Transformations of Linear Systems Under External Equivalence

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ABSTRACT

We consider systems of linear differential and algebraic equations in which some of the variables are distinguished as "external variables." Two systems are called equivalent if the set of solutions for the external variables is the same for both systems. We give an operational form for this definition of equivalence, i.e., we describe a set of system transformations having the property that two systems are equivalent if and only if they can be taken into each other by transformations from that set. Next, an algorithm is described to transform a given system in general form to a system in minimal state-space form. This algorithm differs from existing methods in that it first takes the equations to first-order form, so that each subsequent step can be formulated and interpreted in state-space terms. We also compute the "structure indices" in terms of a state-space description in nonminimal form and use this to prove the minimality of the end result of the algorithm. Finally, an application is shown to the problem of ill-posedness of feedback connections.

1. INTRODUCTION

Two questions that are of fundamental importance in system theory are, firstly, when we call two system descriptions *equivalent* (i.e., when we say that they represent "essentially" the same system) and, secondly, how we can *transform* a system in general form to an equivalent system in a suitable "standard" form. These questions have been discussed since the very beginning of modern system theory. In the early sixties, Kalman [15] and Gilbert [9] criticized the notion of equivalence based on equality of transfer functions, on the grounds that this puts stable and unstable systems into the same equivalence class. Instead, an equivalence relation was stated for systems in

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state-space form, based on similarity transformations in the state space. The notions of controllability and observability were used to describe the relation between the two notions of equivalence, and algorithms were developed to pass from a system description in transfer-matrix form to one in observable and controllable state-space form.

It was soon recognized that it would be desirable to define equivalence of systems for descriptions more general than the state-space form. In 1970, Rosenbrock [26] defined such a concept for systems of higher-order linear differential equations and linear algebraic equations. This notion of equivalence, called "strict system equivalence" by Rosenbrock, was shown to have (after a slight extension; see also [27]) the following properties: (1) every system of the considered type is equivalent to a system in state space form, and (2) two systems in state-space form are equivalent in the sense of Rosenbrock if and only if they are equivalent in the sense of Kalman. The definition of strict system equivalence in [26] is "operational" in nature, i.e., two systems are said to be equivalent if they can be transformed into each other by applying operations of a certain prescribed type (see [26, p. 52]). Later, Wolovich [34] proposed a notion of equivalence in a more "intrinsic" form, based in part on the solution spaces. It was shown by Pernebo in 1977 [24] that the definitions of Wolovich and Rosenbrock give rise to the same equivalence classes.

In all of the abovementioned notions of equivalence (as well as in other versions by Morf [21] and Fuhrmann [7]), the distinction between inputs and outputs is essential. In 1979, J. C. Willems argued that, in many situations, it is neither necessary nor desirable to make such a distinction *a priori*. Rather, one should speak of "external variables," which may or may not be split up into input variables and output variables. From this point of view (which brings us, in fact, closer to the classical theory of differential equations), it is natural to define two systems of equations to be equivalent if the sets of trajectories that they allow for the external variables are the same.

It should be noted that the same notion of equivalence was developed by Blomberg and coworkers as part of a prolonged effort in algebraic system theory during the sixties and seventies (see [3]). However, the definition in [3, p. 92] is restricted to systems in input/output form, so that a certain "regularity" constraint has to be imposed on the describing equations. Although the situation where this condition does not hold is considered also (see p. 89), attention is given almost exclusively to the "regular" case, so that the work by Willems is more general in this respect (while there is also, of course, the difference in viewpoint between "inputs/outputs" and "external variables").

As shown by Willems [32] and by Blomberg and Ylinen [3, p. 173], the notion of "external" equivalence is essentially different from strict system equivalence. When reduced to the category of systems in the usual input/state/output form, systems that differ only in their nonobservable part will be externally equivalent, but not equivalent in the sense of Rosenbrock (or Kalman). On the other hand, not all systems that have the same transfer matrix are externally equivalent, since minimal state-space representations under this equivalence are not necessarily controllable (see [31, 32] and [3, Section 6.4]). Consequently, external equivalence is weaker than transfer equivalence but stronger than strict system equivalence.

The contributions of the present paper are as follows. Firstly, we give an operational form for system equivalence in the sense of Willems; that is, we give a list of operations that take systems to equivalent systems, and show that the list is complete in the sense that if two systems are equivalent, then they can be transformed into each other by operations from the list. Of course, the hard part here is to prove the completeness.

Secondly, we present an algorithm by which one can transform a given system of higher-order linear differential and linear algebraic equations to an equivalent system in minimal state-space form. The novelty about this algorithm is that it avoids polynomial operations. A preliminary step in the algorithm takes the equations to first-order form, by a simple reordering of data that doesn't require numerical processing. All further steps are formulated and can be interpreted in state-space terms, and it turns out that concepts developed in the geometric approach to linear system theory (see, e.g., [36, 19]) are of crucial importance. To the author's best knowledge, all previous algorithms which take a system in general form to state space form rely heavily on polynomial operations. Most authors (for instance [25; 34; 32, p. 596]) start from a reduced form in which no "internal variables" appear ("AR form", in the terminology of [33]); polynomial operations are certainly necessary to arrive at this form (for instance, the Smith canonical form is employed in [32, p. 585]). Rosenbrock starts from the general form, but he uses the Smith form also [26, p. 53]. Wolovich and Guidorzi [35] show how to write down a state-space representation immediately from a certain form which is more general than the "AR" representation; however, to obtain this form, one still needs polynomial operations (reduction to row-column proper form). A disadvantage of the polynomial methods is that they are not easily transported outside the context of linear time-invariant systems. State-space methods have a much better record in this respect, as is evidenced, for instance, by the lively developments that are taking place in the field of nonlinear system theory (see, e.g., [13]). As to the algorithm of the present paper, a generalization of it to the nonlinear case has already been shown in [28]. A second reason to prefer state-space methods could be that they are easier to implement numerically. This is certainly true if one talks about traditional procedures which operate on constant matrices; however, numerical methods for dealing with polynomials are currently in development. The present paper is, in itself, not concerned with issues of numerical robustness.

The presentation of the algorithm is followed by some remarks on invariants under system equivalence. We show how to compute the "structure indices" directly from a state-space description in general (nonminimal) form. As a corollary, we obtain the minimality of the representation that is produced by the algorithm. Finally, it is shown how the theory of the present paper can be used to obtain a better understanding of "ill-posed" feedback connections.

2. OPERATIONAL FORM OF EXTERNAL EQUIVALENCE

We first introduce some notation and terminology, and specify the precise conditions under which we shall work. The set of all infinitely differentiable functions from $\mathbb{R}$ (the time axis) to $\mathbb{R}$ is denoted by $C^\infty(\mathbb{R};\mathbb{R})$. This set is a vector space over $\mathbb{R}$ and the operator of differentiation is a linear mapping of $C^\infty(\mathbb{R};\mathbb{R})$ into itself. Therefore, we can make $C^\infty(\mathbb{R};\mathbb{R})$ into a module over the ring $\mathbb{R}[s]$ of real polynomials in the variable s by the standard definition

$$\left(p_k s^k + \cdots + p_0\right)f = p_k f^{(k)} + \cdots + p_0 f. \tag{2.1}$$

No notational distinction will be made between a polynomial and its associated differential operator. The product $(C^\infty(\mathbb{R};\mathbb{R}))^n$ of n copies of $C^\infty(\mathbb{R};\mathbb{R})$ can be identified in a natural way with $C^\infty(\mathbb{R};\mathbb{R}^n)$, and we shall use the latter notation. In this way, $C^\infty(\mathbb{R};\mathbb{R}^n)$ is also a module over $\mathbb{R}[s]$. A matrix of size $k \times n$ with entries in $\mathbb{R}[s]$ can now be considered as a module homomorphism from $C^\infty(\mathbb{R};\mathbb{R}^n)$ to $C^\infty(\mathbb{R};\mathbb{R}^k)$. The set of all such polynomial matrices is denoted by $\mathbb{R}^{k\times n}[s]$. A square polynomial matrix is said to be *unimodular* if it has a polynomial inverse. It is easily seen that a homomorphism that is represented by a unimodular matrix is, in fact, an isomorphism. The ring $\mathbb{R}[s]$ is a subring of the field of rational functions denoted by $\mathbb{R}(s)$. The rank of a matrix over $\mathbb{R}[s]$ is defined to be equal to its rank over $\mathbb{R}(s)$.

In this paper, we are concerned with submodules of $C^\infty(\mathbb{R};\mathbb{R}^q)$ that are given in the form $Q[\ker P]$ for some $P \in \mathbb{R}^{k\times n}[s]$ and $Q \in \mathbb{R}^{q\times n}[s]$. These submodules are the solution spaces of systems of differential equations of the form

$$P(\mathrm{D})\xi = 0, \tag{2.2}$$

$$Q(\mathrm{D})\xi = w, \tag{2.3}$$

where D denotes derivative, the vector ξ (with n components) represents "internal" variables, and the vector w (with q components) represents "external" variables. (As a rule, internal variables will be denoted by Greek letters and external variables by Latin letters.) A system of the form (2.2)–(2.3) will be abbreviated as $\Sigma(P,Q)$. Two systems $\Sigma(P_1,Q_1)$ and $\Sigma(P_2,Q_2)$ are said to be (externally) *equivalent* [31] if $Q_1[\ker P_1]=Q_2[\ker P_2]$.

We now list a number of transformations under system equivalence.

PROPOSITION 2.1. *Let* $P_1\in\mathbb{R}^{k_1\times n_1}[s]$, $P_2\in\mathbb{R}^{k_2\times n_2}[s]$, $Q_1\in\mathbb{R}^{q\times n_1}[s]$, *and* $Q_2\in\mathbb{R}^{q\times n_2}[s]$. *Under each of the following conditions, the system* $\Sigma(P_1,Q_1)$ *is equivalent to the system* $\Sigma(P_2,Q_2)$.

(1) TC (*addition/deletion of trivially satisfied constraints*):

$$P_2=\begin{pmatrix}P_1\\0\end{pmatrix},\qquad Q_2=Q_1. \tag{2.4}$$

(2) RC (*reformulation of constraints*):

$$P_2=UP_1,\qquad Q_2=Q_1, \tag{2.5}$$

where U is unimodular.

(3) CV (*change of internal variables*):

$$P_2=P_1V,\qquad Q_2=Q_1V, \tag{2.6}$$

where V is unimodular.

(4) IV (*addition/deletion of inactive variables*):

$$P_2=\begin{pmatrix}I&0\\0&P_1\end{pmatrix},\qquad Q_2=\begin{pmatrix}R&Q_1\end{pmatrix}, \tag{2.7}$$

where R is an arbitrary polynomial matrix of compatible size.

(5) IC (*addition/deletion of ineffective constraints*):

$$P_2=\begin{pmatrix}R_1&R_2\\0&P_1\end{pmatrix},\qquad Q_2=\begin{pmatrix}0&Q_1\end{pmatrix}, \tag{2.8}$$

where R_2 is an arbitrary polynomial matrix of compatible size, and R_1 has full row rank.

Proof. All statements are easy to verify. In connection with the transformation IC, it should be noted that a polynomial matrix of full row rank represents a surjective homomorphism. (This is easily proved by reduction to the scalar case via the Smith form; see also [32, Proposition 3.3].) ■

We now proceed to show that this set of transformations under external equivalence is complete in the sense that if two pairs of polynomial matrices (P_1, Q_1) and (P_2, Q_2) give rise to equivalent systems, then these pairs can be transformed into each other by transformations from the above collection. The proof uses two lemmas; the first one shows that every pair (P, Q) can be transformed to a pair (R, I) where R has full row rank and I is the identity matrix, and the second one shows that two pairs of the latter form are equivalent if and only if they are related by a transformation of type RC.

LEMMA 2.2. *Let $P \in \mathbb{R}^{k\times n}[s]$ and $Q \in \mathbb{R}^{q\times n}[s]$. Then there exists a full-row-rank matrix $R \in \mathbb{R}^{l\times q}[s]$ such that $\Sigma(P, Q)$ is equivalent to $\Sigma(R, I)$.*

Proof. We can use IV and CV to do the following transformations:

$$(P,Q) \rightsquigarrow \left(\begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix}, \begin{pmatrix} I & Q \end{pmatrix}\right) \rightsquigarrow \left(\begin{pmatrix} I & -Q \\ 0 & P \end{pmatrix}, \begin{pmatrix} I & 0 \end{pmatrix}\right). \tag{2.9}$$

Now, the matrix $(-Q'\ \ P')'$ can be compressed to full row rank by elementary row operations, i.e., there exists a unimodular matrix U such that

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} -Q \\ P \end{pmatrix} = \begin{pmatrix} Z \\ 0 \end{pmatrix}, \tag{2.10}$$

where Z is of full row rank (see, for instance, [14, p. 375]). So, using transformation RC, we get

$$\left(\begin{pmatrix} I & -Q \\ 0 & P \end{pmatrix}, \begin{pmatrix} I & 0 \end{pmatrix}\right) \rightsquigarrow \left(\begin{pmatrix} U_{11} & Z \\ U_{21} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \end{pmatrix}\right). \tag{2.11}$$

Applications of CV and of IC now lead to the desired form, with U_{21} playing the role of R. ■

The fact that every submodule of the form $Q(\ker P)$ can also be written in the form $\ker R$ for some polynomial matrix R has been proved in [32, Proposition 3.3]. Our proof here has been designed to show explicitly that the transformation can be done by using operations from the list given in Lemma 2.2. In fact, the proof shows that the following result is true, which is interesting enough to state by itself:

COROLLARY 2.3. *Let $P \in \mathbb{R}^{k \times n}[s]$ and $Q \in \mathbb{R}^{q \times n}[s]$, and let U be a unimodular matrix such that*

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} Z \\ 0 \end{pmatrix}, \tag{2.12}$$

where Z is of full row rank. Under these conditions,

$$Q(\ker P) = \ker U_{22}. \tag{2.13}$$

We will also need the following result.

LEMMA 2.4. *Let $P_1 \in \mathbb{R}^{p \times q}[s]$ and $P_2 \in \mathbb{R}^{r \times q}[s]$, and suppose that P_1 is of full row rank. If $\ker P_1 \subset \ker P_2$, then there exists a unique matrix $F \in \mathbb{R}^{r \times p}[s]$ such that $P_2 = FP_1$.*

Proof. Because P_1 has full row rank, it is possible to select p columns from P_1 such that the corresponding $p \times p$ matrix is invertible as a matrix over $\mathbb{R}(s)$. It is no restriction of the generality to assume that $P_1 = (P_{11} \;\; P_{12})$ where P_{11} is invertible over $\mathbb{R}(s)$. Let P_2 be partitioned conformably as $(P_{21} \;\; P_{22})$. Now, note that $\ker P_1 \subset \ker P_2$ implies $\ker P_{11} \subset \ker P_{21}$. (Indeed, suppose that $P_{11}\xi_1 = 0$; then

$$\begin{pmatrix} P_{11} & P_{12} \end{pmatrix} \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} = 0, \tag{2.14}$$

which implies

$$\begin{pmatrix} P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} = 0, \tag{2.15}$$

so that $P_{21}\xi_1 = 0$.)

We claim that it is sufficient to prove that there is a polynomial matrix F satisfying $P_{21} = FP_{11}$. To see this, let $\xi_2 \in C^\infty(\mathbb{R}; \mathbb{R}^{q-p})$. Because P_{11} is nonsingular, there exists $\xi_1 \in C^\infty(\mathbb{R};\mathbb{R}^p)$ such that $P_{11}\xi_1 = -P_{12}\xi_2$, i.e.

$$\begin{pmatrix} P_{11} & P_{12} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0. \tag{2.16}$$

But then also

$$\begin{pmatrix} P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0, \tag{2.17}$$

so that $P_{22}\xi_2 = -P_{21}\xi_1 = -FP_{11}\xi_1 = FP_{12}\xi_2$. It follows that $P_{22} = FP_{12}$, so that, in all, $P_2 = FP_1$.

Proving that P_{21} is a left multiple of P_{11} is the same as showing that P_{11} is a greatest common right divisor of P_{11} and P_{21}. (For the basic facts about gcrd's, see, for instance, [17, p. 35], or [14, pp. 376–380], or [3, Appendix A1]). So, let G be an arbitrary gcrd of P_{11} and P_{21}; then we want to show that there exists a unimodular matrix U such that $P_{11} = UG$. Since G is a right divisor of P_{11}, we already know that there exists a square polynomial matrix V such that $P_{11} = VG$. So all we have to show is that the determinant of V must be a nonzero constant, which will follow if we can prove that $\deg(\det G)$ is equal to $\deg(\det P_{11})$. Note that G must be nonsingular, because it is a square factor of a nonsingular matrix. From this, it follows that $\ker G$ is a finite-dimensional subspace of $C^\infty(\mathbb{R},\mathbb{R}^p)$, and that the dimension of this subspace is, in fact, equal to $\deg(\det G)$ (see, for instance, [4, Theorem 2.3.5.2]). Likewise, $\ker P_{11}$ is a subspace of $C^\infty(\mathbb{R},\mathbb{R}^p)$ of dimension $\deg(\det P_{11})$. We now use the matrix Bézout identity: there exist polynomial matrices X and Y such that $G = XP_{11} + YP_{21}$. Together with the fact that $\ker P_{11} \subset \ker P_{21}$, this implies $\ker P_{11} \subset \ker G$, and so $\deg(\det P_{11}) \leqslant \deg(\det G)$. The reverse inequality is obvious because G divides P_{11}. This completes the proof. ■

The above result has an interesting history. A claim to the effect of the lemma appears already in a 1895 paper by Chrystal [5], but the result is not really proven in that paper. The same claim with the same incomplete argument turns up in the well-known textbook by Ince [12, p. 146]. A closely related statement is presented under the name "Inclusion Lemma" by Lévy et al. in 1977 [16]. Their proof involves manipulation of Laplace transforms and initial conditions. In the book by Blomberg and Ylinen [3], essentially the same statement as the above lemma is given as Theorem 6.2.2. The proof takes several pages. Finally, the statement of the lemma is also given, without proof, by Willems [33, Section 4]. The corresponding statement in the discrete-time case has a straightforward proof if one uses the duality described in [33, 2nd proof of Theorem 5]; see [22]. It seems to be more difficult to construct a similar duality in the continuous-time case.

The following corollary is straightforward.

COROLLARY 2.5. *Let $P_1 \in \mathbb{R}^{p \times q}[s]$ and $P_2 \in \mathbb{R}^{r \times q}[s]$, and suppose that both P_1 and P_2 have full row rank. If $\ker P_1 = \ker P_2$, then $p = r$ and there exists a unimodular matrix U such that $P_2 = UP_1$.*

Proof. By the lemma, there exist polynomial matrices F_1 and F_2 such that $P_2 = F_1 P_1$ and $P_1 = F_2 P_2$. So we have $P_1 = F_2 F_1 P_1$. Because P_1 has full row rank, this implies $F_2 F_1 = I$. Likewise, one has $F_1 F_2 = I$. As a consequence, $p = r$ and F_1 and F_2 are both unimodular. ∎

It is now easy to derive the main result of this section.

THEOREM 2.6. *Let $P_1 \in \mathbb{R}^{k_1 \times n_1}[2]$, $P_2 \in \mathbb{R}^{k_2 \times n_2}[s]$, $Q_1 \in \mathbb{R}^{q \times n_1}[s]$, and $Q_2 \in \mathbb{R}^{q \times n_2}[s]$. The system $\Sigma(P_1, Q_1)$ is equivalent to the system $\Sigma(P_2, Q_2)$ if and only if the pair (P_1, Q_1) can be transformed into the pair (P_2, Q_2) by applying operations from the list given in Proposition 2.1.*

Proof. The "if" part has already been shown in Proposition 2.1. For the "only if" part, suppose that $\Sigma(P_1, Q_1)$ is equivalent to $\Sigma(P_2, Q_2)$. From Lemma 2.2, we know that, using operations from the list of Proposition 2.1, we can transform (P_1, Q_1) into (R_1, I) and (P_2, Q_2) into (R_2, I), where both R_1 and R_2 have full row rank. The equivalence of the systems $\Sigma(R_1, I)$ and $\Sigma(R_2, I)$ implies that $\ker R_1 = \ker R_2$. By Corollary 2.5, there must exist a unimodular matrix U such that $R_2 = UR_1$, i.e., the systems described by (R_1, I) and (R_2, I) are related by a transformation of type RC. Summarizing, operations from the list of Proposition 2.1 allow us to transform (P_1, Q_1) into (R_1, I), then into (R_2, I), and then into (P_2, Q_2). ∎

3. THE ALGORITHM

In this section, we present an algorithm to transform a system in general form

$$P(D)\xi = 0, \tag{3.1}$$

$$Q(D)\xi = w \tag{3.2}$$

to one in minimal state-space form. The algorithm consists of a preliminary step, which takes the system to first-order form by a process which does not involve computation but only a renaming of variables, and three subsequent reduction steps. Such reduction steps are expected to be necessary, since we start from a system in general form. Admittedly, it is possible to obtain a minimal state-space realization immediately from an "AR" description with a row proper matrix, as shown in [33, proof of Theorem 3]; but of course, this requires that the system be already given in a highly developed form. The fact that the representation produced by the algorithm has minimal state-space dimension will be proved in the next section.

So, we start with a system in the general form (3.1)–(3.2). We first take the system to "first-order form," which is the same form as in (3.1)–(3.2) but with the added restrictions that $P(s)$ should be of the form $P_1 s + P_0$, where P_1 and P_0 are constant matrices, and that $Q(s) = Q_0$, a constant matrix.

Preliminary Step

As a temporary abbreviation, define $Z(s) = (P'(s) \;\; Q'(s))'$. Write

$$Z(s) = Z_k s^k + \cdots + Z_1 s + Z_0. \tag{3.3}$$

For $l = k,\ldots,0$, define $Z^l(s)$ by the Horner scheme:

$$Z^k(s) = Z_k, \tag{3.4}$$

$$Z^l(s) = sZ^{l+1}(s) + Z_l \qquad (l = k-1,\ldots,0). \tag{3.5}$$

Note that this implies that $Z^0(s) = Z(s)$. Using operations of the types IV, RC, and CV repeatedly, the following transformations are obtained:

$$Z = Z^0 \rightsquigarrow \begin{pmatrix} I & 0 \\ Z^1 & Z^0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ Z^1 & sZ^1 + Z_0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} I & -s \\ Z^1 & Z_0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} I & 0 & 0 \\ 0 & I & -s \\ Z^2 & sZ^2 + Z_1 & Z_0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} I & -s & 0 \\ 0 & I & -s \\ Z^2 & Z_1 & Z_0 \end{pmatrix} \rightsquigarrow \cdots$$

$$\rightsquigarrow \begin{pmatrix} I & -s & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I & \cdot & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & 0 \\ 0 & & & & 0 & I & -s \\ Z_k & \cdot & \cdot & \cdot & \cdot & Z_1 & Z_0 \end{pmatrix}. \tag{3.6}$$

In other terms, we now have an equivalence $(P,Q) \rightsquigarrow (\hat{P},\hat{Q})$ where

$$\hat{P} = \begin{pmatrix} I & -s & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I & \cdot & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & 0 \\ 0 & & & & 0 & I & -s \\ P_k & \cdot & \cdot & \cdot & \cdot & P_1 & P_0 \end{pmatrix}, \qquad \hat{Q} = \begin{pmatrix} Q_k & \cdots & Q_0 \end{pmatrix}. \tag{3.7}$$

The free parameters in the new matrices are just the coefficients of the polynomial matrices P and Q, so there is no computation involved in this step. Although we have phrased the transformations in the language of Proposition 2.1, the reader will undoubtedly have noticed that what we did was nothing else but the standard trick of replacing higher-order derivatives by new variables.

Step One

After a permutation of columns (transformation CV), our system is now in the form

$$P(s) = \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}, \qquad Q(s) = \begin{pmatrix} H & J \end{pmatrix}. \tag{3.8}$$

(We adopt the convention that all symbols are redefined after a transformation has been completed, to reduce the notational burden.) In what follows, we shall need some definitions and results on weakly unobservable subspaces that we now recall (see [36, 2, 1, 20]). Given a state-space system in standard form $\Sigma(A, B, C, D)$, a subspace V of the state space X is said to be *weakly unobservable* if there exists a feedback matrix F such that $(A + BF)V \subset V$ and $V \subset \ker(C + DF)$. (If a subspace satisfies this property with $F = 0$, then it is called unobservable, whence the terminology.) It is easily seen that the sum of two weakly unobservable subspaces is again weakly unobservable, and that the zero subspace is always weakly unobservable, so that the set of all weakly unobservable subspaces for a given system $\Sigma = \Sigma(A, B, C, D)$ has a maximal element, which is denoted by $V^*(\Sigma)$. This subspace may be computed as the limit of a sequence of subspaces defined recursively by

$$V^0(\Sigma) = X, \tag{3.9}$$

$$V^k(\Sigma) = \{x \in X \mid \exists u \text{ s.t. } Ax + Bu \in V^{k-1}(\Sigma),\ Cx + Du = 0\}$$
$$(k = 1,2,\dots). \tag{3.10}$$

Another way to phrase the definition given above would be to say that a subspace V is weakly unobservable if and only if there exists a feedback matrix F and a decomposition of the state space $X = X_1 \oplus X_2$, in which X_1 equals V, such that the following block matrix representations are obtained:

$$A + BF = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad C + DF = \begin{pmatrix} 0 & C_2 \end{pmatrix}. \tag{3.11}$$

It is easily seen that, if V_1 and V_2 are both weakly unobservable subspaces and V_1 is contained in V_2, then there always exists a feedback matrix F such that $(A + BF)V_i \subset V_i$ and $V_i \subset \ker(C + DF)$ for $i = 1,2$. As a consequence, one obtains the following proposition.

PROPOSITION 3.1. *A weakly unobservable subspace V for a given system $\Sigma(A, B, C, D)$ is maximal (i.e., $V = V^*(\Sigma)$), if and only if in any decomposition of the form* (3.11) *the system $\Sigma_2 = \Sigma(A_{22}, B_2, C_2, D)$ has $V^*(\Sigma_2) = 0$.*

Systems for which the largest weakly unobservable subspace is the zero subspace are sometimes called *strongly observable*. One has the following characterization of this situation (see [10, Theorem 5.1]).

PROPOSITION 3.2. *A system $\Sigma(A, B, C, D)$ with state-space dimension n is strongly observable if and only if*

$$\operatorname{rk}\begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} = n + \operatorname{rk}\begin{pmatrix} -B \\ D \end{pmatrix} \qquad \forall s \in \mathbb{C}. \tag{3.12}$$

These two propositions are useful in the proof of a lemma which will help to take the first step in our algorithm: the elimination of the constraints on the internal variables represented by the C and D matrices in (3.8).

LEMMA 3.3. *Consider a system $\Sigma(A, B, C, D)$. The corresponding system pencil can be written in the form*

$$\begin{pmatrix} sI-A & -B \\ C & D \end{pmatrix} = U(s)\begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & sI-A_{11} & B_1 \end{pmatrix} T, \tag{3.13}$$

where $U(s)$ is unimodular, and T is a constant nonsingular matrix. In fact, we may take A_{11} to be a matrix representation for the restriction of $A+BF$ to $V^*(\Sigma)$ for any F that satisfies $(A+BF)V^* \subset V^*$ and $V^* \subset \ker(C+DF)$, whereas the columns of B_1 form a spanning set for $B[\ker D] \cap V^*$ in V^*

Proof. Note that change of basis in state space and feedback are transformations that correspond to constant nonsingular row and column operations on the system pencil. Using Proposition 3.1, we can therefore write the pencil in the form

$$\begin{pmatrix} sI-A_{11} & -A_{12} & -B_1 \\ 0 & sI-A_{22} & -B_2 \\ 0 & C_2 & D \end{pmatrix}, \tag{3.14}$$

where the 2×2 block in the lower right corner represents a strongly observable system. By constant column operations acting on the rightmost column of the above matrix, the pencil can be rewritten in the form

$$\begin{pmatrix} sI-A_{11} & -A_{12} & -B_{11} & -B_{12} \\ 0 & sI-A_{22} & -B_{21} & 0 \\ 0 & C_2 & D_1 & 0 \end{pmatrix} \tag{3.15}$$

where now $\ker(B_{21}' \ D_1')' = \{0\}$. According to Proposition 3.2, the 2×2 block in the lower middle of the above matrix has full column rank for all $s \in \mathbf{C}$. As is well known (see, for instance, [14, p. 379]), this implies that there exists a unimodular matrix $V(s)$ such that

$$V(s)\begin{pmatrix} sI-A_{22} & -B_{21} \\ C_2 & D_1 \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}. \tag{3.16}$$

So, left multiplication by a suitable unimodular matrix will take our pencil to the form

$$\begin{pmatrix} sI - A_{11} & -A_{12} & -B_{11} & -B_{12} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

Further constant row operations will wipe out $-A_{12}$ and B_{11}. Finally, the form (3.13) is reached by rearrangement of rows and columns. ∎

We are now in a position to proceed with the algorithm. We start from a system description in the form (3.8). The above lemma shows that suitable operations of the types RC and CV will take this to something in the form

$$P(s) = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & sI - A_{11} & B_1 \end{pmatrix}, \qquad Q(s) = \begin{pmatrix} H_1 & H_2 & J_1 \end{pmatrix} \tag{3.18}$$

[note that $Q(s)$ is only affected by the operation of type CV, which corresponds to the matrix T in the lemma, and so it will still be constant after the transformation]. Applying transformations of the types TC and IV, we obtain a system description in the form

$$P(s) = \begin{pmatrix} sI - A & -B \end{pmatrix}, \qquad Q(s) = \begin{pmatrix} H & J \end{pmatrix}. \tag{3.19}$$

We shall call this the *general state-space form*. Although polynomial operations were used to arrive at this form, their only function was to justify a deletion, so that the corresponding computations do not actually have to be carried out. Computationally, the reduction is done on the basis of calculation of the subspace $V^*(\Sigma(A, B, C, D))$, plus some basis transformations.

It is easy to interpret the first step of the algorithm, which has now been completed, if one recalls the interpretation of $V^*(\Sigma)$ as the largest "output-nulling controlled invariant subspace" (see, for instance, [11]). Every solution $\xi(\cdot)$ of the equations

$$\dot{\xi}(t) = A\xi(t) + B\eta(t), \tag{3.20}$$

$$C\xi(t) + D\eta(t) = 0 \tag{3.21}$$

must belong to $V^*(\Sigma)$ for all t, so that we can restrict ourselves to this subspace. Also, the "driving variables" $\eta(t)$ can be restricted to those that do not lead out of this subspace.

Step Two

The purpose of the second step of the algorithm is to transform a system in the general state-space form (3.19) to a description in the same form, but with the special property that the matrix J has full column rank. The significance of this property is the following. A system in general state-space form is described by the equations

$$\dot{\xi}(t) = A\xi(t) + B\eta(t), \tag{3.22}$$

$$w(t) = H\xi(t) + J\eta(t). \tag{3.23}$$

If J is injective, then the driving variables η can be solved from the external variables w, so that the system can be rewritten in a form in which it is driven by (part of) the external variables. This, of course, is absolutely essential if we want to arrive at the standard state-space form, in which the system is indeed driven by the external variables (remember that the external variables are, in the setting of [31], what are outputs *and* inputs in the usual setting).

Again, we need some material from the geometric approach to linear systems; in fact, the concept that will be needed is just the dual of that of a weakly unobservable subspace. Let a standard state-space system $\Sigma = \Sigma(A, B, C, D)$ be given. A subspace T of the state space X is said to be *weakly controllable* if there exists an output injection matrix G such that $(A + GC)T \subset T$ and $\mathrm{im}(B + GD) \subset T$. One way to explain the meaning of this definition is the following. Consider a direct-sum decomposition of the state space $X = X_1 \oplus X_2$ in which $T = X_2$. Correspondingly, we have the system equations

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \tag{3.24}$$

$$\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t), \tag{3.25}$$

$$y(t) = C_1x_1(t) + C_2x_2(t) + Du(t). \tag{3.26}$$

Now, the requirements on T mean that there exists a matrix G_1 such that $A_{12} + G_1C_2 = 0$ and $B_1 + G_1D = 0$. So Equation (3.24) above may be replaced by

$$\dot{x}_1(t) = (A_{11} + G_1C_1)x_1(t) - G_1y(t). \tag{3.27}$$

That is, the x_1-component of the state (i.e., the "system modulo T^*") is driven by the output y.

It is easy to see that the intersection of two weakly controllable subspaces is again weakly controllable, and that the full state space X is always weakly controllable. It follows that there exists, for any given standard state-space system $\Sigma = \Sigma(A, B, C, D)$, a unique minimal weakly controllable subspace, which will be denoted by $T^*(\Sigma)$. By the interpretation given above, one could say that "the part of the system that depends causally on the outputs is the system modulo T^*." Consequently, the purpose of the second step of the algorithm will be to "divide out" $T^*(\Sigma(A, B, H, J))$. We proceed as follows.

Our starting point is the system in general state-space form (3.19). The state space X can be decomposed as $X = X_1 \oplus X_2$ with $X_2 = T$. After a corresponding change of basis (constant transformations RC and CV), the system description can be given in the form

$$P(s) = \begin{pmatrix} sI - A_{11} & -A_{12} & -B_1 \\ -A_{21} & sI - A_{22} & -B_2 \end{pmatrix}, \tag{3.28}$$

$$Q(s) = \begin{pmatrix} H_1 & H_2 & J \end{pmatrix}. \tag{3.29}$$

Transformations of the types IV and CV will take this to the form

$$P(s) = \begin{pmatrix} sI - A_{11} & -A_{12} & -B_1 & 0 \\ -A_{21} & sI - A_{22} & -B_2 & 0 \\ H_1 & H_2 & J & I \end{pmatrix}, \tag{3.30}$$

$$Q(s) = \begin{pmatrix} 0 & 0 & 0 & -I \end{pmatrix}, \tag{3.31}$$

and for some time we shall only be concerned with transformations that do not affect $Q(s)$. We know that there exists G_1 such that $A_{12} + G_1 H_2 = 0$ and $B_1 + G_1 J = 0$. A corresponding row transformation (RC) will transform $P(s)$ to

$$\begin{pmatrix} sI - A_{11} & 0 & 0 & -G_1 \\ -A_{21} & sI - A_{22} & -B_2 & 0 \\ H_1 & H_2 & J & I \end{pmatrix}. \tag{3.32}$$

The 2×2 block in the lower middle represents a system for which T^* is the whole state space X_2 (dual of Proposition 3.1). Let S be a nonsingular matrix such that

$$\begin{pmatrix} S^1 \\ S^2 \end{pmatrix} \begin{pmatrix} H_2 & J \end{pmatrix} = \begin{pmatrix} H_2^1 & J^1 \\ 0 & 0 \end{pmatrix}, \tag{3.33}$$

where $(H_2^1\ \ J^1)$ has full row rank. Using S in a row transformation and following this by row and column permutations, we can take $P(s)$ to the form

$$\begin{pmatrix} sI - A_{22} & -B_2 & -A_{21} & 0 \\ H_2^1 & J^1 & H_1^1 & S^1 \\ 0 & 0 & sI - A_{11} & -G_1 \\ 0 & 0 & H_1^2 & S^2 \end{pmatrix}. \tag{3.34}$$

The 2×2 block in the upper left corner has full row rank for all $s \in \mathbb{C}$ (dual of Proposition 3.2). Therefore, an application of transformation IC will reduce our system to the form

$$P(s) = \begin{pmatrix} sI - A_{11} & -G_1 \\ H_1^2 & S^2 \end{pmatrix}, \tag{3.35}$$

$$Q(s) = (0 \quad -I). \tag{3.36}$$

Now, define a matrix $(T_1\ \ T_2)$ by

$$\begin{pmatrix} S^1 \\ S^2 \end{pmatrix}^{-1} = (T_1 \quad T_2), \tag{3.37}$$

so that $S^2 T_1 = 0$ and $S^2 T_2 = I$. A change of variables defined by this matrix transforms (3.35)–(3.36) into

$$P(s) = \begin{pmatrix} sI - A_{11} & -G_1 T_1 & -G_1 T_2 \\ H_1^2 & 0 & I \end{pmatrix}, \tag{3.38}$$

$$Q(s) = (0 \quad -T_1 \quad -T_2), \tag{3.39}$$

and by a further change of variables this is transformed into

$$P(s) = \begin{pmatrix} sI - A_{11} + G_1 T_2 H_1^2 & -G_1 T_1 & -G_1 T_2 \\ 0 & 0 & I \end{pmatrix}, \tag{3.40}$$

$$Q(s) = (T_2 H_1^2 \quad -T_1 \quad -T_2). \tag{3.41}$$

An application of transformation IV (supported by RC and CV) finally leads to the following form:

$$P(s) = \begin{pmatrix} sI - A_{11} + G_1T_2H_1^2 & -G_1T_1 \end{pmatrix}, \tag{3.42}$$

$$Q(s) = \begin{pmatrix} T_2H_1^2 & -T_1 \end{pmatrix}. \tag{3.43}$$

By definition [see (3.37)], the matrix T_1 has full column rank. We now have a system in the general state-space form (3.19) with the special property that the direct feedthrough matrix J from driving variables to external variables is injective. This implies that there exists a matrix G such that, in the notation of (3.19), $H + GJ = 0$; consequently, $T^*(A, B, H, J) = \{0\}$. We see that we have, indeed, managed to "divide out" the subspace T^*, so that the second step of the algorithm has been completed.

The interpretation of this step is perhaps best shown in a simple example. Consider the system given by

$$P(s) = (s \quad -1), \tag{3.44}$$

$$Q(s) = (1 \quad 0). \tag{3.45}$$

These equations simply say that the first derivative of the external variable w is an arbitrary C^∞ function. When the reduction step that has been described above is applied to this particular system, the result is $P(s) = 1$, $Q(s) = 1$. In other words, this result says that w itself is an arbitrary C^∞ function, which is indeed an alternative description of the original system. So the function of the second step is the removal of integration steps between the driving variables and the external variables. Note that the function space that we work on is of crucial importance here: the same reduction would not be possible if we would use, for instance, the class of continuous functions rather than the C^∞ functions.

If, in (3.19), the matrix J is injective, then it is possible to select a number of rows in J in such a way that the matrix formed from these rows is invertible. The driving variables can then be expressed in terms of the corresponding external variables which we therefore call "inputs," and the system can be written in an input/state/output form. For this reason, we shall call a system of the form (3.19) with J injective a system in *implicit i/s/o form*. This form can be made explicit in the way just described, but the final step may just as well be performed on the implicit form.

Step Three

We start from a system in implicit i/s/o form ([3.19], with J injective). In the final step of the algorithm, we want to remove nonzero trajectories of the internal variables that give rise to zero trajectories of the external variables. Clearly, this operation will be based on the subspace $V^*(A, B, H, J)$. By constant operations of the types CV and RC (effecting change of basis and feedback), the system can be transformed to

$$P(s)=\begin{pmatrix} sI-A_{11} & -A_{12} & -B_1 \\ 0 & sI-A_{22} & -B_2 \end{pmatrix}, \tag{3.46}$$

$$Q(s)=\begin{pmatrix} 0 & H_2 & J \end{pmatrix}, \tag{3.47}$$

where $V^*(A_{22}, B_2, H_2, J)=\{0\}$ (see Proposition 3.1) and A_{11} is the restriction of some mapping $A+BF$ to the subspace $V^*(A, B, H, J)$. An application of operation IC will transform (3.46)–(3.47) into

$$P(s)=\begin{pmatrix} sI-A_{22} & -B_2 \end{pmatrix} \tag{3.48}$$

$$Q(s)=\begin{pmatrix} H_2 & J \end{pmatrix}. \tag{3.49}$$

The end result of the algorithm is a system in the form (3.19), for which both $V^*=\{0\}$ and $T^*=\{0\}$. (For the latter property, note that the matrix J has not been changed in the above reduction, so that it will still be injective.) A corollary of results in the next section (see also [32, Theorem 4.5]) will be that systems of this type have a minimal state-space dimension, i.e., any equivalent system in state-space form will have the same or a larger state-space dimension. Therefore, a system in the general state-space form which satisfies the property $T^*(A, B, H, J)+V^*(A, B, H, J)=\{0\}$ will be said to be in *minimal implicit i/s/o form.*

In the operations of the third step, the property $T^*=\{0\}$ was not used, and this raises the question whether it is possible to interchange the second and the third step. In other words, is the property $V^*=\{0\}$ preserved under the operations of step two? The answer is positive. In order to show this, we need two lemmas that have some independent interest. First, we recall some definitions and results from the theory of singular matrix pencils (see [8, Chapter XII]). Let K and L be constant (real) matrices of the same size. The associated matrix pencil is $sK+L$. Two pencils sK_1+L_1 and sK_2+L_2 are said to be *strictly equivalent* if there exist invertible matrices R and T such that $RK_1T=K_2$ and $RL_1T=L_2$. The *column indices* of the pencil are defined as the degrees of the polynomials in a minimal basis for the nullspace of $sK+L$, taken as a mapping between vector spaces over the field of rational functions. In particular, the pencil has no nonzero column indices if and only if a polynomial $p(s)$ satisfies the equation $(sK + L)p(s) = 0$ only if $(sK + L)p_k = 0$ for all coefficients p_k of $p(s)$. The *finite elementary divisors* of the pencil $sK + L$ are the nontrivial factors (taken to the power with which they appear) in a decomposition into irreducible factors of the polynomials that arise in the Smith form of $sK + L$, considered as a polynomial matrix. In particular, the pencil has no elementary divisors if and only if the rank of the matrix $sK + L$ is constant for all $s \in \mathbb{C}$.

To a state-space system specified by matrices A, B, C, and D, we can associate a "system pencil" as in Lemma 3.3. It is well known that the following sequence of subspaces of the state space X is nonincreasing and converges in a finite number of steps to $V^*(A, B, C, D)$ (see [36, 1]):

$$V^0(A,B,C,D) = X, \tag{3.50}$$

$$V^{j+1}(A,B,C,D) = \{x \mid \exists u \text{ s.t. } Ax + Bu \in V^j(A,B,C,D) \text{ and } Cx + Du = 0\}. \tag{3.51}$$

The following lemma shows how to "lift" this algorithm to the level of pencils; we also identify some invariants under strict equivalence. To alleviate the notation, we shall not write a symbol for the natural imbedding of subspaces of $\mathbb{R}^{n_1}$ into $\mathbb{R}^{n_2}$ $(n_2 > n_1)$.

LEMMA 3.4. *Consider a pencil of $n \times l$ matrices $sK + L$. Define a sequence of subspaces of $\mathbb{R}^n$ by*

$$V^0(K,L) = \operatorname{im} K, \tag{3.52}$$

$$V^{j+1}(K,L) = KL^{-1}V^j(K,L). \tag{3.53}$$

If R and T are invertible matrices, of sizes $n \times n$ and $l \times l$ respectively, then

$$V^j(RKT, RLT) = RV^j(K,L) \tag{3.54}$$

for all j, so that the dimensions of the subspaces $V^j(K, L)$ are invariants under strict equivalence. Moreover, if

$$RKT = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{3.55}$$

and if we write, with a corresponding partitioning,

$$RLT = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.56}$$

then $V^j(A,B,C,D) = RV^j(K,L)$ *for all* j.

Sequences of subspaces similar to the one defined in (3.52)–(3.53) were already used by Dieudonné in a 1946 paper on the Kronecker normal form [6] [see in particular p. 137; note that $g(A_r)$ in the notation of Dieudonné is V^* in the notation used here]. The proof of the lemma is a standard induction argument and is therefore omitted. We now translate the condition "$V^*(A,B,C,D) = \{0\}$" into pencil terms.

LEMMA 3.5. *Let a state space system* $\Sigma(A,B,C,D)$ *be given. The condition* $V^*(A,B,C,D) = \{0\}$ *holds if and only if the associated system pencil has no finite elementary divisors and no nonzero column indices.*

Proof. Let $sK + L$ be a matrix pencil. From the definition, it is clear that the following properties hold (assuming compatibility of dimensions):

$$V^j([0 \quad K],[0 \quad L]) = V^j(K,L), \tag{3.57}$$

$$V^j\left(\begin{bmatrix} 0 \\ K \end{bmatrix},\begin{bmatrix} 0 \\ L \end{bmatrix}\right) = \{0\} \oplus V^j(K,L), \tag{3.58}$$

$$V^j\left(\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix},\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}\right) = V^j(K_1,L_1) \oplus V^j(K_2,L_2). \tag{3.59}$$

This sets the stage for expressing the dimensions of the subspaces $V^j(A,B,C,D)$ in terms of invariants of the associated system pencil, using the canonical form for pencils as derived in [8]. In [8, Chapter XII], it is shown that every pencil is equivalent to a pencil in "quasi-block-diagonal form" [i.e., one that is formed by building blocks of the type appearing in (3.57)–(3.59)] of which the separate blocks each have a canonical form. There are four types of canonical forms, and so it suffices to discuss the behavior of the V^j-algorithm for each of these.

(1) *Blocks corresponding to nonzero column indices.* These blocks are of size $n \times (n+1)$, and their canonical form is $K = (I \;\; 0)$, $L = (0 \;\; I)$. Since K has full row rank, it is clear that for blocks of this type one will have $V^j(K,L) = \mathbb{R}^n$ for all j.

(2) *Blocks corresponding to nonzero row indices.* These blocks are of size $(n+1)\times n$, and their canonical form is $K=(I\ \ 0)'$, $L=(0\ \ I)'$. It is straightforward to compute that, for blocks of this type, $\dim V^j(K,L)=n-j$ for $j\leqslant n$, and $V^j(K,L)=\{0\}$ for $j\geqslant n$.

(3) *Blocks corresponding to finite elementary divisors.* These blocks are square, and they can be brought to a form where $K=I_n$ and L is in any canonical form under similarity (for instance, Jordan form). Since K is surjective, one will have $V^j(K,L)=\mathbb{R}^n$ for all j.

(4) *Blocks corresponding to infinite elementary divisors.* These blocks are also square. In the canonical form, K is zero except for the superdiagonal, where it has ones; and L is the identity matrix. Clearly, $V^j(K,L)=\operatorname{im}K^{j+1}$, and since K is nilpotent, $V^j(K,L)=\{0\}$ for all sufficiently large j.

From the above, and from Lemma 3.4, it is clear that one will have $V^*(A,B,C,D)=\{0\}$ if and only if the canonical form of the associated pencil will have no blocks of the types 1 and 3. But this is what we wanted to prove. ■

REMARK. The proof technique that we have employed is clearly capable of providing explicit expressions for the dimensions of the subspaces $V^k(A,B,C,D)$ in terms of the Kronecker invariants of the associated pencil. By dualization, it is possible to do the same for the subspaces $T^k(A,B,C,D)$, and a combination of the two will also lead to expressions for subspaces such as $V^k\cap T^j$. In this way, one obtains a straightforward method to derive dimensional equalities of the type appearing in [18].

Now, let us come back to our original question: If we start from a state-space system satisfying $V^*=\{0\}$, will the system still have the same property after we have applied the transformations of step two to it? Looking back at these transformations, we see that most of them are quite harmless, consisting of row and column operations and of trivial extensions or deletions which do not affect the property $V^*=\{0\}$. (Note that this property is invariant under feedback.) However, the key step is in the application of transformation IC which takes (3.34), (3.31) to (3.35), (3.36). Stacking the matrices P and Q, we see that the transformation is of the form $M(s)\rightsquigarrow C(s)$, with

$$M(s)=\begin{pmatrix} A(s) & B(s)\\ 0 & C(s)\end{pmatrix}, \tag{3.60}$$

and that it is applied, in step two, in a situation where $A(s)$ has full row rank for all $s\in\mathbf{C}$ [see remark following (3.34)]. Now, whenever $A(s)$ in (3.60) is surjective, one has the equality

$$\operatorname{rk} M(s) = \operatorname{rk} A(s) + \operatorname{rk} C(s), \tag{3.61}$$

and so if $A(s)$ has, in addition, constant rank, then the constancy of the rank of $C(s)$ will follow from the same property for $M(s)$. Furthermore, if a minimal polynomial basis for $\ker M(s)$ consists of constant vectors, then this property is also inherited by $C(s)$. For, let $p(s)$ be a polynomial vector such that $C(s)p(s) = 0$. Because $A(s)$ is of full row rank for all s and hence has a polynomial right inverse, there exists a polynomial $q(s)$ such that $A(s)q(s) + B(s)p(s) = 0$. So

$$M(s)\begin{pmatrix} q(s) \\ p(s) \end{pmatrix} = \begin{pmatrix} A(s) & B(s) \\ 0 & C(s) \end{pmatrix}\begin{pmatrix} q(s) \\ p(s) \end{pmatrix} = 0. \tag{3.62}$$

Therefore, all coefficients of the polynomial $(q(s)' \;\; p(s)')'$ are in the kernel of $M(s)$; but then all coefficients of $p(s)$ are in the kernel of $C(s)$. This means that a minimal polynomial basis for $C(s)$ will consist of constants.

Summarizing, it is now clear from Lemma 3.5 that the property $V^* = \{0\}$ will be preserved under the transformations of step two. Therefore, it is possible to do step three first and follow it by step two, without getting into an iterative loop.

4. COMPUTATION OF THE STRUCTURE INDICES

In this section, we shall derive expressions for the "structure indices," as defined in [33, Section 7], in terms of a general state-space representation. As a corollary, we shall obtain a proof for the minimality of the representation that is produced by the algorithm of the previous section.

So, let us consider a system in general state space-form (3.19). Let the set of trajectories defined by (3.19) (i.e., the submodule $(H\;J)[\ker(sI - A\;B)]$ of $C^\infty(\mathbb{R};\mathbb{R}^q)$) be denoted by $\mathscr{B}$ (for "behavior"). Motivated by the developments in [33], we introduce the following numbers:

$$d_j = \dim\left\{\left(w(0)' \quad \dot{w}(0)' \quad \cdots \quad w^{(j)}(0)'\right)' \middle| w \in \mathscr{B}\right\}. \tag{4.1}$$

Since these numbers are defined directly in terms of the behavior $\mathscr{B}$, it is clear that they are invariants under external equivalence. We are now going to compute the numbers d_j in terms of the matrices A, B, H, and J. One has

$$\begin{aligned} d_0 &= \dim\{H\xi_0 + J\eta_0 \mid \exists \xi, \eta \text{ s.t. } \dot{\xi} = A\xi + B\eta,\ \xi(0) = \xi_0,\ \eta(0) = \eta_0\} \\ &= \dim \operatorname{im}(H \quad J), \end{aligned} \tag{4.2}$$

because, for any ξ_0 and η_0, there exists an $\eta \in C^\infty(\mathbb{R}; \mathbb{R}^m)$ such that $\eta(0) = \eta_0$ [take, for instance, the constant function $\eta(t) = \eta_0$], and the equation $\dot{\xi} = A\xi + B\eta$ will have a corresponding solution in $C^\infty(\mathbb{R}; \mathbb{R}^n)$ that satisfies $\xi(0) = \xi_0$. Since $\dot{w} = H\dot{\xi} + J\dot{\eta} = HA\xi + HJ\eta + J\dot{\eta}$, we get, on the next step,

$$\begin{aligned} d_1 &= \dim\left\{\begin{pmatrix} H\xi_0 + J\eta_0 \\ HA\xi_0 + HJ\eta_0 + J\eta_1 \end{pmatrix} \middle| \exists \xi, \eta \text{ s.t. } \dot{\xi} = A\xi + B\eta,\ \xi(0) = \xi_0,\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. \eta(0) = \eta_0,\ \dot{\eta}(0) = \eta_1 \right\} \\ &= \dim \operatorname{im}\begin{pmatrix} H & J & 0 \\ HA & HB & J \end{pmatrix}, \end{aligned} \tag{4.3}$$

because, for any ξ_0, η_0, and η_1, there exists an $\eta \in C^\infty(\mathbb{R}; \mathbb{R}^m)$ such that $\eta(0) = \eta_0$ and $\dot{\eta}(0) = \eta_1$ [take, for instance, the function $\eta(t) = \eta_0 + \eta_1 t$], and the equation $\dot{\xi} = A\xi + B\eta$ will have a corresponding solution in $C^\infty(\mathbb{R}; \mathbb{R}^n)$ such that $\xi(0) = \xi_0$. Two things are obvious: first, we can go on like this, and second, it is useful to introduce some notation. Write

$$B^k = (A^{k-1}B \quad \cdots \quad AB \quad B), \tag{4.4}$$

$$H^k = \begin{pmatrix} H \\ HA \\ \vdots \\ HA^{k-1} \end{pmatrix}, \tag{4.5}$$

$$J^k = \begin{pmatrix} J & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ HB & J & & & & & \cdot \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & J & 0 \\ HA^{k-2}B & \cdot & \cdot & & \cdot & HB & J \end{pmatrix}. \tag{4.6}$$

We have

$$d_k = \mathrm{rk}\begin{pmatrix} H^k & J^k \end{pmatrix}. \tag{4.7}$$

Also introduce the subspaces

$$V^k = (H^k)^{-1}[\mathrm{im}\, J^k], \tag{4.8}$$

$$T^k = B^k[\ker J^k]. \tag{4.9}$$

It is not difficult to see (cf. [29, p. 356], [23]) that the sequence of subspaces V^k defined here coincides with the one defined in (3.50), (3.51), so that the notation is not ambiguous. Dually, the sequence T^k converges to the subspace T^* that was also discussed in Section 3. Now, note that

$$J^{k+1} = \begin{pmatrix} J^k & 0 \\ H^l B^k & J^l \end{pmatrix}. \tag{4.10}$$

We are going to use this relation in order to obtain a number of dimensional relations. First, a linear mapping ϕ can be defined from $\mathrm{im}\, J^{k+l}$ to $\mathrm{im}\, J^k$ simply by defining the action of ϕ to be the projection on the first coordinate (in a partitioning as in (4.10)). This mapping is clearly surjective, and the dimension of its kernel is

$$\dim\ker\phi = \dim\left\{ J^{k+l}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \middle| J^k\eta_1 = 0 \right\} = \dim\left(H^l B^k[\ker J^k] + \mathrm{im}\, J^l \right). \tag{4.11}$$

So one has the dimensional equality

$$\dim\mathrm{im}\, J^{k+l} = \dim\mathrm{im}\, J^k + \dim(H^l T^k + \mathrm{im}\, J^l). \tag{4.12}$$

Two other equalities can be obtained by letting the mapping $(H^l \; J^l)$ act on the spaces $(T^k + V^l)\oplus\mathbb{R}^{ml}$ and $\mathbb{R}^{n+ml}$ (where m is the number of columns of B and n is the size of A). It should be noted that the kernel of the mapping is the same in each case. One gets

$$\dim\left[(T^k + V^l)\oplus\mathbb{R}^{ml}\right] = \dim\ker\begin{pmatrix} H^l & J^l \end{pmatrix} + \dim(H^l T^k + J^l) \tag{4.13}$$

$$n + ml = \dim\ker\begin{pmatrix} H^l & J^l \end{pmatrix} + \dim\mathrm{im}\begin{pmatrix} H^l & J^l \end{pmatrix}. \tag{4.14}$$

Let us write

$$r_k = \dim \operatorname{im} J^k. \tag{4.15}$$

Then, combining (4.7) with (4.12)–(4.14), one obtains the following relation, which is valid for all k and l:

$$d_l = r_{k+l} - r_k + \operatorname{codim}(T^k + V^l). \tag{4.16}$$

We conclude that the numbers on the right must be invariants under external system equivalence.

A few more manipulations will be needed to arrive at the desired result. The two sequences of subspaces $(V^k)_k$ and $(T^k)_k$ converge in a finite number of steps to limit subspaces which are denoted by V^* and T^*, respectively. Therefore, we get from (4.12)

$$r_{k+1} - r_k = \dim(HT^* + \operatorname{im} J) \overset{\text{def}}{=} m^* \qquad (k \text{ large}). \tag{4.17}$$

Using this in (4.16) with a sufficiently large value of k, we obtain

$$d_l = lm^* + \operatorname{codim}(T^* + V^l). \tag{4.18}$$

This, in turn, leads to

$$d_{l+1} - d_l = m^* \qquad (l \text{ large}). \tag{4.19}$$

It follows that m^* is an invariant under external equivalence. But then it is seen from (4.18) that the numbers $\operatorname{codim}(T^* + V^l)$ are also invariants. We now have enough material to draw the following conclusions.

THEOREM 4.1. *For a system $\Sigma = \Sigma(A, B, H, J)$ in general state-space form* (3.19), *the following statements hold*:

(1) *The minimal state-space dimension in any state-space representation equivalent to Σ is equal to* $\operatorname{codim}[T^*(\Sigma) + V^*(\Sigma)]$.

(2) *The minimal number of driving variables in any state-space representation equivalent to Σ is* $\dim[HT^*(\Sigma) + \operatorname{im} J]$.

(3) *The observability indices in any minimal input/state/output representation equivalent to* Σ *are equal to*

$$\nu_j = \#\left\{ k \,\middle|\, \dim\left(\frac{T^*(\Sigma)+V^k(\Sigma)}{T^*(\Sigma)+V^{k+1}(\Sigma)} \right) \geqslant j \right\}. \tag{4.20}$$

Proof. Let $\hat{\Sigma}$ be any state-space system equivalent to Σ, and let X be the state space of $\hat{\Sigma}$. By what we have seen above,

$$\dim X \geqslant \operatorname{codim}\left[T^*(\hat{\Sigma})+V^*(\hat{\Sigma})\right] = \operatorname{codim}\left[T^*(\Sigma)+V^*(\Sigma)\right]. \tag{4.21}$$

On the other hand, it has been shown in Section 3 that it is possible to find a state-space representation $\hat{\Sigma}$ equivalent to Σ which has $T^*(\hat{\Sigma})+V^*(\hat{\Sigma})=\{0\}$, so that equality can be obtained in (4.21). This proves the first claim.

To show (2), let $\hat{\Sigma}(\hat{A},\hat{B},\hat{H},\hat{J})$ be a state-space representation equivalent to Σ, and suppose that $\hat{\Sigma}$ has $\hat{m}$ driving variables (i.e., $\hat{J}$ is a $q\times\hat{m}$ matrix). We can write

$$\hat{J}^{k+1} = \begin{pmatrix} \hat{J} & 0 \\ \hat{H}^k\hat{J} & \hat{J}^k \end{pmatrix}, \tag{4.22}$$

and the first column block in the partitioned matrix has $\hat{m}$ columns. One has, therefore,

$$\hat{m} \geqslant \operatorname{rk}\hat{J}^{k+1} - \operatorname{rk}\hat{J}^k = \operatorname{rk}J^{k+1} - \operatorname{rk}J^k = \dim\left[HT^*(\Sigma)+\operatorname{im}J\right] \tag{4.23}$$

(where we supposed k to be sufficiently large so that the final equality holds). On the other hand, consider any state-space representation $\hat{\Sigma}$ equivalent to Σ which has $T^*(\hat{\Sigma})=\{0\}$ and $\ker\hat{B}\cap\ker\hat{J}=\{0\}$. (Such a representation is possible by the results of Section 3; the second requirement just means that ineffective driving variables are removed, which can be accomplished by transformations of the types CV and IC.) Taken together, the two requirements imply that $\ker\hat{J}=\{0\}$. So, for such a representation, equality will hold in (4.23).

Finally, the third claim is established immediately by using the fact that the numbers $\operatorname{codim}(T^*+V^k)$ are invariants under external equivalence, so that these may be computed in a minimal input/state/output representation.

In such a representation, it is easily seen that the subspaces V^k coincide with the subspaces $\{x \in X \mid HA^i x = 0,\ i = 0,\dots,k-1\}$, so that (4.20) just expresses the familiar relation between the dimensions of these subspaces and the observability indices. ■

To be completely explicit, let us state the following corollary.

COROLLARY 4.2. *A state-space system $\Sigma(A, B, H, J)$ has minimal state-space dimension under external equivalence if and only if $T^*(\Sigma) + V^*(\Sigma) = \{0\}$. The number of driving variables is minimal if and only if* $\ker J = \{0\}$.

Proof. For the necessity of the condition in the second sentence, compare (4.22) and (4.23). The rest is immediately clear from what has been said above. ■

The statement concerning state-space minimality is also given in [32], but the proof in that paper is not fully detailed. Note, however, that the proof of Theorem 6 in [33] could be taken as an alternative.

5. ILL-POSEDNESS OF FEEDBACK CONNECTIONS

As an application of the material developed in this paper, we shall consider the feedback connection of two systems. Let two systems in input/state/output form be given by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \tag{5.1}$$

$$y_i(t) = C_i x_i(t) + D_i u_i(t) \tag{5.2}$$

($i = 1,2$). When the second system is placed in a feedback loop for the first system, the connected system is described by (5.1), (5.2) and the additional equations (see Figure 1)

$$u_2(t) = y_1(t), \tag{5.3}$$

$$y(t) = y_1(t), \tag{5.4}$$

$$u_1(t) = u(t) + y_2(t). \tag{5.5}$$

In discussions of such connections, the condition $\det(I - D_2 D_1) \neq 0$ is usually imposed as a requisite for "well-posedness" of the connection (see, for instance, [4, p. 144] or [30, p. 100]). Our aim in this section is to analyze the nature of this condition and to see what happens when the condition is not fulfilled.

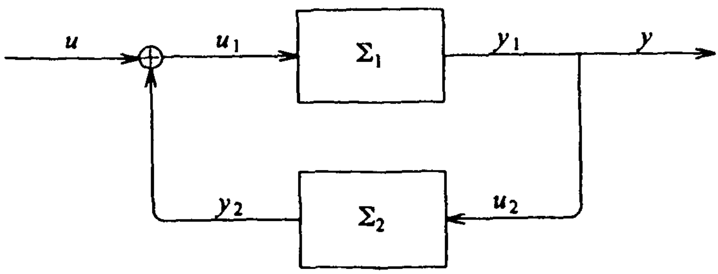

FIG. 1. Feedback connection.

In the notation of (3.1), (3.2), the connected system is described by

$$P(s)=\begin{pmatrix} sI-A_1 & 0 & -B_1 & 0 \\ 0 & sI-A_2 & 0 & -B_2 \\ C_1 & 0 & D_1 & -I \end{pmatrix}, \tag{5.6}$$

$$Q(s)=\begin{pmatrix} C_1 & 0 & D_1 & 0 \\ 0 & -C_2 & I & -D_2 \end{pmatrix}, \tag{5.7}$$

where the external variables are given by $w=(y' \;\; u')'$. We now apply the algorithm of Section 3 to these matrices. The system is already in first-order form, and so we can proceed with step one of the algorithm. In the notation of (3.19), we have

$$A=\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B=\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \tag{5.8}$$

$$C=\begin{pmatrix} C_1 & D_1 \end{pmatrix}, \qquad D=\begin{pmatrix} 0 & -I \end{pmatrix}. \tag{5.9}$$

The matrix D obviously has full row rank, and hence there exists F such that $C+DF=0$. Applying the corresponding transformation as well as the rearrangement of (3.15), we obtain

$$P_1(s)=\begin{pmatrix} sI-A_1 & 0 & -B_1 & 0 \\ -B_2C_1 & sI-A_2 & -B_2D_1 & -B_2 \\ 0 & 0 & 0 & -I \end{pmatrix}, \tag{5.10}$$

$$Q_1(s)=\begin{pmatrix} C_1 & 0 & D_1 & 0 \\ -D_2C_1 & -C_2 & I-D_2D_1 & -D_2 \end{pmatrix}. \tag{5.11}$$

Elimination now leads to a system in state-space form (3.19) with

$$A = \begin{pmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2D_1 \end{pmatrix}, \tag{5.12}$$

$$H = \begin{pmatrix} C_1 & 0 \\ -D_2C_1 & -C_2 \end{pmatrix}, \qquad J = \begin{pmatrix} D_1 \\ I - D_2D_1 \end{pmatrix}. \tag{5.13}$$

This is the result of step one. The matrix J is injective, so step two is redundant in this application, and we could rewrite the system in input/state/output form at this stage. To select the inputs, we have to pick a number of rows from the matrix J such as to form an invertible matrix. We see that *the variable $u(t)$ is an input if and only if the matrix $I - D_2D_1$ is invertible*. So the "well-posedness" condition is a causality condition: It guarantees that one will be able to consider the system as being driven by the "inputs" $u(t)$.

Finally, we apply step three. To do this, we have to compute the subspace $V^*(A, B, H, J)$. One has the following relations, where we use the notation $O(C, A)$ for the unobservable subspace of the pair of output mapping C and state mapping A:

$$\begin{aligned} V^*(A, B, H, J) &= V^*\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 & 0 \\ 0 & -C_2 \end{pmatrix}, \begin{pmatrix} D_1 \\ I \end{pmatrix}\right) \\ &= V^*\left(\begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 & D_1C_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} D_1 \\ I \end{pmatrix}\right) \\ &= O\left(\begin{pmatrix} C_1 & D_1C_2 \end{pmatrix}, \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix}\right). \end{aligned} \tag{5.14}$$

The pair of mappings appearing in (5.14) arises when the two systems

$$\dot{x}_1(t) = A_1x_1(t) + B_1u_1(t), \tag{5.15}$$

$$y_1(t) = C_1x_1(t) + D_1u_1(t) \tag{5.16}$$

and

$$\dot{x}_2(t) = A_2x_2(t), \tag{5.17}$$

$$y_2(t) = C_2x_2(t) \tag{5.18}$$

are connected in series through $u_1(t) = y_2(t)$. An alternative description of $V^*(A, B, H, J)$ is implied by the formula

$$V^k(A, B, H, J) = \ker\begin{pmatrix} C_1^k & D_1^k C_2^k \end{pmatrix}, \tag{5.19}$$

which is easily proven by induction [the definitions of C_1^k and C_2^k are as in (4.5), and of D_1^k as in (4.6)]. It is seen immediately that the unobservable subspaces of the pairs (C_1, A_1) and (C_2, A_2) will both appear in $V^*(A, B, H, J)$, as was to be expected. But even if both pairs are observable, there may still be a nonminimality, due to pole-zero cancellation. States are redundant if they give rise to a zero-input output of the second system in Figure 1 that is at the same time a zero-output input for the first system. One could remove the nonminimality by introducing a new external variable, equal to $y_2(t)$; in fact, this is becoming more and more common as the definition of a feedback connection (with an extra input also added in order to generate the states of the second system, so that the stability of the connected system may be derived from input/output stability; see, for instance, [30, p. 103]). It would take us too far here to analyze the cancellation phenomenon more precisely; we shall be satisfied to draw the following conclusions.

THEOREM 5.1. *Consider two linear systems given by* (5.1), (5.2), *and suppose that these systems are connected through* (5.3)–(5.5) *with* $y(t)$ *and* $u(t)$ *as the new external variables. A state-space representation for the resulting system is given by* (5.12), (5.13), *and this representation is minimal if and only if the pair*

$$\left(\begin{pmatrix} C_1 & D_1C_2 \end{pmatrix}, \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix} \right) \tag{5.20}$$

is observable. The variables $u(t)$ *may be taken as inputs if and only if the matrix* $I - D_2D_1$ *is invertible.*

In other words, our conclusion is that "ill-posed" connections are, from a certain point of view, not seriously ill. In principle, it might have been that the system would be restricted so much by the connection that the only feasible remaining trajectory for the external variables would be the zero trajectory. But such a collapse does not take place; indeed, redundancy in the system is caused only by nonobservability of the subsystems and by pole-zero cancellation, and these phenomena are not related to the issue of ill-posedness.

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