Pergamon

PII: S0005-1098(96)00154-9

Automatica, Vol. 33, No. 2, pp. 193–201, 1997

0005-1098/97 $17.00 + 0.00

A Controllability Test for General First-order Representations*

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A rank test for controllability is presented that applies directly to implicit linear systems. The test is similar to the well-known Kalman test for controllability of standard state-space systems.

Key Words—Controllability; generalized linear systems; implicit systems; rank test; reachable states.

Abstract—We derive a new controllability rank test for general first-order representations. The criterion generalizes the well-known controllability rank test for linear input-state systems as well as a controllability rank test by Mertzios *et al.* for descriptor systems.

1. INTRODUCTION

As is well known (see e.g. Willems, 1991; Aplevich, 1991; Kuijper, 1994), the following is a general form for linear time-invariant dynamical systems:

$$K\sigma x + Lx + Mw = 0, \tag{1}$$

where K, L and M are matrices of sizes $(n+p)\times n$, $(n+p)\times n$ and $(n+p)\times(m+p)$ respectively. (We follow the notation of Kuijper (1994), and therefore denote the parameter matrices for this representation by (K, L, M) rather than (E, F, G).) Specifically, Proposition VII.3 of Willems (1991) states that a system with latent variables x and manifest variables w, over the time azis $\mathbb{Z}$, is a linear time-invariant complete state-space dynamical system if and only if it can be represented in the form (1), with σ denoting the shift. In the continuous-time case one should interpret σ as differentiation. It has been shown by various transformation algorithms that all of the behaviors that are represented by any of the forms used in linear system theory (including matrix fraction descriptions, implicit systems etc.) also admit a representation of the form (1), with appropriate identification of variables; for instance, the external variable w usually denotes a vector consisting of inputs and outputs. In that case the number of inputs is given by m and the number of outputs by p. Although there are various equally general first-order representations besides (1) (see in particular Kuijper (1994) for the extensive discussion of the relations between these representations), the form (1) appears to be particularly suitable for a controllability test as discussed in this paper.

Properties such as observability and controllability can of course be expressed in terms of the matrices K, L and M. In particular (Willems, 1991; Proposition VII.11(v)), (1) is a minimal representation of a controllable external behavior if and only if the following two conditions hold:

(i) $\lambda K + \mu L$ has full column rank for all $(\lambda, \mu) \in \mathbb{C}^2\backslash\{(0, 0)\}$;

(ii) $[\lambda K + \mu L \,|M|$ has full row rank for all $(\lambda, \mu) \in \mathbb{C}^2\backslash\{(0, 0)\}$.

Condition (i) is the observability condition, whereas (ii) is the controllability condition; one readily verifies that these conditions do indeed reduce to the usual ones for the standard state-space case, which is obtained by taking

$$K = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} -A \\ C \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -B \\ I & -D \end{bmatrix}. \tag{2}$$

Under some circumstances, one may, however,

* Received 3 April 1995; revised 3 June 1995; received in final form 7 August 1996. The original version of this paper was presented at the IFAC Conference on System Structure and Control, which was held in Nantes, France during July 1995. The Published Proceedings of this IFAC meeting may be ordered from the Customer Support Departments of the Elsevier Science Regional Sales Offices (see p. ii of *Automatica*) or from Elsevier Science Ltd, The Boulevard, Langford Lane, Kidlington, Oxford OX5 1GB, U.K. This paper was recommended for publication in revised form by Associate Editor J. M. Dion under the direction of Editor R. F. Curtain. Corresponding author Professor J. M. Schumacher. Tel. +31 20 5924090; Fax +31 20 5924199; E-mail hans.schumacher@cwi.nl.

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want to avoid the transformation to standard state-space form. For instance, when the entries of the matrices in the representation (1) are parameter-dependent, different routes to arrive at the standard state-space representation may have to be followed for different parameter values, so that an unattractive case-by-case analysis would be required. Moreover, below we shall develop the controllability criterion for *generalized* representations that may not even be similarity-equivalent to a standard state-space system.

Actually there are several ways to obtain an algebraic test that is capable of deciding whether a generalized state-space system of the form (1) is controllable. One possibility is the computation of all $(n+p)\times(n+p)$ full-size minors of the pencil $[\lambda k+\mu L \mid M]$, followed by the application of a classical 'multiresultant' test due to Macaulay (1903). A controllability rank test different from that presented here is due to Lomadze (1990) (see also Ravi and Rosenthal, 1995); this test involves a matrix of size $n(n+p)\times n(n-1+m\times p)$. The distinguishing feature of the test that we shall present in this paper is that it calls for checking that a certain matrix with n rows has full row rank; moreover, the column space of this matrix can be interpreted as a reachability space (see Section 6), and for this reason we call if the *reachability matrix* of (1). Our test is therefore a direct generalization of the classical Kalman rank test for controllability of standard state-space systems. A first step in this direction has already been taken by Mertzios *et al.* (1988) (see also Helmke, 1993), who developed a Kalman-type test that applies to systems of the form (1) with $p=0$; the present paper generalizes this work to the situation in which p is not necessarily zero.

By duality, the proposed controllability test can also be interpreted as an observability test. As such, it applies to systems of the form

$$\begin{aligned} G\sigma z &= Fz, \\ w &= Hz, \end{aligned} \tag{3}$$

where F, G and H are matrices of sizes $n\times(n+m)$, $n\times(n+m)$ and $q\times(n+m)$ respectively. We emphasize that systems of the form (3) (sometimes called the 'pencil form') have the same description power for smooth behaviors as the representation (1). The pencil form has been used recently in an investigation of 'impulsive–smooth' behaviors (Geerts and Schumacher, 1996a, b), which allow solutions in a space of generalized functions. Because solutions are allowed in a larger space than usual, the resulting minimality conditions are weaker than the standard ones. In fact, the following conditions for minimality are given by Geerts and Schumacher (1996b, Theorem 4.2):

(i) $\lambda G+\mu F$ has full row rank for some $(\lambda,\mu)\in\mathbb{C}^2\setminus\{(0,0)\}$;

(ii) $\begin{bmatrix}\lambda G+\mu F\\ \hline H\end{bmatrix}$ has full column rank for all $(\lambda,\mu)\in\mathbb{C}^2\setminus\{(0,0)\}$.

Condition (ii) is the observability condition, whereas condition (i) might be called an *admissibility condition.* The set of triples (F,G,H) satisfying conditions (i) and (ii), considered modulo similarity equivalence, has the interesting property of being a smooth and compact projective variety (Strømme, 1987; Lomadze, 1990; Helmke, 1993; Ravi and Rosenthal, 1995). Obviously the observability condition (ii) for systems in pencil form is related by duality to the controllability condition for systems in the form (1), and so after simple transposition the controllability test that will be derived below can also be used to test for observability of a triple (F,G,H) in the representation (3).

With an eye towards applications in coding theory, we shall work in this paper over a general base field $\mathbb{F}$. This implies that the field of complex numbers, as the algebraic closure of $\mathbb{R}$, is replaced by the algebraic closure of $\mathbb{F}$, which will be denoted by $\bar{\mathbb{F}}$. All the standard results from the algebraic theory of linear systems go through (see e.g. Kalman *et al.*, 1969, Chap. 10), and will be used without comment. In particular, a triple (K,L,M) will be said to be *controllable* if $[\lambda K+\mu L \mid M]$ has full row rank for all $(\lambda,\mu)\in\bar{\mathbb{F}}^2\setminus\{(0,0)\}$.

The use of the term 'controllability' as above is actually not quite appropriate in a discrete-time context, where one should rather speak of 'reachability'. We shall, however, still speak of a 'controllability test', since this terminology appears to be standard. By way of a concession to the discrete-time terminology, the matrix on which the test is based will be called a reachability matrix, and actually we shall show below (in a discrete-time setting) that the columns of this matrix do in fact span the reachable space.

2. PRELIMINARIES

The purpose of this section is to review some definitions and results on adjoints of matrices. Given an $n\times n$ matrix A, the *adjoint* of A (see

e.g. Wedderburn, 1934, p. 7) is the $n \times n$ matrix defined as

$$\operatorname{adj} A := ((-1)^{i+j} \det A_{ji})_{i,j=1}^{n}. \tag{4}$$

Here $\det A_{kl}$ denotes the $(n-1) \times (n-1)$ minor of A defined by omitting the kth row and lth column from A. The adjoint of a 1×1 matrix is always 1. The following are some basic properties of the adjoint:

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)I_n \tag{5}$$

(see Wedderburn, 1934, p. 7);

$$\operatorname{adj}(A_1 A_2) = (\operatorname{adj} A_2)(\operatorname{adj} A_1) \tag{6}$$

(see Wedderburn, 1934, p. 66). Directly from the definition, we have, for scalar t,

$$\operatorname{adj}(tA) = t^{n-1} \operatorname{adj} A. \tag{7}$$

We now derive some lemmas that will be needed below. The first of these is actually a special case of the result of Mertzios *et al.* (1988); we include a proof that shows the relation to the standard controllability test.

Lemma 2.1. Let $A \in \mathbb{F}^{n \times n}$ and $B \in \mathbb{F}^{n \times m}$ be given. The pair (A, B) is controllable if and only if there is no nonzero constant vector $\xi \in \mathbb{F}^n$ such that $\xi^{\mathrm{T}}\{\operatorname{adj}(\lambda I - A)\}B \in F[\lambda]$ is the zero polynomial in the indeterminate λ.

Proof. Write

$$\{\operatorname{adj}(\lambda I - A)\}B = \sum_{i=1}^{n} \Gamma_i \lambda^{n-i}$$

and

$$\det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n.$$

Clearly, we have $\xi^{\mathrm{T}}\{\operatorname{adj}(\lambda I - A)\}B = 0$ if and only if $\xi^{\mathrm{T}}\Gamma_i = 0$ for all $i = 1, \ldots, n$. Thus it remains to prove that the pair (A, B) is controllable if and only if the matrix $[\Gamma_1 \mid \ldots \mid \Gamma_n]$ has full row rank. As a consequence of (5), we have

$$(\lambda I - A) \sum_{i=1}^{n} \Gamma_i \lambda^{n-i} = \{\det(\lambda I - A)\}B$$
$$= \lambda^n B + a_1 \lambda^{n-1} B + \ldots + a_n B.$$

By equating coefficients, one obtains

$$[\Gamma_1 \mid \Gamma_2 \mid \ldots \mid \Gamma_n] = [B \mid AB \mid \ldots \mid A^{n-1}B] \times \begin{bmatrix} I & a_1 I & \ldots & a_{n-1} I \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 I \\ 0 & \ldots & 0 & I \end{bmatrix}. \tag{8}$$

Obviously the transformation in (8) is invertible, and so the matrix $[\Gamma_1 \mid \ldots \mid \Gamma_{n-1}]$ has full row rank if and only if $[B \mid \ldots \mid A^n{}^{1}B]$ has full row rank. But this is of course just the standard controllability test. □

The following lemma is given for matrices over a general field $\mathbb{K}$; we shall use it later in the case where $\mathbb{K} = \mathbb{F}(s)$, the field of rational functions with coefficients in $\mathbb{F}$.

Lemma 2.2. Let D and N be matrices over a field $\mathbb{K}$, of sizes $p \times p$ and $p \times m$ respectively. If ξ is a p-vector such that $[\xi^{\mathrm{T}} \mid 0]$ belongs to the row span of $[D \mid N]$ then

$$\xi^{\mathrm{T}}(\operatorname{adj} D)N = 0. \tag{9}$$

If D is a nonsingular then the reverse implication holds as well.

Proof. The first claim follows from the relation

$$[D \mid N] \begin{bmatrix} (\operatorname{adj} D)N \\ -(\det D)I \end{bmatrix} = 0,$$

which is immediate from (5). If D is nonsingular then (9) implies that $\xi^{\mathrm{T}} D^{-1} N = 0$, and so $[\xi^{\mathrm{T}} \mid 0] = \eta^{\mathrm{T}}[D \mid N]$ with $\eta^{\mathrm{T}} = \xi^{\mathrm{T}} D^{-1}$. □

Lemma 2.3. Let $A \in \mathbb{F}^{n \times n}$ and $B \in \mathbb{F}^{n \times m}$ be given. If there exists a nonzero vector $x \in \mathbb{F}^n$ such that $x^{\mathrm{T}}A = 0$ and $x^{\mathrm{T}}B = 0$ then $(\operatorname{adj} A)B = 0$.

Proof. The matrix A must be singular. If its rank is less than $n-1$ then $\operatorname{adj} A = 0$, and so certainly $(\operatorname{adj} A)B = 0$. Assume now that $\operatorname{rank} A = n-1$. Because $(\operatorname{adj} A)A = (\det A)I = 0$, all rows of $\operatorname{adj} A$ must be scalar multiples of the row vector x^{T}, and therefore $(\operatorname{adj} A)B = 0$. □

Let $X \in \mathbb{F}^{p \times (m+p)}$ be a matrix with more columns than rows. Let $\mathcal{I}(p, m+p)$ denote the set of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p$ of integers satisfying $1 \le \alpha_1 < \ldots < \alpha_p \le m+p$. For $\alpha \in \mathcal{I}(p, m+p)$ let X_α denote the $p \times p$ submatrix of X formed by the α_1th, ..., α_pth columns of X. Let $\alpha' := \{1, \ldots, m+p\} \backslash \alpha$ denote the complementary index of α and let $X_{\alpha'}$ denote the associated $p \times m$ submatrix of X.

Lemma 2.4. Let $X \in \mathbb{F}^{p \times (m+p)}$. Then $(\operatorname{adj} X_\alpha)X_{\alpha'} = 0$ for all $\alpha \in \mathcal{I}(p, m+p)$ if and only if $\operatorname{rank} X < p$.

Proof. To prove the necessity part, let us suppose that $\operatorname{rank} X = p$. There exists $\alpha \in \mathcal{I}(p, m+p)$ such that the submatrix X_α is

invertible. Without loss of generality, we may assume that $\alpha = (1, \ldots, p)$ and $X = [I_p \mid B]$. If $(\operatorname{adj} X_\alpha)X_{\alpha'} = 0$ for all $\alpha \in \mathcal{I}(p, m+p)$ then $B = 0$. Now consider the multi-index $\beta := (1, \ldots, p-1, p+1)$. Then

$$\operatorname{adj} X_\beta = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 1 \end{bmatrix},$$

and hence $(\operatorname{adj} X_\beta)X_{\beta'} \neq 0$, which is a contradiction. The sufficiency part is immediate from the preceding lemma. □

3. THE CONTROLLABILITY TEST

Consider behaviors represented by

$$K\sigma x + Lx + Mw = 0, \tag{10}$$

where $K, L \in \mathbb{F}^{(n+p)\times n}$ and $M \in \mathbb{F}^{(n+p)\times(m+p)}$ $(n > 0,\ m > 0,\ p \geq 0)$. The system will be called 'admissible' if the rank condition

$$\operatorname{rank}(\lambda K + \mu L) = n \tag{11}$$

holds for some $(\lambda, \mu) \in \bar{\mathbb{F}}^2$. This condition is implied by various forms of observability (Willems, 1991, p. 270). Recall that, in the behavioral setting of Willems (1991), a minimal representation does not necessarily generate a controllable behavior. It is shown (see Willems, 1991, Proposition VII.11) that a minimal representation of the form (10) determines a controllable behavior if and only if the rank condition

$$\operatorname{rank}[\lambda K + \mu L \mid M] = n + p \tag{12}$$

holds for all $(\lambda, \mu) \in \bar{\mathbb{F}}^2 \setminus \{(0, 0)\}$, or equivalently if $[K \mid M]$ has full row rank and $[\lambda K + L \mid M]$ has full row rank for all $\lambda \in \bar{\mathbb{F}}$. Motivated by the application to the formulation of an observability test for pencil-form descriptions of impulsive-smooth behaviors, as discussed in Section 1, we shall here work under the assumption of admissibility, which is implied by but does not imply minimality.

For $p = 0$ a Kalman-type controllability matrix for the system (10) was introduced by Mertzios *et al.* (1988); see also Helmke (1993). It has been shown that the system is controllable if and only if the associated controllability matrix has full rank. Here we seek to extend that construction to the general case, where p is arbitrary.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathcal{I}(p, m+p)$ let M_α denote the $(n+p)\times p$ submatrix of M formed by selecting the α_1th, α_2th, $\ldots$, α_pth columns of M. Let $\alpha' := \{1, \ldots, m+p\} \setminus \alpha$ denote the complementary index and let $M_{\alpha'}$ denote the associated $(n+p)\times m$ submatrix of M. Given any $(n+p)\times p$ submatrix M_α of M, write

$$\operatorname{adj}[\lambda K + \mu L \mid M_\alpha] = \begin{bmatrix} R_\alpha(\lambda, \mu) \\ S_\alpha(\lambda, \mu) \end{bmatrix}, \tag{13}$$

where $R_\alpha(\lambda, \mu)$ and $S_\alpha(\lambda, \mu)$ are formed by the first n and last p rows of $\operatorname{adj}(\lambda K + \mu L \mid M_\alpha)$ respectively; thus $R_\alpha(\lambda, \mu)$ has size $n \times (n+p)$ and $S_\alpha(\lambda, \mu)$ has size $p \times (n+p)$. From the identity

$$\operatorname{adj}[t\lambda K + t\mu L \mid M_\alpha] = \begin{bmatrix} t^{n-1}I & 0 \\ 0 & t^n I \end{bmatrix} \operatorname{adj}[\lambda K + \mu L \mid M_\alpha],$$

we obtain that $R_\alpha(\lambda, \mu)$ and $S_\alpha(\lambda, \mu)$ are matrices of homogeneous polynomials in (λ, μ) of degrees $n-1$ and n respectively. So, in particular,

$$R_\alpha(\lambda, \mu)M_{\alpha'} = \sum_{i=0}^{n-1} \Gamma_{i\alpha}\lambda^i\mu^{n-1-i} \tag{14}$$

for $n \times m$ matrices $\Gamma_{i\alpha}$, $i = 0, \ldots, n-1$. The *reachability matrix* of (K, L, M) is defined as the matrix of size $n \times nm\binom{m+p}{p}$ obtained by putting all matrices $\Gamma_{i\alpha}$ $(i = 0, \ldots, n-1;\ \alpha \in \mathcal{I}(p, m+p))$ next to each other;

$$\mathcal{R}(K, L, M) := [(\Gamma_{0\alpha} \mid \ \ldots \ \mid \Gamma_{n-1,\alpha}) \mid \alpha \in \mathcal{I}(p, m+p)]. \tag{15}$$

We can now state our main result.

Theorem 3.1. An admissible system (K, L, M) is controllable if and only if

$$\operatorname{rank} \mathcal{R}(K, L, M) = n.$$

4. TRANSFORMATIONS

It will be convenient in the proof of the theorem to make use of various transformations on the triple (K, L, M) that do not affect the controllability properties. We begin by studying similarity transformations. Clearly, if T and S are invertible matrices then the triple (K, L, M) is controllable if and only if (TKS^{-1}, TLS^{-1}, TM) is. The effect of such transformations on the matrix $\mathcal{R}(K, L, M)$ is described as follows.

Lemma 4.1. Let T and S be invertible $(n+p)\times(n+p)$ and $n \times n$ matrices respectively. Then

$$\mathcal{R}(TKS^{-1}, TLS^{-1}, TM) = (\det T)(\det S)^{-1}S\mathcal{R}(K, L, M). \tag{16}$$

Proof. For any $\alpha \in \mathcal{I}(p, m+p)$

$$\begin{aligned}(\operatorname{adj}[\lambda TKS^{-1} + \mu TLS^{-1} \mid TM_\alpha])TM_{\alpha'} &= \left\{\operatorname{adj}\left(T[\lambda K + \mu L \mid M_\alpha]\begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix}\right)\right\}TM_{\alpha'} \\ &= \left(\operatorname{adj}\begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix}\right) \\ &\quad \times (\operatorname{adj}[\lambda K + \mu L \mid M_\alpha])(\det T)M_{\alpha'}.\end{aligned}$$

The result follows from the identity

$$\operatorname{adj}\begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} (\det S)^{-1}S & 0 \\ 0 & (\det S)^{-1}I \end{bmatrix}. \quad \square$$

Remark 4.2. The transformations considered above are the natural transformations of system equivalence; the matrix S corresponds to a change of basis in state space, whereas the matrix T gives an invertible linear transformation of the system equations. Even in the generalized context of impulsive–smooth behaviors, the same transformation group is obtained (see Geerts and Schumacher, 1996b, Theorem 4.1). The above proof shows that the row space generated by $\mathcal{R}(K, L, M)$ is invariant under system equivalence. The same proof shows that the matrix $\mathcal{S}(K, L, M)$ that is formed from the coefficients of $S_\alpha(\lambda, \mu)M_{\alpha'}$, in analogy with (15), is transformed as follows:

$$\mathcal{S}(TKS^{-1}, TLS^{-1}, TM) = (\det T)(\det S)^{-1}\mathcal{S}(K, L, M).$$

It follows that the entries of $\mathcal{S}(K, L, M)$ are determined up to one multiplicative constant, or in other words that $\mathcal{S}(K, L, M)$ is a 'projective invariant'.

Apart from the similarity transformations, we shall also use the so-called 'scaling transformations' that are defined as follows. For any invertible 2×2-matrix

$$\Omega = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{F})$$

write

$$(K_\Omega, L_\Omega) := (aK + bL, cK + dL).$$

Note that these transformations actually involve not only rescaling of time, but also rotation; for instance, K and L are interchanged (corresponding to time reversal in the discrete-time interpretation) by the transformation $\Omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It is immediate from the characterization (12) that the triple (K, L, M) is controllable if and only if (K_Ω, L_Ω, M) is. To see how the controllability matrix $\mathcal{R}(K, L, M)$ changes under the scaling transformations, note that

$$\lambda(aK + bl) + \mu(cK + dL) = (a\lambda + c\mu)K + (b\lambda + d\mu)L,$$

so that the effect of $R_\alpha(\lambda, \mu)$ of replacing (K, L) by $(aK + bL, cK + dL)$ is the same as replacing (λ, μ) by $(a\lambda + c\mu, b\lambda + d\mu)$. Let us first consider what the effect of such a transformation is on a scalar homogeneous polynomial

$$p(\lambda, \mu) = \sum_{i=0}^{n-1} p_i \lambda^i \mu^{n-1-i}, \quad p_j \in \mathbb{F},$$

of degree $n-1$ in the variables λ and μ. Carrying out the transformation $(\lambda, \mu) \mapsto (a\lambda + c\mu, b\lambda + d\mu)$ results in a linear transformation of the coefficients $p_0, \ldots, p_{n-1}$. For instance, for $n = 3$,

$$\begin{aligned}\sum_{i=0}^{2} p_i(a\lambda + c\mu)^i(b\lambda + d\mu)^{2-i} &= (d^2p_0 + cdp_1 + c^2p_2)\mu^2 \\ &\quad + \{2bdp_0 + (ad + bc)p_1 + 2acp_2\}\lambda\mu \\ &\quad + (b^2p_0 + adp_1 + a^2p_2)\lambda^2.\end{aligned}$$

Thus the new coefficients are expressed in terms of the old ones by

$$[\tilde{p}_0 \mid \tilde{p}_1 \mid \tilde{p}_2] = [p_0 \mid p_1 \mid p_2]\begin{bmatrix} d^2 & 2bd & b^2 \\ cd & ad + bc & ab \\ c^2 & 2ac & a^2 \end{bmatrix}.$$

We denote the $n \times n$ transformation matrix obtained in this way by $\tau_n(\Omega)$; so for instance it follows from the above that

$$\tau_3\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d^2 & 2bd & b^2 \\ cd & ad + bc & ab \\ c^2 & 2ac & a^2 \end{bmatrix}.$$

Since $\tau_n(\Omega)\tau_n(\Omega^{-1}) = I$, the matrices $\tau_n(\Omega)$ are nonsingular. Now consider a homogeneous polynomial matrix $\Gamma(\lambda, \mu) = \sum_{i=0}^{n-1} \Gamma_i \lambda^i \mu^{n-1-i}$ whose coefficients Γ_i have size $n \times m$. The transformation $(\lambda, \mu) \mapsto (a\lambda + c\mu, b\lambda + d\mu)$ has an entrywise effect on $\Gamma(\lambda, \mu)$, and may therefore expressed in terms of the coefficients by

$$[\tilde{\Gamma}_0 \mid \ \ldots \ \mid \tilde{\Gamma}_{n-1}] = [\Gamma_0 \mid \ \ldots \ \mid \Gamma_{n-1}]\{\tau_n(\Omega) \otimes I_m\},$$

where $\otimes$ denotes the Kronecker product. Finally this transformation applies blockwise to the matrix $\mathcal{R}(K, L, M)$, where the blocks correspond to the selections α, and so we have proved the following.

Lemma 4.3. For each invertible 2×2 matrix Ω

there exists an invertible $n \times n$ matrix $\tau_n(\Omega)$ such that

$$\mathscr{R}(K_\Omega, L_\Omega, M) = \mathscr{R}(K, L, M)\{I_{\binom{m+p}{p}} \otimes (\tau_n(\Omega) \otimes I_m)\}. \quad (17)$$

Remark 4.4. In particular, it follows that the subspace of $\mathbb{F}^n$ spanned by the columns of $\mathscr{R}(K, L, M)$ is invariant under scaling transformations.

Finally, we note that both the property of controllability and the rank of the reachability matrix are invariant under transformations of the type $(K, L, M) \mapsto (K, L, MP)$, where P is an invertible matrix. Such transformations can be interpreted as changes of basis in the space of external variables. Actually we shall only use transformations P that are permutation matrices; these correspond to just renumbering the external variables.

5. PROOF OF THE MAIN RESULT

In this section we prove Theorem 3.1. We first show the sufficiency of the stated condition for controllability. Suppose that the reachability matrix $\mathscr{R}(K, L, M)$ has rank n. This immediately implies that the matrix $[\lambda K + \mu L \mid M]$ must have full row rank for some $(\lambda, \mu) \neq (0, 0)$, because otherwise it follows from Lemma 2.4 that $(\text{adj}\,[\lambda K + \mu L \mid M_\alpha])M_{\alpha'} = 0$ for all (λ, μ) and all α so that $\mathscr{R}(K, L, M)$ is identically zero. Consequently, $[\lambda K + \mu L \mid M]$ has full row rank for almost all (λ, μ). By assumption, we also have that $\lambda K + \mu L$ has full column rank for almost all (λ, μ), so that certainly there will be points (λ, μ) where $[\lambda K + \mu L \mid M]$ and $\lambda K + \mu L$ both have full rank. Because both controllability and the rank of $\mathscr{R}(K, L, M)$ are invariant under scaling transformations, we may assume that this happens at $(\lambda, \mu) = (1, 0)$, so that in this case K has full column rank and $[K \mid M]$ has full row rank. Permuting the columns of M if necessary, we may write $M = [M_1 \mid M_2]$ in such a way that the matrix $[K \mid M_1]$ is invertible. Now using the invariance under similarity action (from the left), we may left-multiply by the inverse of $[K \mid M_1]$ and end up with K, L and M in the following 'output-nulling' (Weiland 1991) form:

$$K = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} -A \\ -C \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -B \\ I & -D \end{bmatrix}. \quad (18)$$

Clearly, the matrix $[\lambda K + \mu L \mid M]$ has full row rank for all $(\lambda, \mu) \neq (0, 0)$ if and only if the matrix $[\lambda I - A \mid B]$ has full row rank for all $\lambda \in \bar{\mathbb{F}}$; that is to say, if and only if the pair (A, B) is controllable.

Assume now that (A, B) is *not* controllable; we want to prove that in this case the matrix $\mathscr{R}(K, L, M)$ cannot have full row rank. By Lemma 2.1, there exists a nonzero vector ξ such that $\xi^{\mathrm{T}}\{\text{adj}\,(\lambda I - A)\}B = 0$. It follows from Lemma 2.2 that there exists a vector $g(\lambda)$ such that $[\xi^{\mathrm{T}} \mid 0] = g^{\mathrm{T}}(\lambda)[\lambda I - A \mid B]$. By the special form of the matrices K, L and M, this implies that

$$[\xi^{\mathrm{T}} \mid 0 \mid 0] = [g^{\mathrm{T}}(\lambda) \mid 0][\lambda K + L \mid M_\alpha \mid M_{\alpha'}]$$

for all selections α. It follows from Lemma 2.2 that

$$[\xi^{\mathrm{T}} \mid 0](\text{adj}\,[\lambda K + L \mid M_\alpha])M_{\alpha'} = 0$$

and consequently

$$\xi^{\mathrm{T}} \sum_{i=0}^{n-1} \Gamma_{i\alpha}\lambda^i = 0$$

for all α. This implies that $\xi^{\mathrm{T}}\Gamma_{i\alpha} = 0$ for all i and all α, so that $\xi^{\mathrm{T}}\mathscr{R}(K, L, M) = 0$.

For the necessity part of the proof, we have to show that the reachability matrix $\mathscr{R}(K, L, M)$ has full row rank if the matrix $[\lambda K + \mu L \mid M]$ has full rank for all $(\lambda, \mu) \neq (0, 0)$. By a suitable scaling transformation, we may assume that K, L and M are in the form (18); the full-rank condition then implies that the pair (A, B) is controllable. We now choose a particular selection α, namely the one for which

$$M_\alpha = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad M_{\alpha'} = \begin{bmatrix} -B \\ -D \end{bmatrix}.$$

After some calculation, we find

$$(\text{adj}\,[\lambda K + L \mid M_\alpha])M_{\alpha'} = -\begin{bmatrix} \{\text{adj}\,(\lambda I - A)\}B \\ C\{\text{adj}\,(\lambda I - A)\}B + \{\det(\lambda I - A)\}D \end{bmatrix}. \quad (19)$$

If $\mathscr{R}(K, L, M)$ were not of full row rank then there would exist a nonzero constant vector ξ such that $\xi^{\mathrm{T}}\mathscr{R}(K, L, M) = 0$. From the above, this would imply in particular that $\xi^{\mathrm{T}}\{\text{adj}\,(\lambda I - A)\}B = 0$. But we know from Lemma 2.1 that this contradicts the controllability of (A, B). The proof is complete. □

Remark 5.1. If $p = 0$ then the set $\mathscr{I}(p, m + p)$ contains just one element, and the controllability matrix $\mathscr{R}(K, L, M)$ can be written as $[\Gamma_0 \mid \ldots \mid \Gamma_{n-1}]$, where the Γ_i are the coefficients of $\{\text{adj}\,(\lambda K + \mu L)\}M$. This is the controllability test of Mertzios *et al.* As we have seen in the proof of Lemma 2.1, in the 'classical' case $(K, L, M) = (I, A, B)$ this is just a similarity transformation away from Kalman's controllability criterion.

Remark 5.2. The calculation of the coefficients of adj $[\lambda K + \mu L \mid M_\alpha]$ can be carried out by an adaptation of Leverrier's algorithm due to Mertzios (1984).

6. THE REACHABLE SPACE

In this section we provide a dynamic interpretation of the matrix $\mathcal{R}(K, L, M)$ that also will justify the name reachability matrix. We shall restrict ourselves to discrete-time systems, so the system dynamics is given by

$$Kx_{t+1} + Lx_t + Mw_t = 0. \tag{20}$$

Definition 6.1. A state vector $\hat{x}$ is said to be a *reachable state* if there exists a sequence of states

$$\Sigma := \{x_i \in \mathbb{F}^n \mid i \in \mathbb{Z}\}$$

having the property that

(i) at most finitely many vectors $x_i \in \Sigma$ are nonzero;

(ii) there is a set of external variables such that (20) is satisfied for all $t \in \mathbb{Z}$;

(iii) $\hat{x} \in \Sigma$.

The set of all reachable states is denoted by $\hat{\mathcal{R}}(K, L, M)$.

The set $\hat{\mathcal{R}}(K, L, M)$ can also be characterized in the following way: $\tilde{x} \in \hat{\mathcal{R}}(K, L, M)$ if and only if there is a vector polynomial

$$x(\lambda) = \sum_{i=0}^{k-1} x_i \lambda^i \in \mathbb{F}^n[\lambda]$$

having $\hat{x}$ as one of its coefficients and a vector polynomial

$$w(\lambda) = \sum_{i=0}^{k} w_i \lambda^i \in \mathbb{F}^{m+p}[\lambda]$$

such that

$$[\lambda K + L \mid M]\begin{bmatrix} x(\lambda) \\ w(\lambda) \end{bmatrix} = 0. \tag{21}$$

Note that this last equation can also be written componentwise in the form

$$\left[\begin{array}{cccc|cccc} K & 0 & \dots & 0 & M & 0 & \dots & 0 \\ L & K & \ddots & \vdots & 0 & M & \ddots & \vdots \\ 0 & L & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & K & \vdots & & \ddots & M \\ 0 & \dots & 0 & L & 0 & \dots & \dots & 0 \end{array}\right.$$

$$\left.\begin{array}{c} M \\ 0 \\ \vdots \\ 0 \end{array}\right]$$

$$\times \begin{bmatrix} x_0 \\ \vdots \\ x_{k-1} \\ w_0 \\ \vdots \\ w_k \end{bmatrix} = 0. \tag{22}$$

The following lemma is now a simple consequence of the description (21).

Lemma 6.2. $\hat{\mathcal{R}}(K, L, M) \subset \mathbb{F}^n$ is a linear subspace.

It is also clear from the description (21) that $\hat{\mathcal{R}}(K, L, M)$ is invariant under transformation of the system equation and under change of basis in the external variables, i.e. we have for $T \in \mathrm{GL}_{n+p}$ and $U \in \mathrm{GL}_{m+p}$ that

$$\hat{\mathcal{R}}(TK, TL, TMU^{-1}) = \hat{\mathcal{R}}(K, L, M). \tag{23}$$

The next lemma states that $\hat{\mathcal{R}}(K, L, M)$ is also invariant under scaling transformations.

Lemma 6.3. For each invertible 2×2 matrix Ω one has

$$\hat{\mathcal{R}}(K_\Omega, L_\Omega, M) = \hat{\mathcal{R}}(K, L, M). \tag{24}$$

Proof. For every fixed positive integer k consider the set of homogeneous polynomial vectors $x(\lambda, \mu) = \sum_{i=0}^{k-1} x_i \lambda^i \mu^{k-1-i}$ whose coefficients x_i satisfy (22) for some set of external variables w_i. The transformation $(\lambda, \mu) \mapsto (a\lambda + c\mu, b\lambda + d\mu)$ has an entrywise effect on $x(\lambda, \mu)$, and may be expressed as in the proof of Lemma 4.3 through

$$[\tilde{x}_0 \mid \dots \mid \tilde{x}_{k-1}] = [x_0 \mid \dots \mid x_{k-1}](\tau_k(\Omega)).$$

But this establishes the invariance. □

We are now in a position to establish the connection between the subspace $\hat{\mathcal{R}}(K, L, M)$ of reachable states and the reachability matrix $\mathcal{R}(K, L, M)$ as introduced in (15).

Theorem 6.4. The vector space $\hat{\mathcal{R}}(K, L, M)$ of reachable states is equal to the column space of the reachability matrix, i.e.

$$\hat{\mathcal{R}}(K, L, M) = \mathrm{colsp}\, \mathcal{R}(K, L, M). \tag{25}$$

Proof. First note that the column space of $\mathcal{R}(K, L, M)$ is certainly invariant under permutation of the external variables. After possible transformations in the internal variables and in the scaling variables and after a possible permutation of the external variables, we can therefore assume that K, L, and M have the special form (18). One readily verifies that in this situation $\hat{\mathcal{R}}(K, L, M)$ is exactly the classical reachability space

$$\mathrm{colsp}\, [B \mid AB \mid \dots \mid A^{n-1}B]. \tag{26}$$

It therefore follows from the identity (19) that $\hat{\mathcal{R}}(K, L, M) \subset \mathrm{colsp}\, \mathcal{R}(K, L, M)$. On the other hand, it follows from the sufficiency part of the

main proof in Section 5 that any vector in the left kernel of (26) is also in the left kernel of $\mathcal{R}(K, L, M)$. But this establishes the proof. □

7. EXAMPLES

In this section we illustrate the theory by two examples. To illustrate that our test also detects lack of controllability 'at infinity' (i.e. the matrix $[K \mid M]$ has less than full row rank), we first consider a simple example in which this occurs.

Example 7.1. Consider the system given by the parameter matrices

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In a continuous-time setting, these parameters correspond to equations $\dot{x}_1 = -x_2 - w_1$, $\dot{x}_2 = x_1 - w_2$ and $x_1 = x_2$. Choosing an input/output assignment, for instance by setting $u = w_1$ and $y = w_2$, and eliminating the algebraic constraint, leads to a standard state-space description in the form $\dot{x} = -x - u$, $y = 2x + u$. However, the differential–algebraic description above also covers impulsive modes that may occur when the constraint $x_1 = x_2$ did not exist for all time but is activated at some instant, for instance by the turning of a switch. A controllability test 'at infinity' is meaningful in connection with such impulsive modes.

In the above example the set $\mathcal{I}(p, m+p) = \mathcal{I}(1, 2)$ has just two elements. Denoting the columns of M by M_1 and M_2, we get

$$\operatorname{adj}[\lambda K + \mu L \mid M_1] = \begin{bmatrix} 0 & -\mu & -\lambda \\ 0 & -\mu & -\mu \\ \mu^2 - \lambda\mu & \mu^2 + \lambda\mu & \mu^2 + \lambda^2 \end{bmatrix},$$

$$\operatorname{adj}[\lambda K + \mu L \mid M_2] = \begin{bmatrix} \mu & 0 & \mu \\ \mu & 0 & -\lambda \\ \mu^2 - \lambda\mu & \mu^2 + \lambda\mu & \mu^2 + \lambda^2 \end{bmatrix}.$$

As predicted by the theory, the first two rows of the resulting matrices have degree 1 and the third rows have degree 2. Denoting the degree-1 parts by $R_1(\lambda, \mu)$ and $R_2(\lambda, \mu)$ respectively, we compute

$$R_1(\lambda, \mu)M_2 = \begin{bmatrix} -\mu \\ -\mu \end{bmatrix}, \quad R_2(\lambda, \mu)M_1 = \begin{bmatrix} \mu \\ \mu \end{bmatrix}.$$

The reachability matrix is now formed from the coefficients in the expressions above:

$$\mathcal{R}(K, L, M) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

Obviously this matrix does not have full row rank, and so the test does indeed show that the triple (K, L, M) is not controllable.

In the second example we take the binary field $\mathbb{F}_2 = \{0, 1\}$ as our base field; this is a common choice in coding theory. In the context of coding, the set of w-trajectories that satisfy a description of the form $Kx_{t+1} + Lx_t + Mw_t = 0$ can be looked at as the set of all possible code messages, and in this way the matrices K, L and M specify a particular code. One may obtain such matrices from a state-space realization of some encoding device given in polynomial form, but recently methods have been advocated that aim at a direct design of the parameter matrices (Rosenthal *et al.*, 1996). In this context, a lack of controllability indicates that a reduction of the state vector is possible. It should be noted that, in contrast to the case in which the base field is $\mathbb{R}$, in the finite-field case controllability is not generic, in the sense that when the parameter matrices are selected 'at random' there is a positive probability that the resulting system is not controllable.

Example 7.2. Consider the binary base field $\mathbb{F} = \mathbb{F}_2 = \{0, 1\}$ and let a collection of code messages be described by the matrices K, L and M given by

$$[\lambda K + \mu L \mid M] := \left[\begin{array}{ccc|cc} \lambda & \mu & 0 & 0 & 1 \\ \lambda + \mu & \lambda & 0 & 1 & 1 \\ \mu & 0 & \lambda & 0 & 0 \\ \lambda & \mu & \mu & 1 & 1 \end{array}\right].$$

Calculation (over $\mathbb{F}_2$) as above shows that the reachability matrix as defined in (15) is given by

$$\mathcal{R}(K, L, M) = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Obviously this matrix does not have full rank, and according to Theorem 6.4 the reachable space $\mathcal{R}(K, L, M)$ is spanned by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

8. CONCLUSIONS

We have presented a rank test for controllability of behaviors described by equations of the form $K\sigma x + Lx + Mw = 0$; similarly, the dual form leads to an observability test for systems in pencil form. The test is in the spirit of Kalman's classical controllability condition; it requires checking that a certain matrix with n rows has full row rank. Moreover, the column span of this matrix has the interpretation of a reachable space.

Acknowledgements—The Research by U. Helmke was partially supported by GIF Grant I 184-078.6/91. J. Rosenthal was supported in part by NSF Grant DMS-94-00965; the research for this paper was carried out in part while he was a visitor at CWI.

REFERENCES

Aplevich, J. D. (1991). *Implicit Linear Systems.* Lecture Notes in Control and Inform. Sciences, Vol. 152, Springer-Verlag, Berlin.

Geerts, A. H. W. and J. M. Schumacher (1996a). Impulsive-smooth behavior in multimode systems. Part I: State-space and polynomial representations. *Automatica,* **32,** 747–758.

Geerts, A. H. W. and J. M. Schumacher (1994b). Impulsive-smooth behavior in multimode systems. Part II: Minimality and equivalence. *Automatica,* **32,** 819–832.

Helmke, U. (1993). The cohomology of moduli spaces of linear dynamical systems. Regensburger Mathematische Schriften 24, Fak. Math. Univ. Regensburg.

Kalman, R. E., P. L. Falb and M. A. Arbib (1969). *Topics in Mathematical System Theory.* McGraw-Hill, New York.

Kuijper, M. (1994). *First-order Representations of Linear Systems.* Birkhäuser, Boston.

Lomadze, V. G. (1990). Finite-dimensional time-invariant linear dynamical systems: algebraic theory. *Acta Appl. Math.,* **19,** 149–201.

Macaulay, F. S. (1983). Some formulae in elimination. *Proc. Lond. Math. Soc.,* **35,** 3–27.

Mertzios, B. G. (1984). Leverrier's algorithm for singular systems. *IEEE Trans. Autom. Control,* **AC-28,** 652–653.

Mertzios, B. G., M. A. Christodoulou, B. L. Syrmos and F. L. Lewis (1988). Direct controllability and observability time domain conditions of singular systems. *IEEE Trans. Autom. Control,* **AC-33,** 788–791.

Ravi, M. S. and J. Rosenthal (1993). A general realization theory for higher order linear differential equations. *Syst. Control Lett.,* **25,** 351–360.

Rosenthal, J., J. M. Schumacher and E. V. York (1996). On behaviors and convolutional codes. To appear in *IEEE Trans. Inf. Theory.*

Strømme, S. A. (1987). On parametrized rational curves in Grassmann varieties. In F. Ghione, C. Peskine and E. Sernesi (Eds). *Space Curves.* Lecture Notes in Mathematics, Vol. 1266, pp. 251–272. Springer-Verlag. Berlin.

Wedderburn, J. H. M. (1934). *Lectures on Matrices.* Coll. Publ. XVII. American Mathematical Society, Providence, RI.

Weiland, S. (1991). Theory of approximation and disturbance attenuation for linear systems. PhD Thesis, University of Groningen.

Willems, J. C. (1991). Paradigms and puzzles in the theory of dynamical systems. *IEEE Trans. Autom. Control,* **AC-36,** 259–294.