APPLICATIONS OF COMPLEMENTARITY SYSTEMS

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Abstract

The class of complementarity systems has been analyzed in considerable detail as a special subclass of hybrid systems. The goal of this paper is to motivate the ongoing research by showing that many physically relevant systems fit in the framework of complementarity systems.

1 Introduction

Technological innovation leads to an increasing interest in systems of a mixed continuous/discrete nature (called 'hybrid systems'). Recently, hybrid systems receive a lot of attention both from the control [2] and computer science community [27]. A subclass of hybrid systems consists of complementarity systems as introduced in [29]. In its most general form a complementarity system is governed by the differential and algebraic equations

$$0 = F(\dot{z}(t), z(t)) \tag{1a}$$

$$y(t) = g(z(t)) \in \mathbb{R}^k \tag{1b}$$

$$u(t) = h(z(t)) \in \mathbb{R}^k \tag{1c}$$

together with the complementarity conditions

$$\{y_i(t) = 0 \text{ or } u_i(t) = 0\},\ y_i(t) \geqslant 0,\ u_i(t) \geqslant 0 \tag{1d}$$

for all $i \in \{1, \dots, k\}$. The complementarity conditions are similar as those appearing in the linear complementarity problem of mathematical programming [8].

A special complementarity system occurs when (1a), (1b) and (1c) are replaced by an "input-output system" of the form

$$\dot{x}(t) = f(x(t), u(t)) \tag{2a}$$

$$y(t) = g(x(t), u(t)). \tag{2b}$$

In this case we speak of "semi-explicit" complementarity systems.

If the system is linear, i.e. $f(x, u) = Ax + Bu$, $g(x, u) = Cx + Du$ for constant matrices A, B, C, D, we speak of a linear complementarity system (LCS).

The class of complementarity systems has been investigated in [15–17, 23, 29, 30]. Several basic issues are studied in these papers: the introduction of a mathematically precise solution concept, existence and uniqueness of solutions, mode selection methods, simulation issues and the study of the particular behaviour of these systems. Current and future research will include stability analysis, development of numerical algorithms to approximate solutions and the inclusion of measurement and control variables. The purpose of this paper to show that the analysis of the class of complementarity systems is motivated by a wide range of applications.

2 Electrical networks with ideal diodes

Consider a linear electrical network consisting of resistors, inductors, capacitors, gyrators, transformers (RLCGT) and of k ideal diodes. To model this system as a LCS, the network is viewed as the interconnection of an RLCGT network with the diodes. More precisely, the RLCGT components form a multiport network described by a state space representation $\dot{x} = Ax + Bu$, $y = Cx + Du$ [1] with state variable x representing voltages over capacitors and currents through inductors. The input/output variables u and y represent the port variables: the pair (u_i, y_i) denotes the voltage-current variables at the i-th port. Interconnection of the i-th port to an (ideal) diode results in the equations

$$u_i = -V_i,\ y_i = I_i \text{ or } u_i = I_i,\ y_i = -V_i,$$

where V_i and I_i are the voltage across and current through the i-th diode, respectively. Finally, the ideal diode characteristic of the i-th diode is given by (see also fig. 1)

$$V_i \leqslant 0,\ I_i \geqslant 0,\ \{V_i = 0 \text{ or } I_i = 0\}. \tag{3}$$
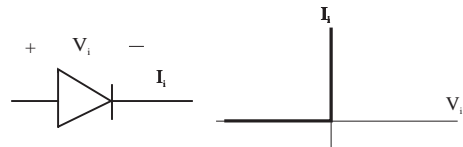

Figure 1: The i-th ideal diode characteristic.

3 Pipelines with one-way valves

Many chemical and hydraulic processes contain valves that only allow flows in one direction. A lid in the pipe can be opened to one side only, which prevents the fluid or gas from streaming back. The situation is shown in fig. 2.
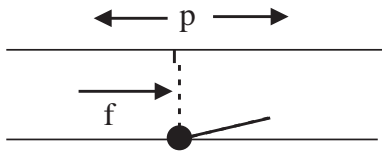

Figure 2: A pipeline with a one-way valve.

The flow in the pipe at time t is denoted by $f(t)$ and the pressure over the valve (lid) by $p(t)$. Ideally, only two situations can happen. The lid is either completely closed (dotted situation) or completely open (solid situation). The closed case occurs only if the pressure on the right is larger than the pressure on the left ($p(t) \geqslant 0$). The flow is then equal to zero ($f(t) = 0$). In the other situation (valve open), the pressure over the valve is zero and the fluid streams in the positive direction ($p(t) = 0$ and $f(t) \geqslant 0$). Hence, flow and pressure are complementarity variables.

4 Constrained mechanical systems

Consider a conservative mechanical system in which q denotes the generalised coordinates and p the generalised momenta. The free motion dynamics can be expressed in terms of the Hamiltonian $H(q,p)$, which has the interpretation of the total energy in the system. The equations are

$$\dot{q} = \frac{\partial H}{\partial p}(q,p) \tag{4a}$$

$$\dot{p} = -\frac{\partial H}{\partial q}(q,p). \tag{4b}$$

The system is subject to the geometric inequality constraints given by

$$C(q) \geqslant 0. \tag{4c}$$

Friction effects are not modelled here. We refer to subsection 5.3 for phenomena like Coulomb friction.

To obtain a complementarity formulation, we introduce (see also [15, 24, 26, 29, 30]) the Lagrange multiplier u generating the constraint forces needed to satisfy the unilateral constraints (4c). According to the rules of classical mechanics, the system can then be written as

$$\dot{q} = \frac{\partial H}{\partial p}(q,p) \tag{5a}$$

$$\dot{p} = -\frac{\partial H}{\partial q}(q,p) + \frac{\partial C^{\top}}{\partial q}(q)u \tag{5b}$$

$$y = C(q) \tag{5c}$$

together with the complementarity conditions (1d). The conditions (1d) express that the Lagrange multiplier u_i

is only nonzero, if the corresponding constraint is active ($y_i = 0$). Vice versa, if the constraint is inactive ($y_i > 0$), the corresponding multiplier u_i is necessarily equal to zero.

The control of these systems is a major research topic. Since most control theories are model-based, adequate modelling of dynamical discontinuities and impact phenomena are necessary. Control applications can be found for instance in the field of robotics [4, 9, 21].

5 Piecewise linear characteristics

In this section we consider a dynamical system in which certain variables are coupled by means of a static piecewise linear (PL) characteristic. The situation is depicted in fig. 3. The variables v, z appear in the dynamics of the system Σ. These variables are related "in closed loop" through a PL relation. As an example one could think of a mechanical system with Coulomb friction or an electrical circuit containing a resistor having a PL behaviour (see e.g. [22]).
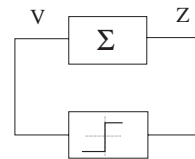

Figure 3: System with a PL relation.

5.1 A simple max-relation

Let v and z be related through $v = \max(0, z)$. See fig. 4. We introduce two auxiliary variables u, y and the algebraic equation $z = u - y$. It is easily verified that adding the complementarity conditions $u \geqslant 0$, $y \geqslant 0$ and $\{y = 0$ or $u = 0\}$, results in $u = v$. Hence, the relation $v = \max(0, z)$ can be replaced by

$$z = u - y \tag{6a}$$

$$v = u \tag{6b}$$

$$u \geqslant 0, \quad y \geqslant 0, \quad \{y = 0 \text{ or } u = 0\} \tag{6c}$$

resulting in a complementarity system. Hence, any system that can be formulated in terms of 'max' operations (think of 'max-plus systems'), can be cast into a complementarity framework due to the fact that $v = \max(w, z) = w + \max(0, z - w)$.
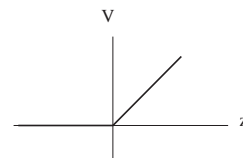

Figure 4: A simple max-relation.

Direct applications of this simple relation are one-sided springs. In fig. 5 a linear spring is attached to a wall, but not to the cart. Let q denote the position of the cart with

respect to the equilibrium of the spring. The spring force $F(q)$ is a nonlinear function of q:

$$F(q) = \begin{cases} -kq, & \text{if } q < 0 \\ 0, & \text{if } q \geqslant 0 \end{cases} \qquad (7)$$

with $k > 0$ denoting the spring constant. The interpretation is clear. Only when the spring is pressed ($q < 0$), the spring exerts a nonzero force $-kq$ on the cart. In the other situation where the cart is on the right of the equilibrium ($q \geqslant 0$), the spring is at rest and the force $F(q)$ is equal to zero. The relation (7) can compactly be written as $F(q) = \max(-kq, 0)$. Systems with one-sided springs are studied in e.g. [19].
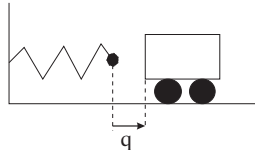

Figure 5: One-sided spring.

As a second example consider the following single input control system $\dot{x} = Ax + Bu$ where the control input u is restricted to take nonnegative values only. In [18] one is interested in the existence of a nonnegative state feedback of the form $u = \max(0, Fx)$ where F is a constant row vector resulting in a stable closed loop system $\dot{x} = Ax + B\max(0, Fx)$.

A max-relation also occurs in application of Pontryagin's maximum principle to optimal control problems with control restraint sets being convex polyhedra. The maximum principle yields a two-point boundary problem containing max-relations as shown in [14].

5.2 Piecewise linear (PL) functions

A dynamical system described by an ordinary differential equation and one or more continuous static PL *functions* can be modelled as a complementarity system. To make this plausible, consider the function between v and z as given by fig. 6. The function consists of three connected branches with slopes r_i, $i = 1, 2, 3$. The offset at $z = 0$ is equal to g and the slope changes at $z = a_i$, $i = 1, 2$. A description of this function in terms of max-relations is given by (8), as is easily verified.
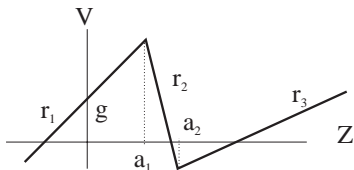

Figure 6: An arbitrary PL characteristic.

$$v = g + r_1 z + (r_2 - r_1)\max(z - a_1, 0) + \\ (r_3 - r_2)\max(z - a_2, 0) \quad (8)$$

Since the max-relation can be rewritten as a complementarity system, it is obvious that this PL characteristic can be rephrased in terms of a complementarity description.
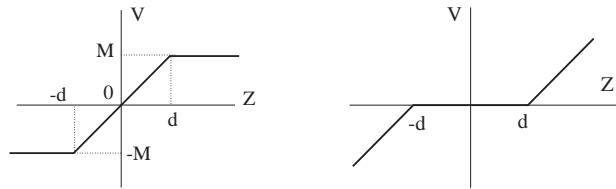

Figure 7: Saturation and deadzone characteristic

Applications are for instance saturation and deadzone characteristics (fig. 7) which occur in many control systems. Furthermore, devices as bipolar transistors, MOSFET's and p-n junction diodes in electrical network theory are often modelled by PL functions [3, 6, 22].

Finally, it is clear that many continuous nonlinear (static) relation can be suitably approximated by PL functions.

5.3 PL relations

Besides the examples given in the previous subsection, there exist many physically relevant models that are given by PL *relations*, but not by PL *functions*. Examples are mechanical systems with Coulomb friction or relay systems (see fig. 8). However, also these systems can be put in a complementarity framework by using an alternative approach. The approach is not given in full detail here, but is sketched by applying it to the example of a Coulomb friction/relay characteristic (see also [20, 23, 26]).
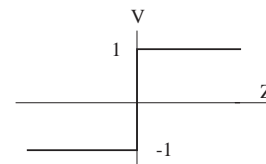

Figure 8: Relay or sgn-relation.

The relay characteristic in fig. 8 can be described by

$$\begin{aligned} v &= 1, & \text{if} \quad z > 0 \\ -1 \leqslant v &\leqslant 1, & \text{if} \quad z = 0 \\ v &= -1, & \text{if} \quad z < 0, \end{aligned} \qquad (9)$$

which is sometimes denoted by $v = \text{sgn}(z)$.

Lemma 5.1 *The PL relation as given in fig. 8 can be described by the equations*

$$u_1 + u_2 = 2 \qquad (10a)$$
$$y_1 - y_2 = z \qquad (10b)$$
$$v = \tfrac{1}{2}(u_2 - u_1) \qquad (10c)$$

together with the complementarity conditions

$$\{u_1 = 0 \ or \ y_1 = 0\}, \ u_1 \geqslant 0, \ y_1 \geqslant 0 \qquad (11)$$
$$\{u_2 = 0 \ or \ y_2 = 0\}, \ u_2 \geqslant 0, \ y_2 \geqslant 0. \qquad (12)$$

Proof Due to the complementarity conditions there are $2^2 = 4$ possibilities.

$u_1 = u_2 = 0$: since (10a) implies that $2 = 0$, this mode is not feasible.

$u_1 = y_2 = 0$: (10a) and (10c) give $v = \frac{1}{2}u_2 = 1$. Eq. (10b) implies $z = y_1 \geqslant 0$. This mode corresponds to the right branch in fig. 8.

$u_2 = y_1 = 0$: Similar to the previous case, we can derive that this mode corresponds to the left branch.

$y_1 = y_2 = 0$: Eq. (10b) implies $z = 0$ and due to (10a) and (10c) it follows that $-1 \leqslant v \leqslant 1$. This corresponds to the middle branch.

Note that in the last mode ($y_1 = y_2 = 0$) the causality between v and z is different then in the other two feasible modes.

The above modelling leads to a complementarity system of the form (1), because the algebraic equations (10a)-(10b) are used. Alternative modelling may lead to a semi-explicit form in case the system Σ (see fig. 3) is represented by $\dot{x} = f(x, v)$ and $z = g(x, v)$. Indeed, take

$$
\begin{aligned}
u_1 &= \tfrac{1}{2}(1 - v) & \text{(13a)} \\
y_2 &= \tfrac{1}{2}(1 + v) & \text{(13b)} \\
z &= y_1 - u_2 & \text{(13c)}
\end{aligned}
$$

together with the complementarity conditions on (u_i, y_i). Similarly as in the previous proof, one can check all the four possibilities to verify that the above equations describe the relay characteristic. By suitable substitutions one gets the semi-explicit form

$$
\begin{aligned}
\dot{x} &= f(x, 1 - 2u_1) & \text{(14a)} \\
y_1 &= g(x, 1 - 2u_1) + u_2 & \text{(14b)} \\
y_2 &= 1 - u_1 & \text{(14c)}
\end{aligned}
$$

Other approaches to PL modelling use absolute value functions [7], extended and generalised complementarity problems [5, 10] or state variables [3, 22]. More complicated examples can also be modelled as complementarity systems. Examples can be found in [22], where a "reversed Z-characteristic" has been put in a complementarity system (left picture in fig. 9) and in [10], where a model has been derived whose characteristic consists of the edges of a square (right picture).
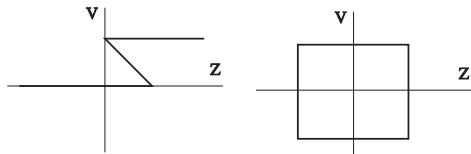

Figure 9: Reversed Z-curve and square

Existence and uniqueness of solutions to dynamical systems with PL characteristics are nontrivial. Such well-posedness issues are studied in [5].

6 Variable structure systems

6.1 Convex definition

Consider a system that switches between two dynamics as a result of inequalities. In fig. 10 the state space is separated into two parts by a hypersurface defined by $\phi(x) = 0$. On one side of the surface $C_+ := \{x \in \mathbb{R}^n \mid \phi(x) > 0\}$ the dynamics $\dot{x} = f_+(x)$ holds, on the opposite side $C_- := \{x \in \mathbb{R}^n \mid \phi(x) < 0\}$ the dynamics $\dot{x} = f_-(x)$ is valid.
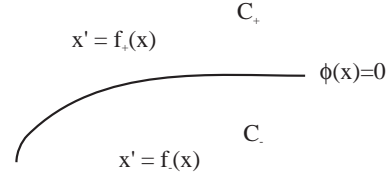

Figure 10: Switching dynamics.

A *sliding mode* occurs when in a state x_0, lying on the hypersurface $\phi(x) = 0$, $f_+(x_0)$ points in the direction of C_- and $f_-(x_0)$ points in the direction of C_+ (fig. 11). Hence, from the initial state x_0 it is impossible to go to C_- or C_+, because the dynamics immediately steers you back to the hypersurface satisfying $\phi(x) = 0$. A kind of sliding solution has been formalized by Filippov [12] by the *convex definition* which corresponds to infinitely fast switching. In brief, it states that the sliding mode is given by taking a convex combination of both dynamics $\dot{x} = \lambda f_+(x) + (1 - \lambda)f_-(x)$, $0 \leqslant \lambda \leqslant 1$ such that x moves along $\phi(x) = 0$.
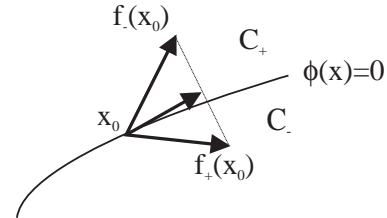

Figure 11: Sliding mode.

Proposition 6.1 *The variable structure system with solutions according to the convex definition can be modelled by*

$$
\dot{x} = \lambda f_+(x) + (1 - \lambda)f_-(x) \tag{15}
$$

and

$$
\begin{aligned}
\lambda = 1, & \quad if \quad \phi(x) > 0 & \text{(16a)} \\
0 \leqslant \lambda \leqslant 1, & \quad if \quad \phi(x) = 0 & \text{(16b)} \\
\lambda = 0, & \quad if \quad \phi(x) < 0, & \text{(16c)}
\end{aligned}
$$

i.e. $\lambda = \frac{1}{2} + \frac{1}{2}sgn(\phi(x))$ *with 'sgn' the relation described by* (9)*. As seen before, this PL relation allows several complementarity reformulations.*

Similar techniques as for a single surface, apply to multiple surfaces splitting up the state space.

6.2 Equivalent control definition

Another solution concept introduced by Filippov is based on the *equivalent control definition* of sliding modes [12]. This definition is related to "switching control systems." The system given by $\dot{x} = f(x, u)$ with x the state variable is controlled by the discontinuous feedback (called the "equivalent control")

$$u = \begin{cases} g_+(x), & \xi(x) > 0 \\ g_-(x), & \xi(x) < 0 \end{cases} \qquad (17)$$

with the function $\xi : \mathbb{R}^n \to \mathbb{R}$ modelling the switching surface. Similar to the previous subsection, a sliding mode occurs when the dynamics $f_+(x) := f(x, g_+(x))$ and $f_-(x) := f(x, g_-(x))$ point outwards C_+ and C_-, respectively. The equivalent control definition of a sliding mode picks a convex combination of the control laws instead of a convex combination of $f_+(x)$ and $f_-(x)$ (note that the definitions are equivalent when $f(x, u)$ is affine in u). Formally, the sliding mode is given by the differential and algebraic equations $\dot{x} = f(x, \lambda g_+(x) + (1 - \lambda)g_-(x))$, $\xi(x) = 0$ and valid as long as $\lambda \in [0, 1]$ is satisfied. Obviously, this system can also be modelled as a system $\dot{x} = f(x, \lambda g_+(x) + (1 - \lambda)g_-(x))$ with a characteristic between λ and $\xi(x)$ as in (16).

Proposition 6.2 *A variable structure system as above with solutions according to the equivalent control definition can be rewritten in terms of a complementarity system.*

7 Optimal control problems with state constraints

An important class of optimal control problems consists of maximizing the criterion $J(x_0, v) := \int_0^T [F(x, v, t)]dt + S(x(T), T)$ by choosing an appropriate control function v subject to the dynamics $\dot{x} = f(x, v, t)$ with initial condition $x(0) = x_0$ and the state constraint $h(x, t) \geqslant 0$ for all $t \in [0, T]$. Additional requirements like control constraints $g(x, v, t) \geqslant 0$ and end-point conditions $a(x(T), T) \geqslant 0$ and $b(x(T), T) = 0$ could be included, but are omitted for brevity.

In the survey [13] Pontryagin's maximum principle [28] is used to obtain necessary conditions for a control input to be optimal.

Introduce the Hamiltonian $H(x, v, \lambda, t) := F(x, v, t) + \lambda^\top f(x, v, t)$. The optimal control v_{opt} satisfies

$$v_{\text{opt}} = \arg\max_v H(x_{\text{opt}}, v, t) \qquad (18a)$$

$$\dot{x}_{\text{opt}} = \frac{\partial H}{\partial \lambda}(x_{\text{opt}}, v_{\text{opt}}, t) \qquad (18b)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x}(x_{\text{opt}}, v_{\text{opt}}, t) - \frac{\partial h}{\partial x}^\top (x_{\text{opt}}, t)u \quad (18c)$$

$$y = h(x_{\text{opt}}, t) \qquad (18d)$$

with complementarity conditions holding between the multiplier u and constraint variables y. The variable λ is called the adjoint or costate variable. There are additional boundary conditions such that the maximum principle results in a two-point boundary problem. It is possible that

jumps occur in the adjoint variable λ. Also for these jumps additional relations are available. We do not specify all the available conditions, but only illustrate that this kind of optimal control problems fit in the class of complementarity systems.

The formulation in [13] is called an informal theorem, because the result is not rigorously established for the general case. It is presented as a kind of recipe to find possible candidates for the optimal controls.

8 Projected dynamical systems

Projected dynamical systems (PDS) have been studied in [11, 25]. These systems are described by differential equations of the form

$$\dot{x}(t) = \Pi_K(x(t), -F(x(t))), \qquad (19)$$

where F is a vector field, K is a closed convex set, and Π_K is a projection operator that prevents the solution from moving outside the constraint set K. Loosely speaking, a PDS obeys an equation of the form $\dot{x} = -F(x)$ as long as x is contained in the interior of K or $-F(x)$ is "pointing inwards K." When $-F(x)$ is pointing outwards and x is at the boundary of K, the operator Π_K projects $-F(x)$ into the direction of K such that the solution stays inside K.

To be precise, the cone of inward normals at $x \in K$ is defined by

$$n(x) = \{\gamma \mid \langle \gamma, x - k \rangle \leqslant 0 \text{ for all } k \in K\}. \qquad (20)$$

Given $x \in K$ and $v \in \mathbb{R}^n$, define the projection of the vector v at x with respect to K by

$$\Pi_K(x, v) = v - \langle v, n^*(x) \rangle n^*(x), \qquad (21a)$$

where

$$n^*(x) \in \arg\max_{n \in n(x), \ \|n\| \leqslant 1} \langle v, -n \rangle. \qquad (21b)$$

Definition 8.1 The $\text{PDS}(K, F)$ is given by

$$\dot{x} = \Pi_K(x, -F(x)). \qquad (22)$$

We consider convex sets K that can be given by finitely many inequalities, i.e. $K = K_h := \{x \in \mathbb{R}^n \mid h(x) \geqslant 0\}$ with $h : \mathbb{R}^n \to \mathbb{R}^p$ a real-analytic function such that the component functions h_i are convex. ∇h_i denotes the gradient of h_i and is considered to be a row vector. The Jacobian $H(x)$ denotes the matrix in which the i-th row is equal to $\nabla h_i(x)$, i.e. the ij-th element of $H(x)$ is equal to $\frac{\partial h_i}{\partial x_j}(x)$. Moreover, F is assumed to be real-analytic as well. Under suitable assumptions (like a rank condition on the Jacobian $H(x)$ and growth conditions on the vector field $F(x)$, see [16] for the details) the following result can be proven.

Proposition 8.2 [16] *For all initial states x_0 both $PDS(K_h, F)$ and the complementarity system given by*

$$\dot{x}(t) = -F(x(t)) + H^\top(x(t))u(t) \qquad (23a)$$

$$y(t) = h(x(t)) \qquad (23b)$$

and the complementarity conditions (1d), have a unique solution defined on $[0, \infty)$. Moreover, the solutions coincide.

PDS are used for studying equilibria of oligopolistic markets, urban transportation networks, traffic systems, international trade, agricultural and energy markets (spatial price equilibria).

9 Conclusions

The class of complementarity systems may seem quite restrictive at first sight. The goal of this paper has been to show that this is not the case: a wide variety of interesting discontinuous dynamical systems can be rewritten in a complementarity formalism. Among the applications of complementarity systems are many examples relevant to the systems and control community. We mentioned the switching control systems (variable structure systems), optimal control problems with state and/or control constraints, systems with discontinuous positive feedback and control systems with relays. Furthermore, many challenging questions are still open in the field of control of complementarity systems. These include characterization of stability, controllability, state/output feedback stabilizability and the development of simulation tools. An incentive to continue this line of research is the range of possible applications: control of mechanical systems with Coulomb friction, unilateral constraints and one-sided springs; control of robots; simulations of crash-tests; regulating landing manoeuvres of spacecraft; feedback control of dynamical systems with saturating actuators or deadzones; control of traffic systems and economical markets; control, design and verification of switching circuits.

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