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System Equivalences and Canonical Forms from a Behavioral Point of View

M.S. Ravi

Department of Mathematics,

East Carolina University, Greenville, NC 27858

e–mail: maravi@ecuvax.cis.ecu.edu

Joachim Rosenthal*

Department of Mathematics, University of Notre Dame

Notre Dame, IN 46556-5683, USA

e–mail: Joachim.Rosenthal@nd.edu

J.M. Schumacher

CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands, and

Tilburg University, CentER and Department of Economics,

P.O. Box 90153, 5000 LE Tilburg, The Netherlands

e–mail: Hans.Schumacher@cwi.nl

Abstract

The construction of canonical forms for linear systems has been studied extensively. Here we give a systematic method to relate polynomial and first-order canonical forms to each other. The discussion is carried out on three levels, corresponding to transfer functions, behaviors, and homogeneous behaviors.

1 Introduction

Canonical forms for linear systems and the global structure of the family of linear systems of a given McMillan degree have been studied extensively in the past, see for instance [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One of the main motivations for this study has been to obtain parametrizations that can be used in system identification procedures. We refer to Chapter 2 of the well-known book by Hannan and Deistler [13] for an exposition of the relevant issues.

To summarize briefly, a *canonical form* is a prescription that assigns to each class of observationally equivalent system representations exactly one element of this class. A *parametrization* is a mapping that establishes a one-to-one correspondence between equivalence classes on the one hand and parameter vectors on the other hand. Often such a parametrization is obtained from a canonical form. Alternatively one can work with overlapping *charts* by parametrizing suitable subsets of the class of all classes of observationally equivalent representations. Whether one works with canonical forms or with charts, an important issue is presented by the *degeneration phenomena* associated with the approach of a boundary of validity of a chosen parametrization.

In this paper we discuss some of the above issues from the perspective that is offered by the behavioral theory of J.C. Willems [14] in connection with the associated realization theory developed by Kuijper and Schumacher [15], and the extensions to homogeneous representations as studied by Ravi and Rosenthal [16]. In particular we shall discuss the various forms of equivalence that may be distinguished from a behavioral point of view, both in the polynomial framework and in the setting of first-order representations.

This discussion is stated in the general framework in which the external variables are not distinguished in inputs and outputs; from the point of view of identification, the presence of a (causal) input-output structure may be viewed as one possible piece of *a priori* information. Then we shall show how polynomial and first-order descriptions may be related to each other in a very simple and direct way. We apply the realization theory to link together polynomial

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and first-order canonical forms at the various levels of equivalence.

2 Equivalence notions

The behavioral theory as developed by J.C. Willems [14] offers a systematic way to discuss *equivalence* and *minimality* of system representations, as follows. Given a representation that in some way specifies a system of differential or difference equations, one can assign to this representation the set of all solutions in some chosen function class; this set of solutions is called the *behavior* associated with the given representation. Two representations of a given type are called *equivalent* if they generate the same behavior.

Among all equivalent representations of a given behavior, one may distinguish those that are *minimal* in an appropriate sense (minimal numbers of equations and auxiliary variables); one may hope to describe the equivalence of minimal representations in an effective way, for instance by means of a group action.

The above approach to defining equivalence of representations of a given type does *not* lead to a unique result however. This is due to the fact that the method depends on the choice of a function class (or a *universum*, in the terminology of Willems). Although the choice of a different function class does not necessarily lead to a new notion of equivalence, it is not true that all choices lead to the same notion. We now discuss this in more detail for linear time-invariant systems, using each time both polynomial and first-order representations. The latter will always be taken of the form

$$\begin{aligned} \sigma G z &= F z \\ w &= H z \end{aligned} \qquad (1)$$

where σ is shift or differentiation, $z(t)$ denotes a vector of auxiliary variables, and $w(t)$ is the vector of external variables; F and G are matrices of size $n\times(n+m)$, and H has size $(p+m)\times(n+m)$. This was called the 'pencil form' in [15]. Other first-order representations that treat all external variables alike are possible (in particular the form $\sigma Kx+Lx+Mw=0$) and would give rise to a similar development. We shall consider polynomial 'kernel' or 'AR' representations of the form

$$P(\sigma)w = 0 \qquad (2)$$

where P is a polynomial matrix of size $p \times (p+m)$.

We shall consider three notions of equivalence which seem to be main ones that can be made to appear by a suitable choice of 'universum', although they are certainly not the only ones.

1. *Transfer equivalence.* This is maybe the most standard and classical notion of equivalence; it says that two linear input-output representations are equivalent if their transfer functions coincide. This can be applied either to polynomial representations

$$[D(\sigma) \;\; -N(\sigma)] \begin{bmatrix} y \\ u \end{bmatrix} = 0$$

(with transfer function $D^{-1}(s)N(s)$) or to standard state space representations. Formulated in this way it seems that transfer equivalence depends on a given i/o structure. However, there is also a behavioral definition. which arises by taking the *quadratically summable* sequences as a universum. One can show that two polynomial matrices P and P' of full row rank which generate the same ℓ_2 behavior are related by a nonsingular rational matrix F such that $P' = FP$; restricted to representations of the form $P = [D \;\; N]$, this comes down to transfer equivalence. For pencil representations (1), it can be shown that these are a minimal representation of their associated ℓ_2 behavior if and only if the triple (F, G, H) satisfies the following requirements:

(i) $sG - tF$ has full row rank for all $(s, t) \neq (0,0)$;

(ii) $\left[\begin{smallmatrix} sG-tF \\ H \end{smallmatrix}\right]$ has full column rank for all $(s,t) \neq (0,0)$.

Moreover, two minimal representations (F, G, H) and (F', G', H') represent the same ℓ_2-behavior if and only if there exist nonsingular matrices S and T such that $(F', G', H') = (SFT^{-1}, SGT^{-1}, HT^{-1})$. Specialized to the case of standard state space systems ($G = [I \mid 0]$, $F = [A \mid B]$, $H = \left[\begin{smallmatrix} C & D \\ 0 & I \end{smallmatrix}\right]$), all this comes down to the usual controllability and observability requirements, and to the standard state space isomorphism. The continuous-time theory is completely analogous. For more information on ℓ_2/L_2 systems, see [17, 18].

2. *Behavioral equivalence.* Although all notions of equivalence that we discuss can be motivated from a behavioral point of view, we shall call 'behavioral equivalence' the one that occurs most frequently in the behavioral approach as developed by Willems. In the discrete-time context, this equivalence comes from considering as a universum *all* sequences on $\mathbb{Z}_+$ (a slightly different notion occurs when considering sequences on $\mathbb{Z}$). In continuous time, the same notion of equivalence comes from employing the C^∞ functions or various other classes of functions or generalized functions. The equivalence that is induced

for polynomial representations is the following: two polynomial matrices P and P' of full row rank are behaviorally equivalent if and only if there exists a unimodular matrix U such that $P' = UP$. A pencil representation is a minimal representation in the sense of behavioral equivalence if and only if

(i) G has full row rank;

(ii) $\left[\begin{smallmatrix} sG-tF \\ H \end{smallmatrix}\right]$ has full column rank for all $(s,t) \neq (0,0)$.

3. *Homogeneous equivalence.* In a study [19, 20] of so-called *impulsive-smooth behaviors* which are defined on $\mathbb{R}_+$, a notion of equivalence was discovered for pencil representations that is even weaker than behavioral equivalence, so that certain representations that would be called equivalent in the (smooth) behavioral framework are no longer equivalent in the impulsive-smooth sense. The ensuing notion of minimality is the following. A triple (F, G, H) is minimal in the impulsive-smooth sense if and only if

(i) $sG - tF$ has full row rank for some (and hence almost all) $(s,t) \neq (0,0)$;

(ii) $\left[\begin{smallmatrix} sG-tF \\ H \end{smallmatrix}\right]$ has full column rank for all $(s,t) \neq (0,0)$.

Moreover, it turns out that there is a natural one-to-one relation between the equivalence classes that are so defined and the equivalence classes of full rank *homogeneous* polynomial matrices modulo left multiplication by homogeneous unimodular matrices [16, 22]. The suggested behavioral interpretation for homogeneous polynomial matrices is given in [23], where in particular discrete-time homogeneous behaviors are defined.

In each of the three cases mentioned above, the set of equivalence classes that one obtains (described either in polynomial or in first-order terms) carries naturally the structure of a smooth manifold of dimension $n(m+p)+mp$; moreover, the manifold that is obtained from transfer equivalence is densely embedded in the one obtained from behavioral equivalence, which is in its turn densely embedded in the manifold of homogeneous systems. A very important fact about the latter manifold is that it is not only smooth but also *compact* [21, 24, 16], which makes it a natural setting for the study of the occurrences that from the point of view of the smaller manifolds would be called 'degeneration phenomena'. In the sequel we will denote this manifold with $\mathcal{H}^n_{p,m}$.

It is in no way implied here that the above three notions of equivalence are the only ones that could be derived from the behavioral approach. Still other function or sequence classes may be used and will give rise to different equivalence notions which may be relevant for particular purposes. For instance the rather small class of fixed-period sequences plays a role in the study of cyclic codes [25]. Another variant of interest is to use stochastic processes rather than vector-valued sequences [26].

3 Some realization theory

We now describe a realization theory that enables one to establish a close relation between polynomial and first-order representations; as will be shown in the next section, this relationship is close enough to allow in a certain sense the 'transfer' of canonical forms between these two types of representations. This realization theory has its roots in the work of Fuhrmann [27, 28], was lifted to the behavioral level by Kuijper and Schumacher [15], and was further extended to the homogeneous level by Ravi and Rosenthal [22] building on the work of Strømme in algebraic geometry [21]. A more algorithmic approach to the homogeneous realization was recently developed by the authors [23]; here we present a further streamlining. It should be noted that it is straightforward to specialize the homogeneous realization algorithm to the 'behavioral' and 'transfer' cases.

With a homogeneous polynomial matrix $P(s,t)$ of size $p \times q$ and of row degrees $\nu_1, \cdots, \nu_p$, we can associate the $\mathbb{R}$-linear space of p-vectors whose i-th entry is a homogeneous polynomial of degree $\nu_i - 1$, or is zero if $\nu_i = 0$. Obviously the dimension of this space, which we shall denote by X_ν, is $n := \sum_{i=1}^p \nu_i$. We define the *canonical basis matrix* $X(s,t)$ as the matrix of size $p \times n$ given by

$$\begin{bmatrix} s^{\nu_1-1} & \cdots & t^{\nu_1-1} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & & 0 & s^{\nu_2-1} & \cdots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & \cdots & & & & & \cdots & t^{\nu_p-1} \end{bmatrix}.$$

The basis of the realization theory is the following proposition.

Proposition 1 *The triple (F, G, H) is a realization of the homogeneous polynomial matrix $P(s,t)$, in the sense that their associated homogeneous behaviors are the same, if the equality*

$$\operatorname{im} \begin{bmatrix} sG - tF \\ H \end{bmatrix} = \ker\,[-X(s,t) \mid P(s,t)] \qquad (3)$$

holds for almost all $(s,t) \neq (0,0)$.

One can show that the minimality properties for pencil representations of homogeneous behaviors are automatically satisfied if the condition in the proposition holds and the number of columns of the matrices F, G, and H is $n+m$, and we shall only consider solutions of this type. Now note that the equation

$$[-X(s,t) \mid P(s,t)] \begin{bmatrix} sG - tF \\ H \end{bmatrix} = 0 \qquad (4)$$

which is implied by (3) can also be written as

$$[tX(s,t) \mid -sX(s,t) \mid P(s,t)] \begin{bmatrix} F \\ G \\ H \end{bmatrix} = 0. \qquad (5)$$

Note also that left multiplication by the matrix $[tX(s,t) \mid -sX(s,t) \mid P(s,t)]$ of a constant vector of length $2n+m+p$ produces a vector with entries that are homogeneous polynomials of degrees $\nu_1, \ldots, \nu_p$. So this matrix can be viewed as a linear mapping from $(2n+m+p)$-dimensional linear space to the space of all such vectors, which has dimension $(\nu_1+1)+\cdots+(\nu_p+1) = n+p$. As it is easily established that the induced mapping is surjective, it follows that it must have an $(n+m)$-dimensional kernel. Consequently, any matrix whose column space spans this kernel presents a solution to the minimal realization problem. The right action on minimal triples, $(F,G,H) \mapsto (FT^{-1}, GT^{-1}, HT^{-1})$, clearly corresponds to the nonuniqueness in the choice of a basis matrix for the kernel. The left action taking (F,G,H) to (SF,SG,H) corresponds to the nonuniqueness in the choice of a basis matrix for the space X_ν.

To translate this realization procedure to the level of behavioral equivalence, it suffices to consider polynomial matrices in row proper form and to set $t=1$ in the above procedure. For transfer equivalence, one should use left coprime polynomial matrices in row proper form. In both situations one will then automatically obtain the appropriate minimality properties for the first-order realizations.

4 Canonical forms

The realization theory of the preceding section allows us to establish a link between polynomial and first-order canonical forms. Let us first suppose that we have some canonical form for homogeneous polynomial matrices under the corresponding action of homogeneous unimodular matrices. Given $P(s,t)$ in canonical form, the corresponding canonical basis matrix is of course uniquely defined and in this way we eliminate the left action on the associated realizations. To eliminate the right action, we can use any method to uniquely define a column basis matrix for a given subspace; for instance one may use the column echelon form, associated with the standard Schubert cell decomposition of the Grassmannian. In this way a canonical form for (F,G,H) triples (satisfying the 'homogeneous' minimality conditions) under similarity action is obtained. In case one works on the 'behavioral' level essentially the same procedure applies, provided that one uses row proper canonical forms for polynomial matrices.

An attractive feature of the approach sketched here can be that the problem of constructing canonical forms for (F,G,H) triples under left and right similarity is split into the (presumably smaller) problems of finding canonical forms for polynomial matrices under the left unimodular action and for full column rank matrices under right similarity. For the same reason it would seem less attractive to invert the procedure and generate canonical forms for polynomial matrices from canonical forms for (F,G,H) triples, although this is possible in principle.

The choice of a canonical representation for $(n+m)$-dimensional subspaces of $(2n+m+p)$-dimensional linear space may be influenced by *a priori* information. A prime example is provided by the causal input-output structure for systems under behavioral equivalence. The presence of an input-output structure means in pencil representations that there is a subdivision of the $(m+p)\times(n+m)$ matrix H into an $m\times(n+m)$-matrix H_u and a $p\times(n+m)$ matrix H_y. Moreover, causality corresponds to the invertibility of the $(n+m)\times(n+m)$-matrix $[{G \atop H_u}]$. In other words, the causal i/o structure indicates an invertible submatrix in the constant matrix appearing in (5), and an obvious normalization would be to set this matrix equal to the identity. This leads to a realization in the form

$$F = [A \mid B], \quad G = [I \mid 0], \quad H = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \qquad (6)$$

which is of course recognized as the standard state space form. Moreover, one easily verifies that the above normalization reduces the left and right similarity actions on triples (F,G,H) to the standard similarity action on quadruples (A,B,C,D). Moreover, we see that in this way we get a direct connection between row proper polynomial canonical forms and canonical forms for the (A, B, C, D)-tuples that determine a standard state space representation. These connections have of course been studied before (see for instance the extensive discussion in [29, Ch. 6]), but we believe that even in the classical case our approach via the choice of a canonical basis matrix helps to systematize. We conclude this section with two examples.

Example 2 Consider systems with $n = 2$, $p = 1$, and $m = 0$; since there are no inputs we are in the 'autonomous' case. On the behavioral level we may normalize realizations (F, G, H) by setting $G = I$ and in this case it is more common to write A rather than F, and C rather than H. The obvious normalization for the polynomial description is of the form $s^2 - as - b$ and then the canonical basis matrix produces

$$A = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}, \quad C = [1 \mid 0], \tag{7}$$

which is the observer canonical form. The same works of course for any n and so we reproduce a standard canonical form. The same example may also be done on the homogeneous level; we leave this for the reader to work out.

Example 3 We consider the set $\mathcal{H}^2_{2,2}$ of homogeneous systems $P(s,t)$ having input number $m = 2$, output number $p = 2$, and McMillan degree $n = 2$. There are two cases to be considered:

a) Assume that $P(s,t)$ has row indices $\nu_1 = \nu_2 = 1$. Without loss of generality we will assume that $P(s,t)$ has the following canonical form:

$$\begin{pmatrix} s + b_1 t & b_2 t & a_3 s + b_3 t & a_4 s + b_4 t \\ d_1 t & s + d_2 t & c_3 s + d_3 t & c_4 s + d_4 t \end{pmatrix}, \tag{8}$$

where a_i, b_i, c_i, d_i are real numbers. Note that the set of numbers representable in the form (8) forms a dense cell of dimension 12 in the manifold $\mathcal{H}^2_{2,2}$.

The canonical basis matrix $X(s,t)$ is in this case

$$X(s,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using the identity (5) one readily computes for every system representable in the form (8) a canonical first order representation given through:

$$\begin{bmatrix} F \\ \hline G \\ \hline H \end{bmatrix} = \left[\begin{array}{cccc} -b_1 & -b_2 & -b_3 & -b_4 \\ -d_1 & -d_2 & -d_3 & -d_4 \\ \hline 1 & 0 & a_3 & a_4 \\ 0 & 1 & c_3 & c_4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

b) Assume now that $P(s,t)$ has row indices $\nu_1 = 0$ and $\nu_2 = 2$. Without loss of generality we will assume that $P(s,t)$ has the following canonical form:

$$\begin{pmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & s^2 + c_2 st + d_2 t^2 & b_3 s^2 + c_3 st + d_3 t^2 & b_4 s^2 + c_4 st + d_4 t^2 \end{pmatrix} \tag{9}$$

In this case the canonical basis matrix $X(s,t)$ is

$$X(s,t) = \begin{bmatrix} 0 & 0 \\ s & t \end{bmatrix}.$$

Using again the identity (5) one readily computes a canonical first order representation given through:

$$\begin{bmatrix} F \\ \hline G \\ \hline H \end{bmatrix} = \left[\begin{array}{cccc} 1 & -c_2 & -c_3 & -c_4 \\ 0 & -d_2 & -d_3 & -d_4 \\ \hline 0 & 1 & b_3 & b_4 \\ 1 & 0 & 0 & 0 \\ \hline 0 & -a_2 & -a_3 & -a_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

5 Conclusions

The behavioral approach provides a framework in which the structure of all linear systems of a given McMillan degree can be profitably studied. It is crucial to distinguish several levels of equivalence which correspond to different but closely related manifolds. On each level of equivalence, one has both polynomial and first-order representations. A simple realization method was shown which enables one to relate these two types of representations in such a way that a transfer of canonical forms becomes possible, and this was illustrated in an example. Among the many topics for further research, we would like to mention the following. (i) A convenient method for constructing canonical forms for i/o systems under transfer equivalence is to construct a mapping from an object that is already in one-to-one correspondence with the equivalence classes (the transfer function) to a specific element of the equivalence class. One might try to do the same on the behavioral and the homogeneous level, replacing the transfer function by the behavior. (ii) The realization method that we discussed is suitable for transferring canonical forms but unfortunately not for transferring charts, since it is tied intimately to the row indices (Kronecker indices). Other realization methods need to be designed which do allow the transfer of charts.

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