frac{\mathbb{P}(J_n j_{n+1})}{\mathbb{P}(J_n)} = \mathbb{P}(j_{n+1}).$$

Then the equation can be further rewritten as

$$\theta_n u'_n(C_n) = \theta_{n+1} \mathbb{E}_n^{\mathbb{P}} [u'_{n+1}(C_{n+1}) R_{n+1}] \quad \text{for } n = 1, \dots, N-1.$$

$C_N$  and  $F_N$  are both  $\mathcal{F}_N$ -measurable and we have

$$\theta_N u'_N(C_N^{J_N}) = \theta_p u'_p(F_N^{J_N}) = -\lambda^{J_N}$$

by taking partial derivatives with respect to  $C_N$  and  $F_N$  and setting them to be zero.

560 We have arrived at a stationary point thanks to the Lagrangian multiplier method; this stationary point is the unique global optimum once we note that the optimization target is a concave function with respect to the decision variables and the feasible set is convex.  $\square$

PROOF OF THEOREM 3.5. The optimization target (3.1) is a parameterized optimization problem of time-additive utility functions:

$$\begin{aligned} & \max_{C_1, \dots, C_N} \mathbb{E}^{\mathbb{P}} \left[ \sum_{n=1}^N \theta_n u_n(C_n) + \theta_p u_p(F_N) \right] \\ & \text{such that } F_n + C_n = X_n + F_{n-1} R_n \quad n = 1, \dots, N, \\ & F_0 = 0. \end{aligned}$$

This optimization problem can be solved by dynamic programming. Add in a new time point  $t_{N+1} = t_N$ , and

$$X_{N+1} \equiv 0, \quad R_{N+1} \equiv 1.$$

Define

$$A_n := X_n + F_{n-1} R_n \quad n = 1, \dots, N+1,$$

which has the interpretation as the total available asset at time  $t_n$  to be divided into the current cash flow and the buffer for later use. Note that by definition  $A_{N+1} = F_N$ . The  $A$ 's are the state variables, the  $C$ 's are the decision variables and the  $X$ 's and  $R$ 's are the risks. Then we shall have the optimization problem formulated as

$$\begin{aligned} & \max_{C_1, \dots, C_N} \mathbb{E}^{\mathbb{P}} \left[ \sum_{n=1}^N \theta_n u_n(C_n) + \theta_p u_p(A_{N+1}) \right] \\ \text{such that } & A_{n+1} = X_{n+1} + (A_n - C_n)R_{n+1}, \quad n = 1, \dots, N, \\ & A_1 = X_1. \end{aligned}$$

Proposition 1.3.1 in [17] tells that in order to solve the problem one needs to define first

$$V_{N+1}(A_{N+1}) = \theta_p u_p(A_{N+1}),$$

and then define backwards, for  $n = 1, \dots, N$

$$V_n(A_n) = \max_{C_n} \mathbb{E}_n^{\mathbb{P}} [\theta_n u_n(C_n) + V_{n+1}(X_{n+1} + (A_n - C_n)R_{n+1})]. \quad (\text{A.1})$$

This can be solved by taking the derivative of

$$\mathbb{E}_n^{\mathbb{P}} [\theta_n u_n(C_n) + V_{n+1}(X_{n+1} + (A_n - C_n)R_{n+1})] \quad (\text{A.2})$$

with respect to  $C_n$  and setting it to zero. We will start from period  $N$  and go backwards in time in order to verify the differentiability of the  $V_n$ 's. For period  $N$ , note that the target (A.2) becomes

$$\theta_N u_N(C_N) + \theta_p u_p(F_N) = \theta_N u_N(C_N) + \theta_p u_p(A_N - C_N).$$

The conditional expectation drops out because of the measurability of  $C_N$  and  $F_N$ . It is continuous and differentiable with respect to  $C_N$ . Take the derivative and set it to zero; we get

$$\theta_N u'_N(C_N^*) = \theta_p u'_p(A_N^* - C_N^*) := L_N^*.$$

Here the star indicates that it is the optimal solution. Next, define

$$\begin{aligned} G_N(x) &:= I_N \left( \frac{x}{\theta_N} \right) + I_p \left( \frac{x}{\theta_p} \right), \\ g_N(x) &:= G_N^{-1}. \end{aligned}$$

Both  $G_N$  and  $g_N$  are well-defined.  $G_N$  is the sum of two strictly decreasing bijective functions thus it is strictly decreasing and bijective from  $\mathbb{R}_{++}$  to  $(\max\{b_N, b_p\}, +\infty)$ , and it follows that  $g_N$  is also strictly decreasing and bijective from  $(\max\{b_N, b_p\}, +\infty)$  to  $\mathbb{R}_{++}$ . The Inada conditions tell

$$\lim_{x \rightarrow 0} G_N(x) = +\infty, \quad \lim_{x \rightarrow +\infty} G_N(x) = \max\{b_N, b_p\}$$

and thus

$$\lim_{x \rightarrow \max\{b_N, b_p\}} g_N(x) = +\infty, \quad \lim_{x \rightarrow +\infty} g_N(x) = 0.$$

$L_N^*$  can then be calculated as

$$L_N^* = g_N(A_N^*)$$

and

$$C_N^* = I_N \left( \frac{L_N^*}{\theta_N} \right), \quad F_N^* = I_p \left( \frac{L_N^*}{\theta_p} \right).$$

The value function is

$$V_N(A_N^*) = \theta_N u_N(C_N^*) + \theta_p u_p(A_N^* - C_N^*),$$

which is a differentiable function of  $A_N^*$  when we regard  $A_N^*$  as its argument:

$$V_N'(A_N^*) = \theta_p u_p'(A_N^* - C_N^*) = \theta_p u_p'(F_N^*) = g_N(A_N^*).$$

Going one period backwards, we have the value function

$$V_{N-1}(A_{N-1}) = \max_{C_{N-1}} \mathbb{E}_{N-1}^{\mathbb{P}} [\theta_{N-1} u_{N-1}(C_{N-1}) + V_N(X_N + (A_{N-1} - C_{N-1})R_N)].$$

The part

$$\begin{aligned} & \mathbb{E}_{N-1}^{\mathbb{P}} [\theta_{N-1} u_{N-1}(C_{N-1}) + V_N(X_N + (A_{N-1} - C_{N-1})R_N)] \\ &= \sum_{j \in \mathcal{J}_{N-1}^N} \mathbb{P}(j) \left[ \theta_{N-1} u_{N-1}(C_{N-1}) + V_N(X_N^j + (A_{N-1} - C_{N-1})R_N^j) \right] \end{aligned}$$

is a differentiable function of  $C_{N-1}$  when we regard  $C_{N-1}$  as its argument. We then take the derivative with respect to  $C_{N-1}$  and set it to zero. Differentiation

and conditional expectation can be interchanged, since we are working on a finite probability space. We have

$$\begin{aligned} & \sum_{j \in \mathcal{J}_{N-1}^N} \mathbb{P}(j) \left[ \theta_{N-1} u'_{N-1}(C_{N-1}) + V'_N(A_N^j)(-R_N^j) \right] \\ &= \theta_{N-1} u'_{N-1}(C_{N-1}) - \mathbb{E}_{N-1}^{\mathbb{P}} V'_N(A_N) R_N = 0, \end{aligned}$$

which leads us to

$$L_{N-1}^* = \theta_{N-1} u'_{N-1}(C_{N-1}^*) = \mathbb{E}_{N-1}^{\mathbb{P}} [g_N(A_N^*) R_N] = \mathbb{E}_{N-1}^{\mathbb{P}} [L_N^* R_N].$$

We then define

$$h_{N-1}(x) := \frac{1}{\theta_N} \mathbb{E}_{N-1}^{\mathbb{P}} [g_N(X_N + x R_N) R_N].$$

Due to the assumption of sequential independence,  $h_{N-1}(x)$  can further be written in the form of an unconditional expectation

$$h_{N-1}(x) = \frac{1}{\theta_N} \mathbb{E}^{\mathbb{P}} [g_N(X_N + x R_N) R_N]$$

since both  $X_N$  and  $R_N$  are independent from  $\mathcal{F}_{N-1}$ . Note that  $h_{N-1}$  is invertible since by definition it is a weighted sum of strictly decreasing functions; thus  $h_{N-1}$  is also a strictly decreasing function with domain  $(d_{N-1}, +\infty)$ , where  $d_{N-1}$  is defined as

$$d_{N-1} = \inf \left\{ d \in \mathbb{R} \mid X_N^j + d R_N^j \geq \max\{b_N, b_p\} \quad \forall j \in \mathcal{J}_{N-1}^N \right\}.$$

Furthermore,  $h_{N-1}$  can be viewed as the marginal utility of a stereotype utility function since

- it is continuous and strictly decreasing,
- it satisfies

$$\lim_{x \rightarrow d_{N-1}} h_{N-1}(x) = +\infty, \quad \lim_{x \rightarrow +\infty} h_{N-1}(x) = 0.$$

Write

$$H_{N-1} := h_{N-1}^{-1}.$$

Then once we combine

$$C_{N-1}^* + F_{N-1}^* = X_{N-1} + F_{N-2}^* R_{N-1} = A_{N-1}^*$$

with

$$L_{N-1}^* = \theta_{N-1} u'_{N-1}(C_{N-1}^*) = \theta_N h_{N-1}(F_{N-1}^*),$$

we have

$$I_{N-1} \left( \frac{L_{N-1}^*}{\theta_{N-1}} \right) + H_{N-1} \left( \frac{L_{N-1}^*}{\theta_N} \right) = A_{N-1}^*.$$

Next, define

$$\begin{aligned} G_{N-1}(x) &:= I_{N-1} \left( \frac{x}{\theta_{N-1}} \right) + H_{N-1} \left( \frac{x}{\theta_N} \right), \\ g_{N-1}(x) &:= G_{N-1}^{-1}. \end{aligned}$$

$G_{N-1}$  and  $g_{N-1}$  are well-defined just as  $G_N$  and  $g_N$ .  $L_{N-1}$  can then be calculated as

$$L_{N-1}^* = g_{N-1}(A_{N-1}^*)$$

and

$$C_{N-1}^* = I_{N-1} \left( \frac{L_{N-1}^*}{\theta_{N-1}} \right), \quad F_{N-1}^* = H_{N-1} \left( \frac{L_{N-1}^*}{\theta_N} \right).$$

For the value function

$$V_{N-1}(A_{N-1}^*) = \theta_{N-1} u_{N-1}(C_{N-1}^*) + \mathbb{E}_{N-1}^{\mathbb{P}} V_N [X_N + (A_{N-1}^* - C_{N-1}^*) R_N],$$

it follows that  $V_{N-1}$  is differentiable when we regard  $A_{N-1}^*$  as the argument and one can calculate  $V'_{N-1}(A_{N-1}^*)$  as follows:

$$V'_{N-1}(A_{N-1}^*) = \mathbb{E}_{N-1}^{\mathbb{P}} [V'_N(A_N^*) \cdot R_N] = \mathbb{E}_{N-1}^{\mathbb{P}} [L_N^* R_N] = L_{N-1}^* = g_{N-1}(A_{N-1}^*).$$

Proceeding one period backwards, we then have the corresponding value function

$$V_{N-2}(A_{N-2}) = \max_{C_{N-2}} \mathbb{E}_{N-2}^{\mathbb{P}} [\theta_{N-2} u_{N-2}(C_{N-2}) + V_{N-1}(X_{N-1} + (A_{N-2} - C_{N-2}) R_{N-1})].$$

To solve the right hand side, note that the expression

$$\theta_{N-2} u_{N-2}(C_{N-2}) + V_{N-1}(X_{N-1} + (A_{N-2} - C_{N-2}) R_{N-1})$$

is a differentiable function of  $C_{N-2}$ . We then take the derivative with regard to  $C_{N-2}$  and set it to zero:

$$\begin{aligned} & \theta_{N-2} u'_{N-2}(C_{N-2}) + \mathbb{E}_{N-2}^{\mathbb{P}} [V'_{N-1}(A_{N-1}) \cdot (-R_{N-1})] \\ &= \theta_{N-2} u'_{N-2}(C_{N-2}) - \mathbb{E}_{N-2}^{\mathbb{P}} [g_{N-1}(A_{N-1}) R_{N-1}] \\ &= \theta_{N-2} u'_{N-2}(C_{N-2}) - \mathbb{E}_{N-2}^{\mathbb{P}} [g_{N-1}(X_{N-1} + (A_{N-2} - C_{N-2}) R_{N-1}) R_{N-1}] = 0. \end{aligned}$$

We can then repeat what has been done in period  $N - 1$ . This recursive procedure can be continued backwards in time until we arrive at the first period. That is, we can always define recursively for  $n = 1, \dots, N - 2$

$$\begin{aligned} h_n(x) &= \mathbb{E}_n^{\mathbb{P}} \left[ \frac{1}{\theta_{n+1}} g_{n+1}(X_{n+1} + x R_{n+1}) R_{n+1} \right] \\ &= \mathbb{E}_n^{\mathbb{P}} \left[ \frac{1}{\theta_{n+1}} g_{n+1}(X_{n+1} + x R_{n+1}) R_{n+1} \right], \\ H_n &= h_n^{-1}, \\ G_n(x) &:= I_n \left( \frac{x}{\theta_n} \right) + H_n \left( \frac{x}{\theta_{n+1}} \right), \\ g_n(x) &:= G_n^{-1}, \end{aligned}$$

and the decision variables are given by

$$\begin{aligned} C_n^* &= I_n \left( \frac{g_n(A_n^*)}{\theta_n} \right) \quad n = 1, \dots, N, \\ F_n^* &= H_n \left( \frac{g_n(A_n^*)}{\theta_{n+1}} \right) \quad n = 1, \dots, N - 1, \\ F_N^* &= I_p \left( \frac{g_N(A_N^*)}{\theta_p} \right). \end{aligned}$$

This will be the unique solution of the optimization problem, as the optimization target is concave with respect to the decision variables and the feasible set is convex.  $\square$

PROOF OF LEMMA 3.8. By definition the function  $g$ 's are all strictly decreasing.

We have

$$\begin{aligned} C_n &= I_n \left( \frac{g_n(X_n + F_{n-1}R_n)}{\theta_n} \right) \quad \text{for } n = 1, \dots, N, \\ F_n &= H_n \left( \frac{g_n(X_n + F_{n-1}R_n)}{\theta_{n+1}} \right) \quad \text{for } n = 1, \dots, N-1, \\ F_N &= I_p \left( \frac{g_N(X_N + F_{N-1}R_N)}{\theta_p} \right), \end{aligned}$$

570 thus both  $C_n$  and  $F_n$  are increasing functions of  $A_n = X_n + F_{n-1}R_n$ .

We only have to consider the case when only one coordinate of  $(X, R) = (X_1, \dots, X_N, R_1, \dots, R_N)$  increases. First consider two trajectories  $J, J^*$  such that there is a time point  $\tau = 1, \dots, N$  such that  $X_\tau^J > X_\tau^{J^*}$  and other random variables from  $(X, R)$  are equal. Since

$$F_n = H_n \left( \frac{g_n(X_n + F_{n-1}R_n)}{\theta_{n+1}} \right) \quad \text{for } n = 1, \dots, N-1$$

then  $F_1^J = F_1^{J^*}$ , and this will lead to  $F_2^J = F_2^{J^*}$ . Doing this recursively we conclude that  $F_n^J = F_n^{J^*}$  for any  $n < \tau$ . Then as

$$X_\tau^J + F_{\tau-1}^J R_\tau^J > X_\tau^{J^*} + F_{\tau-1}^{J^*} R_\tau^{J^*}$$

we have

$$C_\tau^J > C_\tau^{J^*}, \quad F_\tau^J > F_\tau^{J^*},$$

and the latter will tell that  $C_n^J > C_n^{J^*}$  for all  $n > \tau$ . Also  $F_N^J > F_N^{J^*}$ . Then  $\rho^J \not\geq \rho^{J^*}$ .

The cases when only  $R_\tau^J > R_\tau^{J^*}$  follows analogously.  $\square$

575 It is convenient to have the following definition before we continue to the proof of Lemma 3.9.

DEFINITION APPENDIXA.6 (*N-PE Problem.*) An  $N$ -PE problem refers to the 4-tuple  $((X, R), \rho, u', \theta)$  and the corresponding equation systems BC (5.1) and IBE (5.2), where  $(X, R)$  is a vector of random variables,  $\rho$  a vector of decision variables,  $u'$  an  $(N+1)$ -tuple of stereotype marginal utility functions and  $\theta$  a

constant vector, i.e.

$$\begin{aligned}(X, R) &= (X_1, \dots, X_N, R_2, \dots, R_N) \in \mathcal{L}^{2N+1}, \\ \rho &= (C_1, \dots, C_N, F_N) \in \mathcal{L}^{N+1}, \\ u' &= (u'_1, \dots, u'_N, u'_p), \\ \theta &= (\theta_1, \dots, \theta_N, \theta_p) \in \mathbb{R}_{++}^{N+1},\end{aligned}$$

where  $\mathcal{L} := \mathbb{R}^\Omega$  is the space of random variables over the underlying probability space.

PROOF OF LEMMA 3.9. The key point of the proof is that otherwise, the IBE and the BC cannot hold simultaneously.

We use mathematical induction to show this. First consider  $N = 1$ . For a 1-PE problem this is true; we only have two agents including the buffer and there will be only one family of IBE:

$$\theta_1 u'_1(C_1) = \theta_p u'_p(F_1),$$

and the budget constraints are

$$C_1 + F_1 = X_1.$$

580 For any trajectory  $J \in \mathcal{J}_1$ , if  $\theta_1$  increases, then we argue that  $C_1^J$  cannot decrease. Otherwise (i.e.  $C_1^J$  decreases), by the budget constraint  $F_1^J$  will increase, but according to the IBE it will decrease, which is a contradiction. For the same reason  $C_1^J$  cannot stay the same. Thus  $C_1^J$  will increase and  $F_1^J$  has to decrease. As there is a symmetry between  $C_1$  and  $F_1$ , we conclude that the  
585 argument is true for single-period problems.

Assume the statement holds true for an  $N$ -PE problem,  $N > 1$ . Then consider the case of an  $(N + 1)$ -PE problem with the the conventional notations

$$\begin{aligned}(X, R) &= (X_1, \dots, X_{N+1}, R_2, \dots, R_{N+1}), \\ \rho &= (C_1, \dots, C_{N+1}, F_{N+1}), \\ u' &= (u'_1, \dots, u'_{N+1}, u'_p), \\ \theta &= (\theta_1, \dots, \theta_{N+1}, \theta_p).\end{aligned}$$

First consider if some  $\theta_n$  increases,  $n < N + 1$ . Then as we have discussed, this  $(N + 1)$ -PE problem can be converted into an induced  $N$ -PE problem by truncation at time point  $t_N$  and define  $h_N$  as has been defined in Theorem 3.5, that is, the 4-tuple

$$\begin{aligned}(X, R)_{[N]} &= (X_1, \dots, X_N, R_2, \dots, R_N), \\ \rho_{[N]} &= (C_1, \dots, C_N, F_N), \\ u'_{[N]} &= (u'_1, \dots, u'_N, h_N), \\ \theta_{[N]} &= (\theta_1, \dots, \theta_N, \theta_{N+1}).\end{aligned}$$

590 Consider this  $N$ -PE problem. According to the induction assumption, we will have that for any  $J \in \mathcal{J}_{N+1}$ ,  $C_n^J$  will increase if  $\theta_n$  increases, while other cash outflows will decrease. So  $F_N^J$  will decrease and so is  $A_{N+1}^J$ . Note that by definition the function  $g_{N+1}$  will stay the same if  $\theta_n$  increases. Thus  $C_{N+1}^J$  and  $F_{N+1}^J$  will both decrease as they are increasing functions of  $A_{N+1}^J$ .

Now consider the situation if  $\theta_{N+1}$  increases. We will show that  $F_{N+1}^{J_{N+1}}$  will decrease. Otherwise (i.e.  $F_{N+1}^{J_{N+1}}$  either increases or stays the same), by the final period IBE

$$\theta_{N+1} u'_{N+1}(C_{N+1}^{J_{N+1}}) = \theta_p u'_p(F_{N+1}^{J_{N+1}}) = L_{N+1}^{J_{N+1}}$$

we have that  $C_{N+1}^{J_{N+1}}$  has to increase because of the monotonicity of  $u'_{N+1}$  and  $u'_p$ . Then by the budget constraint for that period

$$C_{N+1}^{J_{N+1}} + F_{N+1}^{J_{N+1}} = X_{N+1}^{j_{N+1}} + F_N^{j_{N+1}} R_{N+1}^{j_{N+1}}$$

$F_N^{j_{N+1}}$  also has to increase. This will lead to the fact that  $L_{N+1}^{j_{N+1}^*}$  will not increase for any  $j_{N+1}^* \in \mathcal{J}_N^{N+1}$ . This is because we have

$$L_{N+1}^{j_{N+1}^*} = g_{N+1}(X_{N+1}^{j_{N+1}^*} + F_N^{j_{N+1}^*} R_{N+1}^{j_{N+1}^*})$$

and

$$L_{N+1}^{j_{N+1}^*} = g_{N+1}(X_{N+1}^{j_{N+1}^*} + F_N^{j_{N+1}^*} R_{N+1}^{j_{N+1}^*})$$

595 which shows that  $L_{N+1}^{J_N J_{N+1}}$  and  $L_{N+1}^{J_N J_{N+1}^*}$  should have the same monotonicity property with respect to  $\theta_{N+1}$ . The result is that  $\mathbb{E}_N(L_{N+1}^{J_N} R_{N+1})$  will not increase.

According to the global budget constraint along that trajectory, there has to be at least one  $n$  such that  $C_n^{J_n}$  will decrease. Let the set of such  $n$ 's be denoted by  $\mathcal{T}$ . Consider first the situation that  $\max\{\mathcal{T}\} = N$ . Then  $L_N^{J_N} = \theta_N u'_N(C_N^{J_N})$  will increase. On the other hand,  $\mathbb{E}_N(L_{N+1}^{J_N} R_{N+1})$  will not increase. We then arrive at a contradiction by noting that by IBE we should have

$$L_N^{J_N} = \mathbb{E}_N(L_{N+1}^{J_N} R_{N+1}).$$

Then consider more generally that  $\tau = \max\{\mathcal{T}\} < N$ . Then as  $F_N^{J_N}$  will increase and  $C_N^{J_N}$  will not decrease, by budget constraint we know  $F_{N-1}^{J_{N-1}}$  will increase. Repeat this reasoning until we get that  $F_\tau^{J_\tau}$  will have to increase. Then by analogy as above we will have that  $\mathbb{E}_\tau(L_{\tau+1}^{J_\tau} R_{\tau+1})$  will not increase. However,  $L_\tau = \theta_\tau u'_\tau(C_\tau^{J_\tau})$  will increase as  $C_\tau^{J_\tau}$  decreases. The IBE will then not hold. We conclude that  $F_{N+1}^{J_{N+1}}$  will decrease and  $L_{N+1}^{J_{N+1}}$  will increase. According to

$$L_n = \mathbb{E}^{\mathbb{P}}[L_{n+1} R_{n+1}]$$

we know that for any  $n < N + 1$ , along the trajectory  $J_n$  which is the up-to-time- $t_n$  part of  $J_{N+1}$ ,  $L_n^{J_n}$  will increase. Then  $C_n^{J_n}$  will decrease since

$$L_n^{J_n} = \theta_n u'_n(C_n^{J_n}).$$

Finally, consider the global budget constraint (2.3) along the trajectory  $J_{N+1}$ . It must be that  $C_{N+1}^{J_{N+1}}$  will have to increase since all the other  $C$ 's and  $F_{N+1}$  will decrease.  
600

The case when only  $\theta_p$  increases follows analogously as there is symmetry between  $C_{N+1}$  and  $F_{N+1}$ . This completes the proof.  $\square$

## AppendixB. Proofs for Section 5

Please note that some of the proofs in this section make use of the mapping  
605  $\varphi$  defined in Section 6.

DEFINITION APPENDIXB.1 (*N-PEFF Problem.*) An  $N$ -PEFF problem refers to the 4-tuple  $((X, R), \rho, u', v)$  and the corresponding equation systems (5.1), (5.2) and (5.3), where  $(X, R)$  is a vector of random variables,  $\rho$  a vector of decision variables,  $u'$  an  $(N + 1)$ -tuple of stereotype marginal utility functions and  $v$  a value profile vector, i.e.

$$\begin{aligned}(X, R) &= (X_1, \dots, X_N, R_2, \dots, R_N) \in \mathcal{L}^{2N+1}, \\ \rho &= (C_1, \dots, C_N, F_N) \in \mathcal{L}^{N+1}, \\ u' &= (u'_1, \dots, u'_N, u'_p), \\ v &= (v_1, \dots, v_N, v_p) \in \mathcal{V}.\end{aligned}$$

The set  $\mathcal{V}$  is totally determined by  $(X, R)$  and  $u'$  according to Expression (4.3).

DEFINITION APPENDIXB.2 (*Hilbert metric on  $\mathbb{R}_{++}^n$ .*) The Hilbert metric defines a distance as

$$d(x, y) = \log \frac{\max_i \{x_i/y_i\}}{\min_i \{x_i/y_i\}}$$

for any  $x, y \in \mathbb{R}_{++}^n$ . It is not a real metric as

$$d(x, y) = 0 \quad \Leftrightarrow \quad \exists c \in \mathbb{R}^+ \quad \text{such that} \quad y = cx.$$

It will become a true metric if restricted on e.g. the open unit simplex in  $\mathbb{R}_{++}^n$ .

LEMMA APPENDIXB.3 *If  $\phi : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}^n$  is homogeneous and strongly monotone, then  $\phi$  is contractive with respect to the Hilbert metric.*

610 PROOF See for instance Lemma 4.5 in Pazdera et al [15]. □

Any contractive mapping  $\phi$  can only have one fixed point. Suppose there are two, namely  $x$  and  $y$  with  $d(x, y) > 0$ . Then by contractiveness we have

$$d(x, y) = d(\phi(x), \phi(y)) < d(x, y)$$

which is contradictory. Then  $d(x, y) = 0$ . Note that the uniqueness is in the sense of Hilbert metric.

The following lemma is the key part of proving the uniqueness of the PEFF solution.

LEMMA APPENDIXB.4 *The mapping  $\varphi_1$  defined in Section 6 is strictly increasing, i.e. for any trajectory  $J \in \mathcal{J}_N$ , we have that*

$$L_N^J(\theta') \geq L_N^J(\theta'') \quad \forall \theta' \geq \theta''.$$

615 PROOF To show this we only need to show that  $L_N^J$  is strictly increasing with respect to any one of the coordinates of  $\theta$ . We can utilize Lemma 3.9.

Consider first that only  $\theta_n$  increases while the other  $\theta$ 's stay the same,  $n = 1, \dots, N$ . Then according to Lemma 3.9,  $F_N^J$  will decrease thus

$$L_N^J = \theta_p u'_p(F_N^J)$$

will increase. The case when only  $\theta_p$  increases follows analogously as there is symmetry between  $C_N$  and  $F_N$ .  $\square$

LEMMA APPENDIXB.5 *(The uniqueness of the PEFF rule.) For any given value*  
 620 *profile  $v = (v_1, \dots, v_N, v_p) \in \mathcal{V}$ , the corresponding PEFF risk-sharing rule will be unique if it exists.*

PROOF The main point of this proof is to show that  $\varphi$  defined in Section 6 is homogeneous and strictly monotone thus by Lemma AppendixB.3 it can only have one fixed point (up to normalization) if it has.

The mapping is homogeneous by definition thus we only have to consider monotonicity. First, according to Lemma AppendixB.4  $\varphi_1$  is strictly increasing with respect to  $\theta$  along all possible trajectories. Then  $L_n^J$  is also increasing since

$$L_n = \mathbb{E}_n^{\mathbb{P}}[L_{n+1}R_{n+1}].$$

Now consider  $\theta' \geq \theta''$ . Then for any  $J \in \mathcal{J}_N$  we have that  $L_N^J(\theta') > L_N^J(\theta'')$ . For any possible  $n$ , the  $n$ -th coordinate of  $\varphi_2$ :  $\varphi_{2(n)}(L_N) = [\mathbb{E}^{\mathbb{Q}} I_n \left( \frac{L_n}{\theta_n} \right)]^{-1}(v_n)$  will lead to that  $\varphi_{2(n)}(L_N(\theta')) > \varphi_{2(n)}(L_N(\theta''))$ . This is because  $\varphi_2$  will always require that

$$\mathbb{E}^{\mathbb{Q}} C_n = \mathbb{E}^{\mathbb{Q}} I_n \left( \frac{L_n}{\theta_n} \right) = \sum_{J \in \mathcal{J}_n} \mathbb{Q}(J) I_n \left( \frac{L_n^J}{\theta_n} \right) = v_n.$$

If  $L_n^J$  increases for all  $J \in \mathcal{J}_n$ , then  $\theta_n$  also will increase according to this  $\varphi_2$ .  
The result is that

$$\varphi(\theta') = \varphi_2(L_N(\theta')) > \varphi_2(L_N(\theta'')) = \varphi(\theta''),$$

625 i.e.  $\varphi$  is strictly increasing with respect to  $\theta$ . □

PROOF OF THEOREM 5.1. The proof uses mathematical induction. Note that we can always fix  $\theta_p = 1$  as a normalization to the  $\theta$ 's unless specified otherwise.

For any 1-PEFF problem, there is only one random variable  $X_1$  to be shared. One needs to solve

$$\begin{aligned} C_1 + F_1 &= X_1, \\ \theta_1 u'_1(C_1) &= \theta_p u'_p(F_1), \\ \mathbb{E}^{\mathbb{Q}} C_1 &= v_1. \end{aligned}$$

For any given  $\theta_1$ , the equations of BC and IBE will jointly produce a certain risk sharing rule according to the mapping  $\Phi$  in Theorem 3.5. However, the third FF equation may not hold. We need to show that there will exist some  $\theta_1$  such that the FF equation will hold. We define

$$w(\theta_1) = \mathbb{E}^{\mathbb{Q}} C_1 = \sum_{J \in \mathcal{J}_1} \mathbb{Q}(J) C_1^J.$$

It is a continuous function of  $\theta_1$  which follows as a property of the mapping  $\Phi$ . Next we will show that the value of the function  $w$  can be both above and below  
630  $v_1$ , so that there exists some  $\theta_1^*$  such that  $w(\theta_1^*) = v_1$  since  $w$  is continuous. This will be done by taking  $\theta_1$  to the limits.

First consider  $\lim_{\theta_1 \rightarrow 0} w(\theta_1)$ . Then along any trajectory  $J \in \mathcal{J}_1$  it must be that  $C_1^J \rightarrow b_1 < v_1$ . Otherwise, suppose there exists some sequence of  $\theta_1$ , say  $\{\hat{\theta}^{[m]}\}$  with  $\hat{\theta}^{[m]} \rightarrow 0$  as  $m \rightarrow \infty$ , such that

$$\lim_{m \rightarrow \infty} C_1^J(\hat{\theta}^{[m]}) \geq b_1 + \varepsilon$$

for some trajectory  $J$  and some  $\varepsilon \in \mathbb{R}_{++}$ . If  $b_1 = -\infty$  then this is interpreted as bounded from below. Then according to the IBE

$$\theta_1 u'_1(C_1^J) = \theta_p u'_p(F_1^J)$$

the left hand side will go to zero as  $u'_1(C_1^J)$  will be bounded. As a result,  $F_1^J$  will have to go to  $+\infty$  which is not possible if we take into consideration the budget constraint. We conclude that  $C_1^J \rightarrow b_1 < v_1$  along all the  $J$ 's if we let  
635  $\theta_1 \rightarrow 0$ .

Next consider  $\lim_{\theta_1 \rightarrow \infty} w(\theta_1)$ . Now we drop the normalization constraint  $\theta_p = 1$ . Taking into consideration the freedom of choosing a way of normalization, it follows that the following two statements are equivalent:

- fix  $\theta_p$  and let  $\theta_1 \rightarrow +\infty$ ;
- 640 • fix  $\theta_1$  and let  $\theta_p \rightarrow 0$ .

Then following the analogy above we have  $F_1^J \rightarrow b_p$  for all  $J \in \mathcal{J}_1$  as  $\theta_p \rightarrow 0$ . Thus  $\lim_{\theta_1 \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} F_1 = b_p$  and according to the budget constraint

$$\lim_{\theta_1 \rightarrow \infty} w(\theta_1) = \lim_{\theta_1 \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} C_1 = v_1 + v_p - b_p.$$

Then since

$$v_p - b_p > 0$$

must hold, we have

$$v_1 < v_1 + v_p - b_p.$$

By a simple intermediate value theorem we know that there will exist some  $\theta_1^*$  such that  $w(\theta_1^*) = v_1$ . Then we have found a weight vector  $\theta$  (i.e.  $(\theta_1^*, \theta_p = 1)$ ) that leads to a PEFF solution to the system. This indicates that the fixed points of the mapping  $\varphi$  will exist; the fixed point must be unique according  
645 to Lemma AppendixB.5, i.e. the vector  $\theta$  is unique. The uniqueness is up to normalization.

Let's assume that there always exists a unique solution for an  $N$ -PEFF problem,  $N > 1$ . Consider an  $(N + 1)$ -PEFF problem using our conventional

notations

$$\begin{aligned}(X, R) &= (X_1, \dots, X_{N+1}, R_2, \dots, R_{N+1}), \\ \rho &= (C_1, \dots, C_{N+1}, F_{N+1}), \\ u' &= (u'_1, \dots, u'_{N+1}, u'_p), \\ v &= (v_1, \dots, v_N, v_{N+1}, v_p)\end{aligned}$$

Consider the corresponding  $(N + 1)$ -PE problem with some given weight  $\theta$ . Use  $\theta_p = 1$  as a normalization. As we have discussed, the whole system will degrade to an induced  $N$ -PEFF problem with  $F_N$  now being the “final” buffer whose risk aversion is characterized by  $h_N$  given by Theorem 3.5. That is,

$$\begin{aligned}(X, R)_{[N]} &= (X_1, \dots, X_N, R_2, \dots, R_N), \\ \rho_{[N]} &= (C_1, \dots, C_N, F_N), \\ u'_{[N]} &= (u'_1, \dots, u'_N, h_N), \\ v_{[N]} &= (v_1, \dots, v_N, \mathbb{E}^{\mathbb{Q}}F_N)\end{aligned}$$

where  $\mathbb{E}^{\mathbb{Q}}F_N$  can be calculated according to the global budget constraint of the induced  $N$ -PEFF problem.

For any given  $\theta_{N+1}$ , according to the assumption, the degraded system has a unique PEFF solution with coefficients  $(\theta_1, \dots, \theta_N)$ . This solution, together with the  $\theta_{N+1}$  and  $\theta_p = 1$ , satisfies all the equations except the following one<sup>2</sup>

$$\mathbb{E}^{\mathbb{Q}}C_{N+1} = \sum_{J \in \mathcal{J}_{N+1}} \mathbb{Q}(J)C_{N+1}^J = v_{N+1}.$$

Next we will show that there exists  $\theta_{N+1}$  such that the equation above will hold.

650 Then by Theorem AppendixB.5 the solution  $\theta_{N+1}$  will be unique.

Define

$$w(\theta_{N+1}) = \mathbb{E}^{\mathbb{Q}}C_{N+1} = \sum_{J_{N+1} \in \mathcal{J}_{N+1}} \mathbb{Q}(J_{N+1})C_{N+1}^{J_{N+1}}.$$

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<sup>2</sup>The equation  $\mathbb{E}^{\mathbb{Q}}F_{N+1} = v_p$  will also not hold. However, as we have discussed, we don't have to consider this equation, since it will be automatically satisfied if other FF conditions hold.

Note that  $C_{N+1}^{J_{N+1}}$  is a continuous function of  $\theta_{N+1}$  for any  $J$  which follows from Theorem 3.5 and so is  $w$  itself. Next we will show that the value of the function  $w$  can be both above and below  $v_{N+1}$ , so that there exists  $\theta_{N+1}^*$  such that  $w(\theta_{N+1}^*) = v_{N+1}$  since  $w$  is continuous. This will be done by taking  $\theta_{N+1}$  to the limits.

First consider  $\lim_{\theta_{N+1} \rightarrow 0} w(\theta_{N+1})$ . We will distinguish between the following two cases.

**A.** The lower bounds of the utility functions  $b_n$  are all finite. We will then have

$$\lim_{\theta_{N+1} \rightarrow 0} C_{N+1}^{J_{N+1}} = b_{N+1} \quad \forall J_{N+1} \in \mathcal{J}_{N+1}.$$

Otherwise, suppose there exists a sequence of  $\theta_{N+1}$ , say  $\{\hat{\theta}^{[m]}\}$  with  $\hat{\theta}^{[m]} \rightarrow 0$  as  $m \rightarrow \infty$ , such that

$$\lim_{m \rightarrow \infty} C_{N+1}^{J_{N+1}}(\hat{\theta}^{[m]}) \geq b_{N+1} + \varepsilon$$

for some trajectory  $J_{N+1}$  and some  $\varepsilon \in \mathbb{R}_{++}$ . Then according to the final period IBE

$$\theta_{N+1} u'_{N+1}(C_{N+1}^{J_{N+1}}) = \theta_p u'_p(F_{N+1}^{J_{N+1}})$$

the left hand side will go to zero as  $u'_{N+1}(C_{N+1}^{J_{N+1}})$  will be bounded. As a result,  $F_{N+1}^{J_{N+1}}$  will have to go to  $+\infty$  which is not possible when all the  $C$ 's can only be finite.

**B.** Consider when  $b_n = -\infty$  for some  $n$ .  $\mathcal{T}$  denotes the set of all such  $n$ 's. We will show that still

$$\lim_{\theta_{N+1} \rightarrow 0} C_{N+1}^{J_{N+1}} = b_{N+1} \quad \forall J_{N+1} \in \mathcal{J}_{N+1}.$$

We then only have to show this for a special  $J'$  which satisfies that for any  $n$ ,  $X_n^{J'} = \max_{J \in \mathcal{J}_{N+1}} X_n^J$  and  $R_n^{J'} = \max_{J \in \mathcal{J}_{N+1}} R_n^J$ , i.e.  $(X^{J'}, R^{J'})$  is the attainable “upper bound” of all trajectories. This is possible because the number of trajectories is finite, the condition (2.1) holds and the risk stream is sequentially independent. Once we show that  $\lim_{\theta_{N+1} \rightarrow 0} C_{N+1}^{J'} = b_{N+1}$ , by Lemma 3.8, the limit

of  $C_{N+1}$  of all other trajectories cannot be larger than  $b_{N+1}$ , and also cannot be smaller than  $b_{N+1}$ .

Otherwise, suppose there exist a sequence of  $\theta_{N+1}$ , say  $\{\hat{\theta}^{[m]}\}$  with  $\hat{\theta}^{[m]} \rightarrow 0$  as  $m \rightarrow \infty$ , and  $\varepsilon > 0$ , such that

$$\lim_{m \rightarrow \infty} C_{N+1}^{J'}(\hat{\theta}^{[m]}) \geq b_{N+1} + \varepsilon.$$

If  $b_{N+1} = -\infty$  then the equation above is interpreted as that the sequence  $\{C_{N+1}^{J'}(\hat{\theta}^{[m]})\}$  is bounded from below.

Then by final period IBE

$$\theta_{N+1} u'_{N+1}(C_{N+1}^{J'}) = \theta_p u'_p(F_{N+1}^{J'})$$

670 we have that  $F_{N+1}^{J'}$  will have to go to  $+\infty$  since  $u'_{N+1}(C_{N+1}^{J'})$  will be bounded. Consider the global budget constraint: now since  $C_{N+1}^{J'} + F_{N+1}^{J'} \rightarrow +\infty$ , there will exist  $\tau \in \mathcal{T}$  such that  $C_{\tau}^{J'} \rightarrow -\infty$ . By the definition of  $J'$ , we have that  $C_{\tau}^J \rightarrow -\infty$  for any other possible  $J \in \mathcal{J}_{N+1}$ , thus the value profile condition for  $C_{\tau}$  will not hold. This is a contradiction.

To conclude: we have shown that

$$\lim_{\theta_{N+1} \rightarrow 0} C_{N+1}^J = b_{N+1} \quad \forall J \in \mathcal{J}_{N+1}$$

whatever the value of  $b_{N+1}$  is. Thus

$$w(\theta_{N+1}) = \mathbb{E}^{\mathbb{Q}} C_{N+1} = \sum_{J \in \mathcal{J}_{N+1}} \mathbb{Q}(J) C_{N+1}^J \rightarrow b_{N+1}.$$

675 Next consider  $\lim_{\theta_{N+1} \rightarrow \infty} w(\theta_{N+1})$ . Now we drop the normalization constraint  $\theta_p = 1$ . Taking into consideration the freedom of choosing a way of normalization, we can conclude that the following two statements are equivalent:

- fix  $\theta_p$  and let  $\theta_{N+1} \rightarrow +\infty$ ;
- fix  $\theta_{N+1}$  and let  $\theta_p \rightarrow 0$ .

Then following the analogy we have  $F_{N+1}^J \rightarrow b_p$  for all  $J \in \mathcal{J}_{N+1}$  as  $\theta_p \rightarrow 0$ . Thus according to the budget constraint for the last period, we conclude that

$$\lim_{\theta_{N+1} \rightarrow \infty} w(\theta_{N+1}) = \lim_{\theta_{N+1} \rightarrow \infty} \sum_{J \in \mathcal{J}_{N+1}} \mathbb{Q}(J) C_{N+1}^J = v_{N+1} + v_p - b_p.$$

Then since

$$v_p - b_p > 0$$

must hold, we have

$$v_{N+1} < v_{N+1} + v_p - b_p.$$

680 By a simple intermediate value theorem we know that there will exist some  $\theta_{N+1}^*$  such that  $w(\theta_{N+1}^*) = v_{N+1}$ . Then we have found a weight vector  $\theta$  that leads to a PEFF solution to the system. This indicates that the fixed points of the mapping  $\varphi$  will exist; the fixed point must be unique according to Theorem AppendixB.5, i.e. the solution  $\theta$  is unique. The uniqueness is up to normalization. This finishes  
685 the proof. □

### AppendixC. Proofs for Section 6

LEMMA APPENDIXC.1 *When  $(\mathcal{X}, d)$  is a locally compact and connected metric space, and  $f : \mathcal{X} \rightarrow \mathcal{X}$  is a contractive mapping with fixed point  $x^* \in \mathcal{X}$ , then for every  $x \in \mathcal{X}$  the sequence of iterates  $\{f^{(n)}(x) | n \in \mathbb{N}_+\}$  converges to  $x^*$ .*

690 PROOF See Thm. 1 by Nadler [20]. □

PROOF OF THEOREM 6.1. Lemma AppendixB.5 has shown that the mapping  $\varphi$  is contractive with respect to the Hilbert metric. The theorem then is a direct result of Lemma AppendixC.1. □

### AppendixD. Proofs for Section 7.2

PROOF OF THEOREM 7.1. The proof of the theorem is actually a process of calculation. First we need the following preparations. By Theorem 3.5, for any given  $\theta$  we can define  $f_n(\cdot)$  such that

$$C_n = f_n(X_n + F_{n-1}R_n).$$

By the IBE for the last period we have

$$\theta_p u'_p(X_N + F_{N-1}R_N - C_N) = \theta_N u'_N(C_N),$$

which will translate into

$$\theta_p \alpha_p \exp[-\alpha_p(X_N + F_{N-1}R_N - C_N)] = \theta_N \alpha_N \exp[-\alpha_N C_N].$$

Take the logarithm on both sides and after rearranging the items we get

$$C_N = \frac{\alpha_p}{\alpha_p + \alpha_N}(X_N + F_{N-1}R_N) + \frac{1}{\alpha_p + \alpha_N} \ln \frac{\theta_N \alpha_N}{\theta_p \alpha_p}.$$

Take the  $\mathbb{Q}$ -expectation and we shall have

$$\mathbb{E}^{\mathbb{Q}} C_N = v_N$$

which gives us

$$C_N = \frac{\alpha_p}{\alpha_p + \alpha_N} [X_N + F_{N-1}R_N] + (v_N - \frac{\alpha_p}{\alpha_p + \alpha_N} w_N),$$

where

$$w_N := \mathbb{E}^{\mathbb{Q}} A_N = \mathbb{E}^{\mathbb{Q}}(C_N + F_N) = v_N + v_p.$$

695 Next we show that for any possible  $n$ ,  $f_n$  should be linear. We will show this by first showing that if  $f_{n+1}$  is linear with positive slope, then so is  $f_n$ .

By IBE

$$\theta_n u'_n(C_n) = \theta_{n+1} \mathbb{E}_n^{\mathbb{P}} [u'_{n+1}(C_{n+1})R_{n+1}]$$

we have

$$\frac{\theta_n}{R_{n+1}\theta_{n+1}} u'_n(f_n(x)) = \sum_{j \in \mathcal{J}_n^{n+1}} \mathbb{P}(j) u'_{n+1} [f_{n+1}(X_{n+1}^j + (x - f_n(x))R_{n+1})]$$

where  $x$  is the variable standing for the available assets. Assume that  $f_{n+1}(x) = a_{n+1}x + e_{n+1}$  with  $a_{n+1} > 0$ . We have

$$\begin{aligned} \frac{\theta_n}{R_{n+1}\theta_{n+1}} \exp(-\alpha_n f_n(x)) &= \sum_{j \in \mathcal{J}_n^{n+1}} \mathbb{P}(j) \exp \left\{ -\alpha_{n+1} [a_{n+1}(X_{n+1}^j + (x - f_n(x))R_{n+1}) + e_{n+1}] \right\} \\ &= \exp \left\{ -\alpha_{n+1} a_{n+1} R_{n+1} (x - f_n) \right\} \cdot \left\{ \sum_{j \in \mathcal{J}_n^{n+1}} \mathbb{P}(j) \exp \left[ -\alpha_{n+1} (a_{n+1} X_{n+1}^j + e_{n+1}) \right] \right\} \end{aligned}$$

Take the logarithm on both sides:

$$\ln \left( \frac{\theta_n}{R_{n+1}\theta_{n+1}} \right) - \alpha_n \cdot f_n = \ln \kappa_{n+1} - \alpha_{n+1} a_{n+1} R_{n+1} (x - f_n),$$

finally

$$f_n(x) = \frac{a_{n+1}\alpha_{n+1}R_{n+1}}{\alpha_n + a_{n+1}\alpha_{n+1}R_{n+1}}x + \frac{1}{\alpha_n + a_{n+1}\alpha_{n+1}R_{n+1}} \ln \left( \frac{\theta_n}{R_{n+1}\theta_{n+1}} \frac{1}{\kappa_{n+1}} \right)$$

where

$$\kappa_{n+1} = \frac{\mathbb{E}u'_{n+1}(f_{n+1}(X_{n+1}))}{\alpha_{n+1}}.$$

It follows that all  $f_n$  should be linear with positive slope since  $f_N$  is. The slope satisfies

$$a_n = \frac{a_{n+1}\alpha_{n+1}R_{n+1}}{\alpha_n + \alpha_{n+1}a_{n+1}R_{n+1}}.$$

By recursion we know that if we start with  $a_N = \frac{\alpha_p}{\alpha_p + \alpha_N}$ , then all the  $a_n$ 's can be calculated. Hence

$$C_n = f_n(X_n + F_{n-1}R_n) = a_n(X_n + F_{n-1}R_n) + \text{constant}.$$

Taking the expectation under  $\mathbb{Q}$  immediately gives the constant part and finally we have

$$C_n = a_n(X_n + F_{n-1}R_n) + (v_n - a_n w_n),$$

where  $w_n = \mathbb{E}^{\mathbb{Q}}A_n$  can be recursively calculated according to the relationship

$$A_{n+1} = X_{n+1} + (A_n - C_n)R_{n+1}.$$

□

## AppendixE. Tables for Section 7

Tables E.3 – E.6 shows the distributions of the decision variables for Section

700 7.1.

$X_1$	$\mathbb{P}$ -prob.	$\mathbb{Q}$ -prob.	$C_1$ : OEB	$C_1$ : CEB	$C_1$ : au- tarky
1.2	0.6	0.5	1.0507	1.0704	1.2
0.8	0.4	0.5	0.9493	0.9296	0.8

Table E.3: Distributions of payments for agent 1

$X_1$	$X_2$	$\mathbb{P}$ -prob.	$\mathbb{Q}$ -prob.	$C_2$ : OEB	$C_2$ : CEB	$C_1$ : au- tarky
1.2	1.2	0.36	0.25	1.1197	1.1741	1.2
0.8	1.2	0.24	0.25	1.0181	1.0376	1.2
1.2	0.8	0.24	0.25	0.9818	0.9632	0.8
0.8	0.8	0.16	0.25	0.8801	0.8251	0.8

Table E.4: Distributions of payments for agent 2

$X_1$	$X_2$	$X_3$	$\mathbb{P}$ - prob.	$\mathbb{Q}$ - prob.	$C_3$ : OEB	$C_3$ : CEB	$C_1$ : au- tarky
1.2	1.2	1.2	0.216	0.125	1.2157	1.3556	1.2
0.8	1.2	1.2	0.144	0.125	1.1167	1.2327	1.2
1.2	0.8	1.2	0.144	0.125	1.0832	1.1665	1.2
0.8	0.8	1.2	0.096	0.125	0.9844	1.0452	1.2
1.2	1.2	0.8	0.144	0.125	1.0157	0.9556	0.8
0.8	1.2	0.8	0.096	0.125	0.9167	0.8327	0.8
1.2	0.8	0.8	0.096	0.125	0.8832	0.7665	0.8
0.8	0.8	0.8	0.064	0.125	0.7844	0.6452	0.8

Table E.5: Distributions of payments for agent 3

$X_1$	$X_2$	$X_3$	$\mathbb{P}$ -prob.	$\mathbb{Q}$ -prob.	$F_3$ : OEB	$F_3$ : CEB
1.2	1.2	1.2	0.216	0.125	1.2157	1
0.8	1.2	1.2	0.144	0.125	1.1167	1
1.2	0.8	1.2	0.144	0.125	1.0832	1
0.8	0.8	1.2	0.096	0.125	0.9844	1
1.2	1.2	0.8	0.144	0.125	1.0157	1
0.8	1.2	0.8	0.096	0.125	0.9167	1
1.2	0.8	0.8	0.096	0.125	0.8832	1
0.8	0.8	0.8	0.064	0.125	0.7844	1

Table E.6: Distributions of end buffer