M. A. Kaashoek, J. H. van Schuppen, A. C. M. Ran (eds.)
Realization and Modelling in System Theory
(Proceedings of the International Symposium MTNS-89, Vol. I). Birkhäuser, Boston, 1990.

ANOTHER LOOK AT THE BRAYTON-MOSER NETWORK EQUATIONS

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The problem of reducing a general set of linear algebraic and differential equations to state form has been discussed in system theory by Rosenbrock, Luenberger, J. C. Willems, and others. Recently, the author proposed an algorithm which is able to deal with all kinds of singularities. Of course, the problem of writing state equations is a classical one. Brayton and Moser presented a solution of this problem for LCR networks in 1964. Their method works under a certain parametrization condition, and they show that the resulting state equations have a gradient structure. In this paper, it will be shown that the Brayton-Moser algorithm for linear networks can be obtained in a natural way from the general linear reduction algorithm, with full account of the gradient structure. The notion of 'redundancy type' is introduced in order to classify the various forms of redundancy that can occur in systems of linear algebraic and differential equations, and characteristics are given of the first few redundancy classes for LC networks with ports.

1. Introduction

A mathematical model for a physical system is often obtained by writing constitutive equations and connection constraints. This procedure leads to a system of mixed algebraic and differential equations. Although the option exists of simulating such a mixed system directly, it may be useful for several purposes to rewrite the equations in state form. This problem has been discussed in system theory for instance by Rosenbrock [8, 9], Luenberger [6, 7], and J. C. Willems [13, 14]. Recently, the author has proposed a method to perform the reduction by algorithms gleaned from the 'geometric approach' to linear systems [12].

Of course, the problem of writing state equations for systems described by constitutive equations and element connections is of central importance in electrical and mechanical engineering, and there are well-known methods in these areas to obtain state equations. These methods have been developed with particular applications in mind and so there is often a special structure (Hamiltonian, gradient) which plays a role. In this paper, we shall consider in particular the method proposed by Brayton and Moser [3] (see also [2]) for nonlinear LCR networks. They show that state equations in a gradient form can be given for such networks, and give an explicit form for the equations under a certain 'parametrization condition'. In this paper we shall consider only linear networks; moreover, we shall limit ourselves to networks without resistors (for our purposes here, this is just a matter of technical convenience). The Brayton-Moser theory can be extended to networks with ports (see [10]) and we shall incorporate this feature so that, effectively, we will be discussing LCP networks. It will be shown that the algorithm proposed in [12] allows the definition of a *redundancy type* for systems of linear algebraic and differential equations. The redundancy type is given by a pair of nonnegative integers. For the case of LCP networks, the first nontrivial redundancy class is described exactly by the Brayton-Moser parametrization condition. We shall also discuss the characteristics of larger redundancy classes, and give explicit state equations in gradient form for systems in these classes.

Due to space limitations, no proofs are included in this paper. A more complete account will be given elsewhere.

2. Notation

If X and V are vector spaces (all vector spaces will be over the reals), then i_X will denote the natural embedding of X into the product $X \times V$, and π_X will denote the natural projection from $X \times V$ onto X. This notation will be used even when it is ambiguous, as is the case when we are dealing with a product of more than two vector spaces; it will be clear from the context which embedding or projection is intended. If $T: X \to Y$ is any mapping between vector spaces and Y_0 is a subspace of Y, then, as usual, the set of all $x \in X$ for which $TX \in Y_0$ will be denoted by $T^{-1}Y_0$. For the type of symplectic algebra we need here, in particular the definition and basic properties of Lagrangian subspaces, see [1, §5.3].

3. The 'general linear' algorithm

We shall now briefly describe the algorithm in [12] for reduction of a general system of linear algebraic and differential equations to state form. Actually, this algorithm can be cast into different forms depending on the way in which the system one starts with is written (although there is no essential difference, of course), and we shall present here a derivation that is adapted to the form in which the equations for LCP networks appear. Every system of linear algebraic and differential equations in 'internal variables' $z(t)$ and 'external variables' $w(t)$ can be written in the form

$$G\dot{z} = Fz \tag{3.1}$$

$$\begin{bmatrix} z \\ w \end{bmatrix} \in \mathsf{L} \tag{3.2}$$

where G and F are mappings from the 'internal variable space' Z to an 'equation space' V, G is surjective, L is a subspace of $Z \times W$, and the time argument has been suppressed (as will also always be done below). We consider such a system as defining a set of trajectories of the external variables, and two systems will be considered 'equivalent' if the sets of trajectories defined by them are equal (cf. [13]). A system (3.1-3.2) can be reduced to input/state/output form in three steps, which will now be briefly described.

3.1 First step: elimination of static constrains

A system of the form (3.1-3.2) is said to have *static constraints* if $\pi_Z \mathsf{L}$ is smaller than Z. An equivalent system in which the static constraints have been removed can be constructed in the following way. Define sequences of subspaces of Z and V as follows:

$$Z^1 = \pi_Z \mathsf{L}, \qquad V^1 = GZ^1 \tag{3.3}$$

$$Z^k = F^{-1}V^{k-1} \cap Z^1, \qquad V^k = GZ^k \qquad (k > 1). \tag{3.4}$$

The sequence $(Z^k)_k$ is non-increasing and so it must reach a limit after a finite number of steps; the resulting limit subspace will be denoted by Z^*. We define V^* similarly and, of course, we have $GV^* = Z^*$. By the way that Z^* and V^* are defined, it follows that F maps Z^* into V^* and we can replace the system (3.1-3.2) by one in which all mappings and spaces are restricted to Z^* and V^*. It can be verified that the resulting system is equivalent to the original one, and that it has no static constraints.

3.2 Second step: Removal of redundant integrations

A system of the form (3.1-3.2) is said to have *redundant integrations* if $\ker G$ intersects $i_Z^{-1} \mathcal{L}$ nontrivially; see [12] for an explanation of this terminology. Redundant integrations have to be removed in order to understand the system as being driven by some of the external variables ('inputs'). For this, we have the following procedure. Define a sequence of subspaces of Z as follows:

$$Q^1 = \ker G \cap i_Z^{-1} \mathcal{L} \tag{3.5}$$

$$Q^k = G^{-1} F Q^{k-1} \cap i_Z^{-1} \mathcal{L} \qquad (k > 1). \tag{3.6}$$

This sequence is non-decreasing, and so it will reach a limit after a finite number of steps. The limit subspace will be denoted by Q^*. In the given system (3.1-3.2), we can now replace Z and V respectively by the quotient spaces Z/Q^* and V/FQ^*, while replacing F, G and $\mathcal{L}$ by their induced versions. The resulting system is equivalent to the original one and is free of redundant integrations. Moreover, if the original system had no static constraints, then the system resulting from this reduction step will also have no static constraints.

3.3 Third step: Selection of inputs, outputs, and states

We now suppose we have a system in the form (3.1-3.2) which is free of static constraints and of redundant integrations; so not only G is surjective, but also $\pi_Z \mathcal{L} = Z$ and $\ker G \cap i_Z^{-1} \mathcal{L} = \{0\}$. When dealing with a system in this form, it is convenient to add as a standing assumption that $i_W^{-1} \mathcal{L} = \{0\}$. This means that there are no external variables for which there are no equations at all; these variables would be trivial to describe, of course.

We will now first select inputs and outputs. Define a subspace W^0 of W as follows:

$$W^0 = \{ w \in W \mid \exists z \in \ker G \colon \begin{bmatrix} z \\ w \end{bmatrix} \in \mathcal{L} \}. \tag{3.7}$$

Now, let (U, Y) be a pair of subspaces of W that parametrizes W^0; that is, Y and U are complementary subspaces and for every $u \in U$ there is a unique $y \in Y$ such that $u + y \in W^0$. Note that there are many such pairs: Y may be any complement of W^0, and U may then be any complement of Y.

Next, we shall call a vector space X a *state space* for the system (3.1-3.2) if there exists a surjective mapping $S : Z \to X$ such that $\ker S = \ker G$. (Obvious choices for the state space could be V or the quotient space $Z/(\ker G)$, but, as we shall see in the application to LCP networks, the circumstances in a particular situation may suggest other choices.) It can then be verified that for every $x \in X$ and $u \in U$ there exist a unique $z \in Z$ and $y \in Y$ such that

$$\begin{bmatrix} z \\ y + u \end{bmatrix} \in \mathcal{L}, \quad \text{and} \quad Sz = x. \tag{3.8}$$

By applying F and G to the vector z that is defined in this way, we get two mappings that take the pair (x, u) to vectors in V. Note that Gz does not depend on u since $\ker S = \ker G$. With z and y satisfying (3.8), define E, A, B, C, and D by

$$Gz = Ex \tag{3.9}$$

$$Fz = Ax + Bu \tag{3.10}$$

$$y = Cx + Du. \tag{3.11}$$

It can be verified that the system

$$E\dot{x} = Ax + Bu \tag{3.12}$$

$$y = Cx + Du \tag{3.13}$$

is equivalent to the original system. Moreover, it is easy to check that E is invertible, so that the above equations can be taken to standard input/state/output form if desired.

We see that a reduction to input/state/output form can be achieved by two algorithms, which each are completed in a finite number of steps. We shall say that a system of the form (3.1-3.2) has *redundancy type* (k, j) if the static constraints are removed in k steps (that is, $Z^k = Z^*$) and subsequently the redundant integrations are removed in j steps ($Q^j = Q^*$). We shall also use the term *redundancy class* (k, j) to denote the set of all systems of the form (3.1-3.2) that are of redundancy type (k, j). Note that the definition is such that the redundancy class (k, j) is contained in the redundancy class (k', j') for all $k' \geqslant k$ and $j' \geqslant j$.

4. LCP NETWORKS

For the description of linear LCP networks, we employ the vectors of currents through and voltages across inductors, capacitors, and ports, to be denoted respectively by i_L, i_C, i_P, V_L, V_C, and V_P (this notation will be changed shortly). There is a natural duality between the current and the voltage variables, which allows us to define a symplectic structure on the product space. The element constitutive equations are

$$M_L \frac{\mathrm{d}}{\mathrm{d}t} i_L = V_L \tag{4.1}$$

$$M_C \frac{\mathrm{d}}{\mathrm{d}t} V_C = i_C \tag{4.2}$$

where M_L and M_C are symmetric matrices. In the natural coordinate system M_L and M_C will in fact often be diagonal; however, the condition of symmetry is invariant under canonical coordinate transformations and allows coupled inductors. Kirchhoff's current and voltage laws state that there exists a mapping N from the space of current variables to another vector space such that

$$i \in \ker N \quad \text{and} \quad V \in \operatorname{im} N^{\mathsf{T}}, \tag{4.3}$$

where i denotes the vector with components i_L, i_C, and i_P, and V is the vector with as components the corresponding voltage variables. It should be noted that the product $\ker N \times \operatorname{im} N^{\mathsf{T}}$ is a Lagrangian subspace of the product space of i- and V-variables with its natural symplectic structure.

We shall now rewrite the above equations in a slightly more abstract framework, in order to make the notation somewhat more compact. Let X denote the vector with components i_L and V_C, and let v have components V_L and i_C. Denote by z the vector with components x and v so that z contains all the 'internal' variables, and write w for the vector of the port variables i_P and V_P. The vector spaces corresponding to these variables will be indicated by capital letters, so that for instance $Z = X \times V$. The element constitutive equations can now be summarized as

$$G\dot{z} = Fz, \qquad G = [M \quad 0], \qquad F = [0 \quad I], \tag{4.4}$$

where $M : X \to V$ is a symmetric mapping formed from M_L and M_C. If we write L for the Lagrangian subspace of $Z \times W$ that is specified by Kirchhoff's laws, then (4.3) can be written as

$$\begin{bmatrix} z \\ w \end{bmatrix} \in \mathsf{L}. \tag{4.5}$$

This brings our equations into the form (3.1-3.2). Our goal will be to rewrite the equations in the form (3.12-3.13), with E being invertible and with the following extra requirements which come from the special structure of the network equations:

(i) the spaces X and V are dual;

(ii) U and Y are both Lagrangian subspaces of W, so that there is a natural induced duality between U and Y;

(iii) the mapping $E : X \to V$ is symmetric;

(iv) the mapping

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : X \times U \to V \times Y$$

is also symmetric.

Explicitly, the i/s/o form is given by

$$\dot{x} = E^{-1}Ax + E^{-1}Bu \tag{4.6}$$

$$y = Cx + Du. \tag{4.7}$$

It is easily verified that, under the conditions above, this system is a linear gradient system as defined in [10] with respect to the quadratic form on X given by $[x_1, x_2] = \langle Ex_1, x_2 \rangle$, where $\langle \cdot , \cdot \rangle$ denotes the duality between X and V. Also, the transfer matrix $C(sE - A)^{-1}B + D$ is clearly symmetric.

Actually, one might like to have more than the condition (ii) above: from each pair of port variables, we would like to let one be an input and one be an output. This is possible if the subspace W^0 that was discussed in the previous section is Lagrangian, according to the following proposition.

PROPOSITION 4.1 *Let W be a symplectic space that appears as the product of n two-dimensional symplectic spaces W_i, which in turn are each a product of two one-dimensional spaces W_i^0 and W_i^1 that are dual to eachother:*

$$W = \prod_{i=1}^{i=n} (W_i^0 \times W_i^1). \tag{4.8}$$

Let L be a Lagrangian subspace of W. Then it is possible to choose for every i a $k_i \in \{0, 1\}$ in such a way that L is parametrized by the pair $(\prod_{i=1}^{n} W_i^{k_i}, \prod_{i=1}^{n} W_i^{1-k_i})$.

We shall work under the following two *standing assumptions*:

(i) $\dim W > 0$;

(ii) $i_W^{-1} \mathsf{L} = \{0\}$.

The first assumption says that the network has ports; since we define equivalence of descriptions as equality of port behavior, the problem would be trivial without this assumption. The second standing assumption states that there are no ports that are disconnected from the rest of the network.

5. Redundancy classes of LCP networks

The next theorem summarizes the extent of the first redundancy classes of linear LCP networks.

Theorem 5.1 *Consider the equations (4.4-4.5). Under the assumptions stated above, the following holds.*

(i) *The system (4.4-4.5) is never of redundancy type* $(0, k)$, *for any* $k \geqslant 0$.

(ii) *The system (4.4-4.5) is of redundancy type* $(1, 0)$ *if and only if the following two equivalent conditions hold:*

$$\pi_X L = X \tag{5.1}$$

$$i_V^{-1} L = \{0\}. \tag{5.2}$$

(iii) *The system (4.4-4.5) is of redundancy type* $(1, j)$, *for any* $j \geqslant 0$, *if and only if it is of redundancy type* $(1, 0)$.

(iv) *The system (4.4-4.5) is of redundancy type* $(2, 0)$ *if and only if the following two equivalent conditions hold:*

$$X = \pi_X L \oplus M^{-1} i_V^{-1} L \tag{5.3}$$

$$V = i_V^{-1} L \oplus M \pi_X L. \tag{5.4}$$

(v) *The system (4.4-4.5) is of redundancy type* $(2, 1)$ *if the following four equivalent conditions hold:*

$$X = \pi_X L + M^{-1} i_V^{-1} \pi_Z L \tag{5.5}$$

$$i_V^{-1} L \cap M \pi_X i_Z^{-1} L = \{0\} \tag{5.6}$$

$$\pi_X i_Z^{-1} L \cap M^{-1} i_V^{-1} L = \{0\} \tag{5.7}$$

$$V = i_V^{-1} \pi_Z L + M \pi_X L. \tag{5.8}$$

For generic values of M, the system (4.4-4.5) is of redundancy type $(2, 0)$.

The redundancy class $(1, 0)$ is the one that is found in most textbooks (for instance [5, § 12.4]): the corresponding condition says that the capacitor voltages and inductor currents must form a state for the system. This is also the 'parametrization condition' used by Brayton and Moser [3]. For certain network topologies this condition does not hold, and a method due to Bryant [4] can be used to arrive at state equations. For certain special parameter values, even this method may break down. As we shall see in the next section, the procedure discussed here allows one to give an explicit expression for the state equations in gradient form even in such cases.

6. Setting up state equations

Let us consider the most general situation encountered in the previous section, the one covered by the condition (v) of Thm. 5.1. Application of the algorithms of § 3 leads, after elimination of the static constraints and the redundant integrations, to a description of the form (3.1-3.2) with

$$'Z' = \{ \begin{bmatrix} x \\ v \end{bmatrix} \in \pi_Z L \mid v \in M \pi_X L / (M \pi_X L \cap i_V^{-1} L) \} \tag{6.1}$$

and

$$`V' = M\pi_X L/(M\pi_X L \cap i_V^{-1} L) \tag{6.2}$$

and with induced versions of F, G and L. For instance, the induced version of G is defined by

$$`G': x \mapsto Mx \mod M\pi_X L \cap i_V^{-1} L. \tag{6.3}$$

It can be shown that the subspace W^0 is Lagrangian, so that it is possible to assign inputs and outputs in the way discussed in § 3. Moreover, if we introduce a space 'X' by

$$`X' = \pi_X L/(\pi_X L \cap M^{-1} i_V^{-1} L) \tag{6.4}$$

and a mapping S from 'Z' to 'X' by

$$S: \begin{pmatrix} x \\ v \end{pmatrix} \mapsto x \mod \pi_X L \cap M^{-1} i_V^{-1} L, \tag{6.5}$$

then it can be shown that S satisfies the requirements mentioned in § 3, so that the space defined in (6.4) will serve as a state space. The final steps of the algorithm of § 3 lead to a definition of E as an induced version of M which is still symmetric. The definition of the mappings A, B, C, and D comes down to requiring that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} v \\ y \end{bmatrix} \quad \text{iff} \quad \begin{bmatrix} x \\ v \\ y \\ u \end{bmatrix} \in `L' \tag{6.6}$$

where 'L' is an induced version of L. It follows from [1, Prop. 5.3.10] that 'L' is Lagrangian, and this implies that the "parametrization mapping" appearing in (6.6) is symmetric. So, we have state equations which are in the desired form.

7. An example

Consider a network consisting of one port and two capacitors, all connected in parallel. Let the values of the capacitors be C_1 and C_2. By the procedures given above, it is straightforward to verify that the system of equations that appears when one writes down the element constitutive relations and Kirchhoff's laws is of redundancy type (2, 0) as long as $C_1 + C_2 \neq 0$. If $C_1 + C_2 = 0$ (note that a capacitor with a negative value may be realized using active elements), then the system is of type (2, 1) and we may still write down state equations following the development in the previous section. It turns out that in this case the state space becomes trivial and the relation between the port variables reduces to a static one; the two parallel capacitors with equal but opposite values are equivalent to an open circuit.

8. Conclusions

In this note, it has been shown that the procedure given in [12] for the reduction of a general system of linear algebraic and differential equations to input/state/output form can be applied to the equations that appear in the modeling of LCP networks, and that one is able to reproduce the special structure of the resulting state equations emphasized in [3]. The approach is clearly different from the standard one, which is based on consideration of trees and cotrees in the graph associated with the network. As shown above, the method presented here provides formulas for the state equations even in special cases where Bryant's method fails.

The results we obtained call for further development. It should be no problem to generalize the discussion to RCLP networks, but it will require more work to carry the ideas over to the nonlinear case. (Note that a nonlinear generalization of the algorithms in [12] is already available in [11].) We haven't had room here to discuss the meaning of the 'mixed potential', introduced by Brayton and Moser, in the present framework, nor have we explicitly discussed the role of energy. It should prove worthwile to work out the relation with the traditional approach which is based on an analysis of the graph associated with the network. The extension to non-reciprocal networks also calls for attention; although algorithms on the 'general linear' and the (local) 'general nonlinear' level have already been given in [12] and [11], one would probably like to retain some of the special structure of electrical networks in the state equations, even when the network contains non-reciprocal elements.

Three more subjects for further research are the following. The classification of the redundancy classes of LCP networks as given here is not complete. An open question is whether there exists a *maximal redundancy* for LCP networks in the sense that there is a redundancy class to which every LCP network belongs. More generally, one can ask for a complete description of all different redundancy classes of LCP networks. Moreover, one would like to show that for all these classes the state equations can be given in gradient form. A second question relates to the choice of inputs and outputs. We have shown here that, at least for the redundancy classes we considered, it is possible to select one input and one output from each pair of port variables in such a way that the resulting state equations are in the standard (causal) form. One could ask what remains of the gradient structure if the inputs and output variables are prescribed rather than free to be chosen, so that a causal description may no longer be possible. Finally, there is an intuitive relation between redundancy and approximation which remains to be explored. If a system of algebraic and differential equations is 'close' in a suitable sense to one of a high redundancy type, then one would expect that the given system can be represented to a good degree of approximation by state equations of relatively low order. This rather vague idea should be made more precise, possibly with the help of techniques from singular perturbation theory.

REFERENCES

1. R. A. ABRAHAM, J. E. MARSDEN (1978). *Foundations of mechanics* (2nd ed.), Benjamin/Cummings, Reading, Mass.
2. R. K. BRAYTON (1971). Nonlinear reciprocal networks. *SIAM-AMS Proc.* (Vol. III), AMS, Providence, 1-15.
3. R. K. BRAYTON, J. K. MOSER (1964). A theory of nonlinear networks. *Quart. Appl. Math. 22*, 1-33 (Part I), 81-104 (Part II).
4. P. R. BRYANT (1962). The explicit form of Bashkow's *A* matrix. *IRE Trans. Circuit Th. CT-9*, 303-306.
5. C. A. DESOER, E. S. KUH (1969). *Basic Circuit Theory*, McGraw-Hill, New York.
6. D. G. LUENBERGER (1977). Dynamic equations in descriptor form. *IEEE Trans. Automat. Contr. AC-22*, 312-321.
7. D. G. LUENBERGER (1978). Time-invariant descriptor systems. *Automatica 14*, 473-480.
8. H. H. ROSENBROCK (1970). *State Space and Multivariable Theory*, Wiley, New York.
9. H. H. ROSENBROCK, A. C. PUGH (1974). Contributions to a hierarchical theory of systems. *Int. J. Contr. 19*, 845-867.
10. A. J. VAN DER SCHAFT (1984). *System Theoretic Descriptions of Physical Systems*, CWI Tract 3, CWI, Amsterdam.
11. A. J. VAN DER SCHAFT (1987). On realization of nonlinear systems described by higher-order differential equations. *Math. Syst. Th. 19*, 239-275. (Correction: *Math. Syst. Th. 20* (1987), 305-306.)
12. J. M. SCHUMACHER (1988). Transformations of linear systems under external equivalence. *Lin. Alg. Appl. 102*, 1-34.
13. J. C. WILLEMS (1983). Input-output and state-space representations of finite-dimensional linear time-invariant systems. *Lin. Alg. Appl. 50*, 581-608.
14. J. C. WILLEMS (1986). From time series to linear system. Part I: Finite dimensional linear time invariant systems. *Automatica 22*, 561-580.