INT. J. CONTROL, 1984, VOL. 39, NO. 4, 779–805

A nine-fold canonical decomposition for linear systems

H. ALING† and J. M. SCHUMACHER‡

The zero structure for non-minimal proper systems in state-space form is investigated. The approach is 'geometric', and a complete characterization in geometric terms is given of the invariant, decoupling, system and transmission zeros, as defined by Rosenbrock. The first main result is a formula for the transmission zeros. Second, a 'canonical' lattice diagram is presented of a decomposition of the state space which can be viewed as the 'product' of the Kalman canonical decomposition and the Morse canonical decomposition. This decomposition gives a straightforward characterization of all zeros just mentioned in terms of spectral properties of subspaces under a certain class of feedback and injection mappings. Via this diagram a number of equivalent formulae for the transmission zeros are derived. The freedom in pole assignment leads to new characterizations for the invariant and system zeros in terms of greatest common divisors of characteristic polynomials. Finally, the relation is demonstrated between certain subspaces and some structural invariants, i.e. the zeros at infinity and the minimal indices of a polynomial basis for the kernel of the transfer function.

1. Introduction

In the past decade there has been a great deal of interest in the zero structure of linear multivariable systems. Many definitions have been proposed, some of which were defined in a state-space context, others in input/output terms (MacFarlane and Karcanias 1976, Francis and Wonham 1975). Most of them are covered by the work of Rosenbrock (1970, 1974), who characterized different kinds of zeros in terms of polynomial system matrices, showed how they were related and what their interpretations were. His definitions are now standard. Of central importance were the Smith form and the Smith–McMillan form, notions which were defined for general polynomial and rational matrices respectively, and which were not only used for definitions, but also as powerful instruments. At approximately the same time the geometric approach was introduced, and soon (Moore and Silverman 1974, Hosoe 1975, Anderson 1975) a geometric interpretation of transmission zeros for strictly proper minimal systems was available. The key was given by the 'Morse canonical' decomposition based on the supremal output-nulling controlled invariant subspace and its dual counterpart. Implicitly, much earlier the same decomposition had been made by Kronecker in 1890 (see Gantmacher 1959), but its system-theoretic meaning was only made clear by Morse (1973) and, independently, by Thorp (1973). Related research is concerned with the pole/zero structure at infinity. Using polynomial system matrices, van der Weiden and Bosgra (1979, 1980) extended and completed Rosenbrock's theory of system zeros.

Received 4 July 1983.

† Delft University of Technology, Department of Mechanical Engineering, Laboratory for Measurement and Control, Mekelweg 2, Delft, the Netherlands.

‡ Erasmus University, Econometric Institute, Burg. Oudlaan 50, Rotterdam, the Netherlands.

Zeros at infinity have also been studied in a different context, i.e. system invertibility, by Silverman (1969). Although the geometric interpretation of zeros at infinity was already implicit in Morse (1973) (see also Morse (1976)), this characterization obtains a more natural interpretation in terms of almost invariant subspaces (Willems 1981)—see Commault and Dion (1982).

In this paper we consider systems of the form

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \text{where } u(t) \in \mathscr{U} \cong R^m \\ & \qquad\quad\; y(t) \in \mathscr{Y} \cong R^p \\ y(t) = Cx(t) + Du(t) & \qquad\quad\; x(t) \in \mathscr{X} \cong R^n \end{cases} \tag{1}$$

Minimality is not assumed. In § 2 we give a summary of the definitions of zeros and the relevant geometric concepts, and we derive a formula for the transmission zeros.

A lattice diagram, which contains all subspaces relevant to the zero structure is presented in § 3, as well as five alternative formulae for the transmission zeros and a block matrix representation. Section 4 contains results on pole assignment and new formulae for the invariant and system zeros in such terms. The orders of the zeros at infinity and the minimal indices of a polynomial basis for the kernel of the transfer function are shown in § 5 to be in one–one relation to the dimensions of certain subspaces, which completes a characterization of all edges in the lattice diagram of § 3.

Notation

Sets and subspaces are denoted by script symbols. The controllable subspace is written $\langle A|\mathscr{B}\rangle = \mathscr{B} + A\mathscr{B} + \ldots + A^{n-1}\mathscr{B}$, where $\mathscr{B} = \operatorname{Im} B$. Dually, the unobservable subspace is $\langle \mathscr{K}|A\rangle = \mathscr{K} \cap A^{-1}\mathscr{K} \cap \ldots \cap A^{1-n}\mathscr{K}$, where $\mathscr{K} = \operatorname{Ker} C$. For a mapping M, $M^{-1}\mathscr{V}$ denotes the set $\{x | Mx \in \mathscr{V}\}$. The spectrum of a matrix A is written $\sigma(A)$. For output-nulling controlled invariant subspaces $\mathscr{V}$ (see § 2) we denote $\mathscr{F}(\mathscr{V}) = \{F | (A+BF)\mathscr{V} \subset \mathscr{V} \subset \operatorname{Ker}(C+DF)\}$ and dually $\mathscr{G}(\mathscr{S}) = \{G | (A+GC)\mathscr{S} \subset \mathscr{S} \supset \operatorname{Im}(B+GD)\}$. The characteristic polynomial of a matrix A is denoted $\chi(A)$. The restriction of a mapping A to an A-invariant subspace $\mathscr{V}$ is written $A|\mathscr{V}$. $\oplus$ is the direct sum for linear subspaces. For a polynomial $p(s)$, deg (p) is its degree. For a subspace $\mathscr{V}$, dim $(\mathscr{V})$ is its dimension. If $\mathscr{V}$ and $\mathscr{W}$ are both invariant for A, and $\mathscr{W} \subset \mathscr{V}$, then the mapping induced by A on the quotient space $\mathscr{V}/\mathscr{W}$ is denoted by $A|(\mathscr{V}/\mathscr{W})$. In case any confusion could arise over which system a subspace $\mathscr{V}$ is defined, $\mathscr{V}$ is denoted by $\mathscr{V}(\Sigma)$, where Σ is the relevant system. For notational convenience sI is abbreviated as s.

2. Transmission zeros

We shall first summarize the definitions of zeros as introduced by Rosenbrock (1970, 1974) and discuss some relationships between various kinds of zeros.

(*a*) *Transmission zeros*: The zeros of the numerator polynomials not equal to zero in the Smith–McMillan form of the transfer function $G(s) = C(s-A)^{-1}B + D$. These zeros have the following dynamical interpretation (Desoer and

Schulman 1974): Assume that $G(s)$ has maximal rank for almost any $s \in C$, and for the system (1), $x(0)=0$, and

(1) $m \leqslant p$: If z is a transmission zero of order k, then there exist polynomial vectors $g(s)$ and $m(s)$ such that the input $\hat{u}_\sigma(s)=g(s)(s-z)^{-(\sigma+1)}+m(s)$ produces an output

$$y(t)=\begin{cases}0 & (\forall t>0,\quad \sigma=0\ \dots\ k-1)\\ a\exp(zt) & (\forall t>0,\quad \sigma=k\ ;\ a\neq 0)\end{cases}$$

(2) $m \geqslant p$: If z is a transmission zero of order k, then there exists a linear combination $\zeta(t)$ of $y(t)$ and its derivatives, such that $\zeta(t)=0$ ($\forall t>0$, $\sigma=0\ \dots\ k-1$) for all inputs of the form $u_\sigma(t)=g(t^\sigma/\sigma!)\exp(zt)+\sum_\alpha m_\alpha \delta^{(\alpha)}(t)$, where g is an arbitrary constant vector and where the m_α are functions of g. For $\sigma=k$, $\zeta(t)$ is proportional to $\exp(zt)$ for all $t>0$ and non-zero for almost all g.

Roughly speaking, (1) says that some input of 'frequency' z is compensated by an initial condition, which the state has been kicked into by a singular input at $t=0$. For the case $m \geqslant p$, (2) says that the same is true for an arbitrary input of 'frequency' z where the output is taken to be the output of a polynomial postcompensator. Note that transmission zeros depend only on the transfer function.

(*b*) *Input decoupling zeros*: The zeros of the GCD of the minors of order n of $[A-s\ B]$ which are not identical to the zero polynomial. They correspond to uncontrollable modes of the system.

(*c*) *Output decoupling zeros*: Analogous to (*b*) with (A, B) replaced by (A^T, C^T). Output decoupling zeros correspond to unobservable modes.

(*d*) *Input/output decoupling zeros*: The output decoupling zeros which disappear when the input decoupling zeros are eliminated by the procedure of Rosenbrock (1970, pp. 60–63).

(*e*) *System zeros*: Consider the system matrix

$$P(s)=\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}$$

and let k be the largest integer such that there is a non-zero minor of order $n+k$ which is formed by rows and columns of P, in such a way that the first n rows and columns are included. The system zeros are defined as the zeros of the GCD of these minors.

(*f*) *Invariant zeros*: The zeros of the non-zero polynomials on the diagonal of the Smith form of $P(s)$.

These definitions can also be applied to a general system matrix

$$\begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}$$

Rosenbrock (1973, 1974) showed that $\{e\}=\{a, b, c\}-\{d\}$ which means that the set of system zeros is the union, counting multiplicities, of the set of transmission zeros and the set of decoupling zeros, and that $\{f\} \subset \{e\}$, which means that every invariant zero is a system zero, counting multiplicity.

Earlier (1970, pp. 60–63) he presented a procedure by which a polynomial system matrix $P(s)$ is reduced to a least-order polynomial system matrix $P_0(s)$ with the same transfer function, in such a way that the minors of $P(s)$ differ from the corresponding minors of $P_0(s)$ by factors $(s-s_i)$, where the s_i are decoupling zeros. Therefore, the invariant zeros of $P(s)$ differ from the invariant zeros of $P_0(s)$ by a set of decoupling zeros. As the invariant zeros of the least-order system matrix $P_0(s)$ are the transmission zeros of $P_0(s)$ (or $P(s)$), it is clear that the set of transmission zeros is contained in the set of invariant zeros, or $\{a\} \subset \{f\}$. These relations are summarized in Fig. 1.

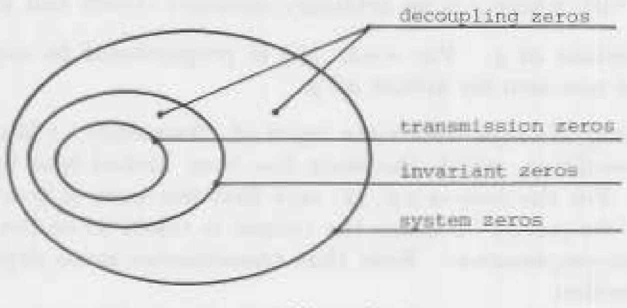

Figure 1.

The definition of input/output decoupling zeros (i.o.d. zeros) may seem somewhat asymmetric with respect to the input decoupling zeros (i.d. zeros) and the output decoupling zeros (o.d. zeros), but the following can be established (Rosenbrock 1970, p. 83)

$$\left.\begin{aligned} \text{i.o.d. zeros} &= \sigma(A \,|\, (\langle \mathscr{K} \,|\, A\rangle + \langle A \,|\, \mathscr{B}\rangle)/\langle A \,|\, \mathscr{B}\rangle) \\ \text{i.d. zeros} &= \sigma(A \,|\, \mathscr{X}/\langle A \,|\, \mathscr{B}\rangle) \\ \text{o.d. zeros} &= \sigma(A \,|\, \langle \mathscr{K} \,|\, A\rangle) \end{aligned}\right\} \tag{2}$$

For more elaborate descriptions of these zeros, see van der Weiden and Bosgra (1979). A general survey has been presented by MacFarlane and Karcanias (1976).

For strictly proper systems, a formula for the transmission zeros is known by the paper of Moore and Silverman (1975). Anderson (1975) generalized the geometric concepts to systems of the form (1). We shall briefly summarize his definitions and main results :

(1) A subspace $\mathscr{V}$ is an output-nulling controlled invariant subspace iff $(\exists F)$ such that $(A+BF)\mathscr{V} \subset \mathscr{V} \subset \operatorname{Ker}(C+DF)$. The set of o.n.c.i. subspaces is non-empty and closed under subspace addition, so that it has a maximal element $\mathscr{V}^*$.

(2) A subspace $\mathscr{R}$ is an output-nulling controllability subspace iff

$$(\forall x_0, x_1 \in R)(\exists u(\cdot), \quad \exists t \geqslant 0)\, x(0)=x_0, \quad x(t)=x_1, \quad y(\cdot) \equiv 0.$$

There is a maximal element, which is given by

$$\mathscr{R}^* = \langle A+BF \mid \mathscr{V}^* \cap B(\text{Ker } D)\rangle \quad \text{with } F \in \mathscr{F}(\mathscr{V}^*) \tag{3}$$

(3) The dual concepts are 'input-containing' subspaces $\mathscr{S}$:

$$(\exists G)(A+GC)\mathscr{S} \subset \mathscr{S} \supset \text{Im } (B+GD),$$

and 'unobservability' subspaces with minimal element

$$\mathscr{N}^* = \langle \mathscr{S}^* + C^{-1}(\text{Im } D) \mid A+GC\rangle \quad \text{with } G \in \mathscr{G}(\mathscr{S}^*) \tag{4}$$

(4) The following relations, which reduce to results of Morse (1973) for $D=0$, hold

$$\mathscr{R}^* = \mathscr{V}^* \cap \mathscr{S}^* \quad \mathscr{N}^* = \mathscr{V}^* + \mathscr{S}^* \tag{5}$$

(5) $\sigma(A+BF|\mathscr{R}^*)$ and $\sigma(A+GC|\mathscr{X}/\mathscr{N}^*)$ can be assigned arbitrarily by $F \in \mathscr{F}(\mathscr{V}^*)$, resp. $G \in \mathscr{G}(\mathscr{S}^*)$; while for these F and G, $\sigma(A+BF|\mathscr{V}^*/\mathscr{R}^*)$ and $\sigma(A+GC|\mathscr{N}^*/\mathscr{S}^*)$ remain fixed.

(6) The transmission zeros are given by $\sigma(A+BF|\mathscr{V}^*/\mathscr{R}^*)$, with $F \in \mathscr{F}(\mathscr{V}^*)$. In proving this, Anderson implicitly assumed minimality, as the transmission zeros were set equal to the invariant zeros. For non-minimal systems pole–zero cancellations will occur in the transfer function, so in this case this formula merely represents the invariant zeros (cf. also van der Weiden and Bosgra (1979)).

Algorithms for the computation of $\mathscr{V}^*$ as a limit of subspaces $\mathscr{V}^i$ are available (Anderson 1975, Molinari 1976, 1978). These subspaces $\mathscr{V}^i$ are defined recursively as follows

$$\mathscr{V}^0 = \mathscr{X}, \quad \mathscr{V}^i = \{x \mid (\exists u \in \mathscr{U}) Ax+Bu \in \mathscr{V}^{i-1}, Cx+Du=0\} \tag{6}$$

For a discrete-time system $\Sigma_d : x_{t+1} = Ax_t + Bu_t$, $y_t = Cx_t + Du_t$, this definition has a natural interpretation

$$\bar{\mathscr{V}}^0 = \mathscr{X}, \quad \bar{\mathscr{V}}^i = \{x \mid (\exists u_0, \ldots, u_{i-1}) y_0 = \ldots = y_{i-1} = 0 \quad \text{for } x_0 = x\} \tag{7}$$

That these definitions are equivalent can be proved by induction as follows.

$$\begin{aligned}
(i=1) \quad \mathscr{V}^1 &= \{x \mid (\exists u \in \mathscr{U}) Ax+Bu \in \mathscr{V}^0 = \mathscr{X}, Cx+Du=0\} \\
&= \{x \mid (\exists u_0 \in \mathscr{U}) y_0 = Cx_0 + Du_0 = 0 \quad \text{for } x_0 = x\} = \bar{\mathscr{V}}^1 \\
(i \to i+1) \quad \mathscr{V}^{i+1} &= \{x \mid (\exists u_0 \in \mathscr{U}) Ax + Bu_0 \in \mathscr{V}^i, Cx+Du_0 = 0\} \\
&= \{x \mid (\exists u_0 \in \mathscr{U}) x_1 \in \mathscr{V}^i = \bar{\mathscr{V}}^i, y_0 = 0 \quad \text{for } x_0 = x\} \\
&= \{x \mid (\exists u_0 \in \mathscr{U})(\exists u_1, \ldots, u_i \in \mathscr{U}) y_1 = \ldots = y_i = 0, \quad y_0 = 0 \quad \text{for } x_0 = x\} \\
&\qquad \text{as } x_1 \in \bar{\mathscr{V}}^i \Leftrightarrow (\exists u_1, \ldots, u_i) y_1 = \ldots = y_i = 0
\end{aligned}$$

So $\mathscr{V}^{i+1} = \bar{\mathscr{V}}^{i+1}$.

Definition (7) was first proposed by Molinari (1976 a), who proved the equivalence with the definition of Anderson. The dual definitions are

$$\mathscr{S}^0 = \{0\}, \quad \mathscr{S}^i = \{x \mid (\exists u \in \mathscr{U})(\exists \bar{x} \in \mathscr{S}^{i-1}) A\bar{x} + Bu = x, \quad C\bar{x} + Du = 0\} \tag{8}$$

$$\bar{\mathscr{S}}^0 = \{0\}, \quad \bar{\mathscr{S}}^i = \{x \mid (\exists u_0, \ldots, u_{i-1}) x_0 = 0, \quad x_i = x, \quad y_0 = \ldots = y_{i-1} = 0\} \tag{9}$$

Note that definitions (6) and (8) are both independent of state feedback and output injection. As a special case the same holds for $\mathscr{V}^*$ and $\mathscr{S}^*$. As a consequence, we may calculate $\mathscr{V}^i$ and $\mathscr{S}^i$ just as well for the system which has been constructed as follows.

If we decompose $\mathscr{U}=\mathscr{U}_1\oplus\mathscr{U}_2$ and $\mathscr{Y}=\mathscr{Y}_1\oplus\mathscr{Y}_2$ with $\mathscr{U}_1=\operatorname{Ker} D$ and $\mathscr{Y}_2=\operatorname{Im} D$, then we can write

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & D_{22} \end{bmatrix} \tag{10}$$

where D_{22} is invertible. Choosing

$$G=[0 \quad -B_2 D_{22}^{-1}] \quad F=\begin{bmatrix} 0 \\ -D_{22}^{-1} C_2 \end{bmatrix}$$

and selecting suitable bases in $\mathscr{U}_2$ and $\mathscr{Y}_2$, we obtain the matrix representation

$$\begin{bmatrix} A+BF+GC+GDF & B+GD \\ C+DF & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & 0 \\ C_1 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{11}$$

where $A_1 = A - B_2 D_{22}^{-1} C_2$.

So if we define $\Sigma=(A, B, C, D)$ and $\tilde{\Sigma}=(A_1, B_1, C_1, 0)$, then $\mathscr{V}^*(\Sigma)=\mathscr{V}^*(\tilde{\Sigma})$ and $\mathscr{S}^*(\Sigma)=\mathscr{S}^*(\tilde{\Sigma})$. In some cases this allows us to use familar results for the strictly proper case, as will be done in § 5.

After these preliminaries we are in a position to derive a formula for the transmission zeros in geometric terms. This will be done by calculating the transmission zeros for a minimal system with the same transfer function, which will now be constructed. As in (1), we have the system $\Sigma=(A, B, C, D)$. Define new systems $\bar{\Sigma}=(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ and $\bar{\bar{\Sigma}}=((\bar{\bar{A}}, \bar{\bar{B}}, \bar{\bar{C}}, \bar{\bar{D}})$ in the following ways

$$\begin{aligned} &\bar{X}=\langle A \mid B\rangle, \quad i:\bar{X}\to X \text{ (natural imbedding)} \\ &\bar{\bar{X}}=\bar{X}/i^{-1}(\langle K \mid A\rangle), \quad \pi:\bar{X}\to\bar{\bar{X}} \text{ (canonical projection)} \end{aligned}$$

and the remaining mappings such that the diagram in Fig. 2 commutes.

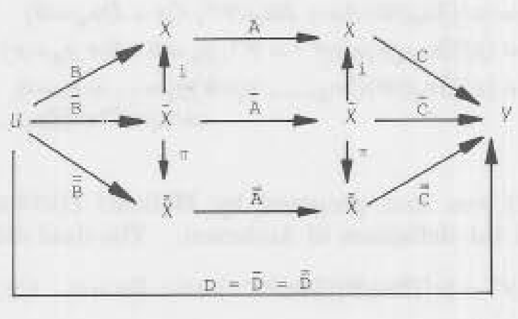

Figure 2.

Note, that these mappings are well-defined, because $\operatorname{Im} i = \langle A|\mathscr{B}\rangle$ is A-invariant, $\operatorname{Ker}\pi = i^{-1}(\langle\mathscr{K}|A\rangle)$ is $\bar{A}$-invariant, $\operatorname{Im} B \subset \operatorname{Im} i$ and $\operatorname{Ker}\pi \subset \operatorname{Ker}\bar{C}$.

The Morse decompositions of $\bar{\Sigma}$ and $\bar{\bar{\Sigma}}$ are related to that of Σ in a very natural way by the following proposition, which is proved in the Appendix.

Proposition 1

$$\mathscr{V}^*(\bar{\bar{\Sigma}}) = \pi i^{-1}\mathscr{V}^*(\Sigma)$$
$$\mathscr{S}^*(\bar{\bar{\Sigma}}) = \pi i^{-1}\mathscr{S}^*(\Sigma)$$
$$\mathscr{R}^*(\bar{\bar{\Sigma}}) = \pi i^{-1}\mathscr{R}^*(\Sigma)$$
$$\mathscr{N}^*(\bar{\bar{\Sigma}}) = \pi i^{-1}\mathscr{N}^*(\Sigma)$$

By the results of Anderson (1975), and because the transfer functions of Σ and $\bar{\bar{\Sigma}}$ are the same, the transmission zeros of the system Σ are given by $\sigma(\bar{\bar{A}} + \bar{\bar{B}}\bar{\bar{F}}\,|\,\mathscr{V}^*(\bar{\bar{\Sigma}})/\mathscr{R}^*(\bar{\bar{\Sigma}}))$, with $\bar{\bar{F}} \in \mathscr{F}(\mathscr{V}^*(\bar{\bar{\Sigma}}))$. Of course we want a formula in terms of Σ. Formula (12) is one of several possibilities, and alternatives will be given later.

Theorem 1

The transmission zeros of the system $\Sigma = (A, B, C, D)$ are given by

$$\sigma(A + BF\,|\,(\mathscr{V}^*(\Sigma)\cap\langle A|\mathscr{B}\rangle)/(\mathscr{R}^*(\Sigma) + (\langle\mathscr{K}|A\rangle\cap\langle A|\mathscr{B}\rangle))) \tag{12}$$

for any $F \in \mathscr{F}(\mathscr{V}^*(\Sigma))$ such that $F|\langle\mathscr{K}|A\rangle = 0$.

Proof of Theorem 1

During the course of the proof we shall state and prove various lemmas, starting with the following.

Lemma 1

Let F be as in (12). Define $\bar{\bar{F}}$ by $\bar{\bar{F}}\pi = Fi$. Then $\bar{\bar{F}} \in \mathscr{F}(\mathscr{V}^*(\bar{\bar{\Sigma}}))$.

Proof

Define $\bar{\mathscr{V}} = i^{-1}\mathscr{V}^*(\Sigma)$ and $\bar{F} = Fi$. As $\langle\mathscr{K}|A\rangle \subset \operatorname{Ker} F$, we have $\operatorname{Ker}\pi \subset \operatorname{Ker}\bar{F}$, so $\bar{\bar{F}}$ is well-defined. Because i is monic, we have

$$\begin{aligned}(\bar{A} + \bar{B}\bar{F})\bar{\mathscr{V}} &= i^{-1}(i(\bar{A} + \bar{B}\bar{F})\bar{\mathscr{V}})\\ &= i^{-1}((A + BF)ii^{-1}(\mathscr{V}^*(\Sigma))) \subset i^{-1}((A + BF)\mathscr{V}^*(\Sigma))\\ &\subset i^{-1}(\mathscr{V}^*(\Sigma)) = \bar{\mathscr{V}},\end{aligned}$$

so

$$(\bar{\bar{A}} + \bar{\bar{B}}\bar{\bar{F}})(\mathscr{V}^*(\bar{\bar{\Sigma}})) = (\bar{\bar{A}} + \bar{\bar{B}}\bar{\bar{F}})\pi\bar{\mathscr{V}} = \pi(\bar{A} + \bar{B}\bar{F})\bar{\mathscr{V}} \subset \pi\bar{\mathscr{V}} = \mathscr{V}^*(\bar{\bar{\Sigma}}).$$

Further,

$$\begin{aligned}(\bar{\bar{C}} + \bar{\bar{D}}\bar{\bar{F}})(\mathscr{V}^*(\bar{\bar{\Sigma}})) &= (\bar{\bar{C}} + \bar{\bar{D}}\bar{\bar{F}})\pi i^{-1}(\mathscr{V}^*(\Sigma)) = (\bar{C} + \bar{D}F)i^{-1}(\mathscr{V}^*(\Sigma))\\ &= (C + DF)ii^{-1}(\mathscr{V}^*(\Sigma)) \subset (C + DF)\mathscr{V}^*(\Sigma) = \{0\},\end{aligned}$$

so $\bar{\bar{F}} \in \mathscr{F}(\mathscr{V}^*(\bar{\bar{\Sigma}}))$. □

Define $A_0 = A + BF$, $\bar{A}_0 = \bar{A} + \bar{B}\bar{F}$ and $\bar{\bar{A}}_0 = \bar{\bar{A}} + \bar{\bar{B}}\bar{\bar{F}}$. We shall show that $A_0|(\mathscr{V}^*(\Sigma)\cap\langle A|\mathscr{B}\rangle)/(\mathscr{R}^*(\Sigma)+(\langle\mathscr{K}|A\rangle\cap\langle A|\mathscr{B}\rangle))$ and $\bar{\bar{A}}_0|\mathscr{V}^*(\bar{\bar{\Sigma}})/\mathscr{R}^*(\bar{\bar{\Sigma}})$ are similar, using Fig. 3. Here $\tilde{i}$ and i' are natural imbeddings, $\tilde{\pi}$ and π' are canonical projections. We shall prove that Fig. 3 commutes, and that T is an isomorphism. Thus the theorem will have been proved, as

$$(\tilde{i})^{-1}(\mathscr{R}^*(\Sigma)+\langle\mathscr{K}|A\rangle) = (\tilde{i})^{-1}(\mathscr{R}^*(\Sigma)+(\langle\mathscr{K}|A\rangle\cap\langle A|\mathscr{B}\rangle))$$

by Lemma 2 below.

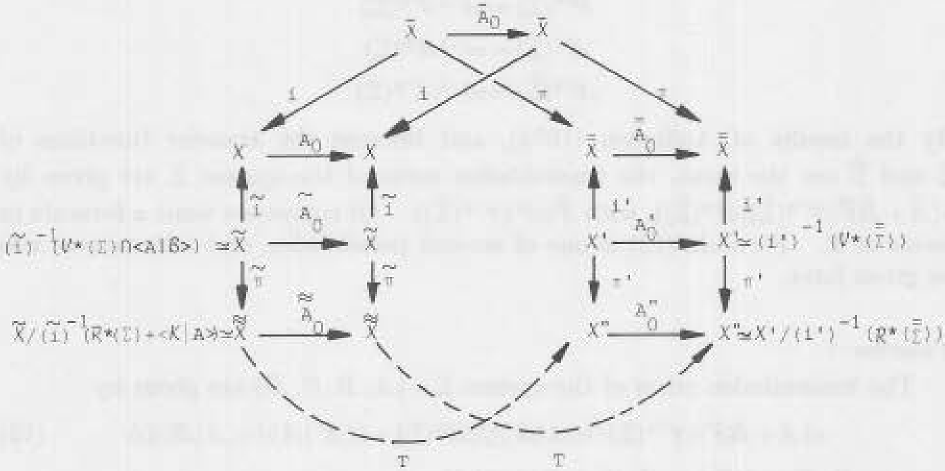

Figure 3.

Lemma 2

Let $\mathscr{V}\subset\mathscr{X}$ and $i:\mathscr{V}\to\mathscr{X}$ the natural injection. Then for $\mathscr{W}\subset\mathscr{X}$, the following hold

$$ii^{-1}\mathscr{W}=\mathscr{W}\cap\mathscr{V},\quad i^{-1}\mathscr{W}=i^{-1}(\mathscr{W}\cap\mathscr{V}).$$

We recall the following rules from linear algebra. The first one is a special case of Lemma A 1 in the Appendix.

Lemma 3

Let $\phi:\mathscr{X}\to\mathscr{Y}$ be a linear mapping, let $\mathscr{V}$, $\mathscr{W}\subset\mathscr{Y}$, and $\mathscr{V}+\mathscr{W}\subset\operatorname{Im}\phi$. Then $\phi^{-1}(\mathscr{V}+\mathscr{W})=\phi^{-1}(\mathscr{V})+\phi^{-1}(\mathscr{W})$.

Lemma 4

Let $\phi:\mathscr{X}\to\mathscr{Y}$ be monic, ϕ^+ such that $\phi^+\phi=1_{\mathscr{X}}$, and let $\mathscr{W}\subset\mathscr{Y}$ such that $\mathscr{W}\subset\operatorname{Im}\phi$. Then $\phi^+\mathscr{W}=\phi^{-1}\mathscr{W}$.
We shall need the following isomorphism.

Lemma 5

$$(\mathscr{V}^*(\Sigma)\cap\langle A|\mathscr{B}\rangle)/(\mathscr{R}^*(\Sigma)+(\langle\mathscr{K}|A\rangle\cap\langle A|\mathscr{B}\rangle))\cong\mathscr{V}^*(\bar{\bar{\Sigma}})/\mathscr{R}^*(\bar{\bar{\Sigma}}).$$

Proof

$$(\mathscr{V}^*(\Sigma)\cap\langle A\,|\,\mathscr{B}\rangle)/(\mathscr{R}^*(\Sigma)+(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle))$$
$$\cong\{(\mathscr{V}^*(\Sigma)\cap\langle A\,|\,\mathscr{B}\rangle)/(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle)\}/\{(\mathscr{R}^*(\Sigma)+(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle))/(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle)\}$$

For the numerator we have

$$(\mathscr{V}^*(\Sigma)\cap\langle A\,|\,\mathscr{B}\rangle)/(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle)=ii^{-1}(\mathscr{V}^*(\Sigma))/ii^{-1}(\langle\mathscr{K}\,|\,A\rangle)$$
$$\cong i^{-1}(\mathscr{V}^*(\Sigma))/i^{-1}(\langle\mathscr{K}\,|\,A\rangle)\cong\pi i^{-1}p\mathscr{V}^*(\Sigma)=\mathscr{V}^*(\bar{\bar{\Sigma}})$$

The denominator is worked out similarly

$$\begin{aligned}&(\mathscr{R}^*(\Sigma)+(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle))/(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle)\\ &=ii^{-1}(\mathscr{R}^*(\Sigma)+\langle\mathscr{K}\,|\,A\rangle)/ii^{-1}(\langle\mathscr{K}\,|\,A\rangle)\\ &\cong i^{-1}(\mathscr{R}^*(\Sigma)+\langle\mathscr{K}\,|\,A\rangle)/i^{-1}(\langle\mathscr{K}\,|\,A\rangle)\\ &\cong\pi(i^{-1}(\mathscr{R}^*(\Sigma))+i^{-1}(\langle\mathscr{K}\,|\,A\rangle))\\ &=\pi i^{-1}(\mathscr{R}^*(\Sigma))=\mathscr{R}^*(\bar{\Sigma})\end{aligned}$$

□

We define the following mappings

$T_1=(\bar{\pi})^+$, where $(\bar{\pi})^+$ is a right inverse of $\bar{\pi}$

$T_2=(i')^+\pi i^+\bar{i}$, where $(i')^+$ and i^+ are left inverses of i' (resp. i),

$T_3=\pi'$, and $T=T_3T_2T_1$

Then T is an isomorphism by Lemma 5 and the following lemma.

Lemma 6

T is epic.

Proof

First,

$$\begin{aligned}&T_2(\bar{i})^{-1}(\mathscr{R}^*(\Sigma)+\langle\mathscr{K}\,|\,A\rangle)\\ &\quad=(i')^+\pi i^+\bar{i}(\bar{i})^{-1}(\mathscr{R}^*(\Sigma)+\langle\mathscr{K}\,|\,A\rangle)\\ &\quad=(i')^+\pi i^+(\mathscr{R}^*(\Sigma)+(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle))\\ &\quad=(i')^+\pi i^+(\mathscr{R}^*(\Sigma))+(i')^+\pi i^+(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle)\\ &\quad=(i')^+\pi i^{-1}(\mathscr{R}^*(\Sigma))+(i')^+\pi i^{-1}(\langle\mathscr{K}\,|\,A\rangle\cap\langle A\,|\,\mathscr{B}\rangle)=(i')^+\mathscr{R}^*(\bar{\bar{\Sigma}})\\ &\quad=(i')^{-1}\mathscr{R}^*(\bar{\Sigma})\end{aligned}$$

so $T_3T_2(\bar{i})^{-1}(\mathscr{R}^*(\Sigma)+\langle\mathscr{K}\,|\,A\rangle)=\{0\}$. But then

$$T_3T_2T_1\bar{\bar{\mathscr{X}}}=T_3T_2(T_1(\bar{\bar{\mathscr{X}}})+(\bar{i})^{-1}(\mathscr{R}^*(\Sigma)+\langle\mathscr{K}\,|\,A\rangle))$$

which, as $\bar{\pi}T_1=1_{\bar{\bar{\mathscr{X}}}}$ and $(\bar{i})^{-1}(\mathscr{R}^*(\Sigma)+\langle\mathscr{K}\,|\,A\rangle)=\operatorname{Ker}\bar{\pi}$, equals

$$\begin{aligned}T_3T_2\bar{\mathscr{X}}&=\pi'(i')^+\pi i^+\bar{i}\bar{\mathscr{X}}\\ &=\pi'(i')^+\pi i^+(\mathscr{V}^*(\Sigma)\cap\langle A\,|\,\mathscr{B}\rangle)=\pi'(i')^+\pi i^{-1}(\mathscr{V}^*(\Sigma)\cap\langle A\,|\,\mathscr{B}\rangle)\\ &=\pi'(i')^+\pi i^{-1}(\mathscr{V}^*(\Sigma))=\pi'(i')^+\mathscr{V}^*(\bar{\bar{\Sigma}})=\pi'(i')^{-1}\mathscr{V}^*(\bar{\Sigma})=\bar{\bar{\mathscr{X}}}\end{aligned}$$

□

Commutativity is proved as follows. Because $\bar{\imath}\bar{\mathscr{X}} \subset i\bar{\mathscr{X}}$, $\bar{\imath}$ and i are monic and the subdiagrams commute, we have $i^+\bar{\imath}\tilde{A}_0 = \bar{A}_0 i^+\bar{\imath}$. Because also $\pi(i^+\bar{\imath}\bar{\mathscr{X}}) \subset i'\mathscr{X}'$ and i' is monic, we have

$$A_0'(i')^+\pi(i^+\bar{\imath}) = (i')^+\pi\bar{A}_0(i^+\bar{\imath}) = (i')^+\pi i^+\bar{\imath}\tilde{A}_0$$

or $A_0'T_2 = T_2\tilde{A}_0$. By definition, $T_3A_0' = A_0''T_3$.

Now for $\tilde{\bar{x}} \in \tilde{\bar{\mathscr{X}}}$ there holds $T_1\tilde{\bar{A}}_0\tilde{\bar{x}} - \tilde{A}_0T_1\tilde{\bar{x}} \in (\bar{\imath})^{-1}(\mathscr{R}^*(\Sigma) + \langle \mathscr{K} | A \rangle)$, since

$$\bar{\pi}T_1\tilde{\bar{A}}_0\tilde{\bar{x}} - \bar{\pi}\tilde{A}_0T_1\tilde{\bar{x}} = \tilde{\bar{A}}_0\tilde{\bar{x}} - \tilde{\bar{A}}_0\bar{\pi}T_1\tilde{\bar{x}} = 0$$

We had already noted that $(\bar{\imath})^{-1}(\mathscr{R}^*(\Sigma) + \langle \mathscr{K} | A \rangle) \subset \operatorname{Ker}(T_3T_2)$. Altogether, we have for $\tilde{\bar{x}} \in \tilde{\bar{\mathscr{X}}}$

$$T_3T_2T_1\tilde{\bar{A}}_0\tilde{\bar{x}} = T_3T_2\tilde{A}_0T_1\tilde{\bar{x}} = T_3A_0'T_2T_1\tilde{\bar{x}} = A_0''T_3T_2T_1\tilde{\bar{x}}$$

So $T\tilde{\bar{A}}_0 = A_0''T$ and $\tilde{\bar{A}}_0$ and A_0'' are similar, which completes the proof. ■

3. Complete characterization in terms of subspaces

In this section we shall combine the Kalman canonical decomposition and the Morse canonical decomposition into one canonical lattice diagram, which gives a complete characterization of the zeros mentioned in § 2 in geometric terms. In order to make $\mathscr{V}^*$ and $\mathscr{S}^*$ invariant and to maintain the invariance of $\langle A | \mathscr{B} \rangle$ and $\langle \mathscr{K} | A \rangle$, we take $F \in \mathscr{F}(\mathscr{V}^*)$ such that $\langle \mathscr{K} | A \rangle \subset \operatorname{Ker} F$ and, dually, $G \in \mathscr{G}(\mathscr{S}^*)$ such that $\operatorname{Im} G \subset \langle A | \mathscr{B} \rangle$. This invariance with respect to $A + BF + GC + GDF$ is then verified as follows

$$\left.\begin{aligned}
&(A+BF+GC+GDF)\mathscr{V}^* = (A+BF)\mathscr{V}^* \subset \mathscr{V}^* \\
&(A+BF+GC+GDF)\mathscr{S}^* \subset (A+GC)\mathscr{S}^* \\
&\qquad\qquad + (B+GD)F\mathscr{S}^* \subset \mathscr{S}^* \\
&(A+BF+GC+GDF)\langle \mathscr{K} | A \rangle = (A+GC)\langle \mathscr{K} | A \rangle \\
&\qquad\qquad = A\langle \mathscr{K} | A \rangle \subset \langle \mathscr{K} | A \rangle \\
&(A+BF+GC+GDF)\langle A | \mathscr{B} \rangle \subset A\langle A | \mathscr{B} \rangle + BF\langle A | \mathscr{B} \rangle \\
&\qquad\qquad + G(C+DF)\langle A | \mathscr{B} \rangle \subset \langle A | \mathscr{B} \rangle
\end{aligned}\right\} \tag{13}$$

Note that $(A+BF+GC+GDF)|\langle \mathscr{K} | A \rangle = A|\langle \mathscr{K} | A \rangle$ so the o.d. zeros are the spectrum of either of these maps. A similar statement holds for the i.d. zeros.

Because of the invariance (13) all sums, intersections etc. of the subspaces $\mathscr{V}^*$ etc. are invariant too, so now we can talk about the spectrum on each of the edges in the lattice diagram of Fig. 4. This lattice diagram contains all information on the zero structure of the system, as will be explained below. In the figure, the acronyms ∞, k.i. and c.i. stand for zeros at infinity, kernel indices and co-range indices of the transfer function, respectively; these will be discussed in § 5. Decoupling zeros which are not invariant zeros have been given the addition 'sys.', decoupling zeros which are invariant zeros but not i.o.d. zeros have been given the addition 'inv.'. Further, 'trm.' stands for transmission zeros.

We summarize the results so far in the following statement, which gives a complete geometric characterization of the zero structure of a proper linear system.

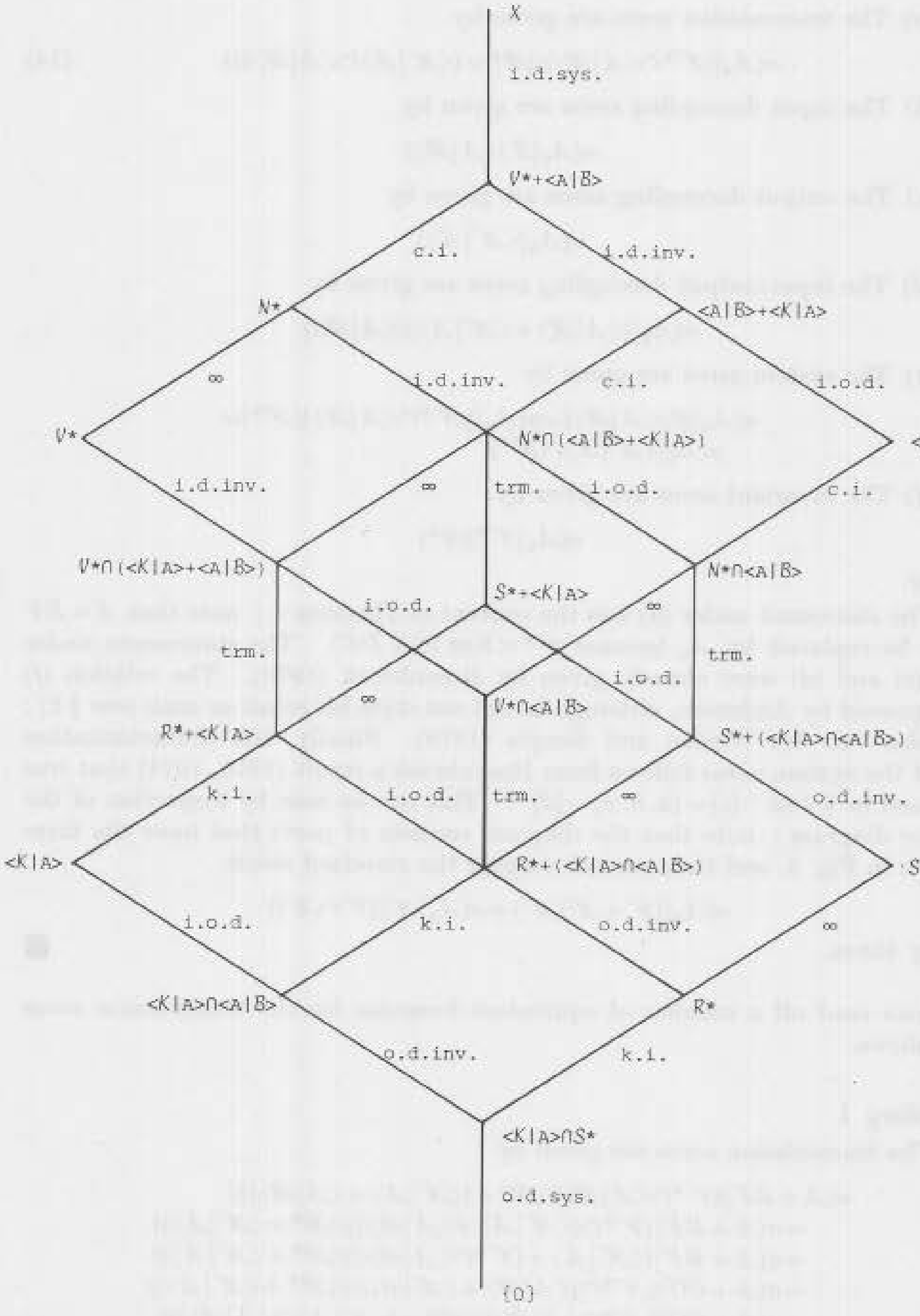

Figure 4.

Theorem 2

Let the system (1) be given. Take any $F \in \mathscr{F}(\mathscr{V}^*)$ such that Ker $F \supset \langle \mathscr{K} | A \rangle$, and $G \in \mathscr{G}(\mathscr{S}^*)$ such that Im $G \subset \langle A | \mathscr{B} \rangle$. Write $A_0 = A + BF + GC + GDF$. The following statements hold with respect to the zeros of the system (A, B, C, D) (cf. § 2):

(*a*) The transmission zeros are given by

$$\sigma(A_0 | (\mathscr{V}^* \cap \langle A | \mathscr{B} \rangle) / (\mathscr{R}^* + (\langle \mathscr{K} | A \rangle \cap \langle A | \mathscr{B} \rangle))) \tag{14}$$

(*b*) The input decoupling zeros are given by

$$\sigma(A_0 | \mathscr{X} / \langle A | \mathscr{B} \rangle)$$

(*c*) The output decoupling zeros are given by

$$\sigma(A_0 | \langle \mathscr{K} | A \rangle)$$

(*d*) The input/output decoupling zeros are given by

$$\sigma(A_0 | (\langle A | \mathscr{B} \rangle + \langle \mathscr{K} | A \rangle) / \langle A | \mathscr{B} \rangle)$$

(*e*) The system zeros are given by

$$\sigma(A_0 | \mathscr{X} / \langle A | \mathscr{B} \rangle) \cup \sigma(A_0 | (\mathscr{N}^* \cap \langle A | \mathscr{B} \rangle) / \mathscr{S}^*) \cup \sigma(A_0 | \langle \mathscr{K} | A \rangle \cap \mathscr{S}^*)$$

(*f*) The invariant zeros are given by

$$\sigma(A_0 | \mathscr{V}^* / \mathscr{R}^*)$$

Proof

The statement under (*a*) was the content of Theorem 1; note that $A + BF$ may be replaced by A_0 because $\mathscr{V}^* \subset \text{Ker}\,(C + DF)$. The statements under (*b*), (*c*) and (*d*) were already given by Rosenbrock (1970). The relation (*f*) was proved by Anderson, although he did not state his result as such (see § 2); cf. also van der Weiden and Bosgra (1979). Finally, the characterization (*e*) of the system zeros follows from Rosenbrock's result (1973, 1974) that was phrased in § 2 as '$\{e\} = \{a, b, c\} - \{d\}$'. This can be seen by inspection of the lattice diagram; note that the diagram consists of parts that have the form shown in Fig. 5, and that one can employ the standard result

$$\sigma(A_0 | (\mathscr{V} + \mathscr{K}) / \mathscr{V}) = \sigma(A_0 | \mathscr{K} / (\mathscr{V} \cap \mathscr{K}))$$

many times. ■

We can read off a number of equivalent formulae for the transmission zeros as follows.

Corollary 1

The transmission zeros are given by

$$\begin{aligned}
&\sigma(A + BF | (\mathscr{V}^* \cap \langle A | \mathscr{B} \rangle) / (\mathscr{R}^* + (\langle \mathscr{K} | A \rangle + \langle A | \mathscr{B} \rangle)))\\
&\quad = \sigma(A + BF | (\mathscr{V}^* \cap (\langle \mathscr{K} | A \rangle + \langle A | \mathscr{B} \rangle)) / (\mathscr{R}^* + \langle \mathscr{K} | A \rangle))\\
&\quad = \sigma(A + BF | (\langle \mathscr{K} | A \rangle + (\mathscr{V}^* \cap \langle A | \mathscr{B} \rangle)) / (\mathscr{R}^* + \langle \mathscr{K} | A \rangle))\\
&\quad = \sigma(A + GC | (\mathscr{N}^* \cap (\langle A | \mathscr{B} \rangle + \langle \mathscr{K} | A \rangle)) / (\mathscr{S}^* + \langle \mathscr{K} | A \rangle))\\
&\quad = \sigma(A + GC | (\mathscr{N}^* \cap \langle A | \mathscr{B} \rangle) / (\mathscr{S}^* + (\langle \mathscr{K} | A \rangle \cap \langle A | \mathscr{B} \rangle)))\\
&\quad = \sigma(A + GC | (\mathscr{N}^* \cap \langle A | \mathscr{B} \rangle) / ((\mathscr{S}^* + \langle \mathscr{K} | A \rangle) \cap \langle A | \mathscr{B} \rangle))
\end{aligned} \tag{15}$$

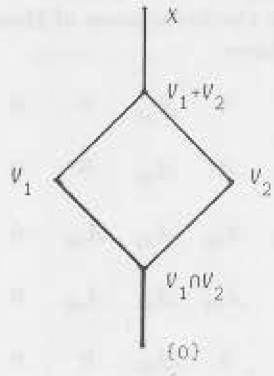

Figure 5.

Proof

This follows from Fig. 4. Note that $A+BF+GC+GDF$ reduces to $A+BF$ 'below' $\mathscr{V}^*$, as $\mathscr{V}^* \subset \operatorname{Ker}(C+DF)$, and, dually, to $A+GC$ 'above' $\mathscr{S}^*$.

Remark

After all, the equations (15) admit a 'direct' interpretation. Knowing that the zeros of the transfer matrix are invariant for output injection and state feedback, we can find the transmission zeros by applying state feedback and/or output injection such that maximal pole/zero cancellation takes place, or, equivalently, the McMillan degree is minimized. The latter means that the system is made as unobservable and as uncontrollable as possible, i.e. that output injection and state feedback are applied to make $\mathscr{V}^*$ and $\mathscr{S}^*$ invariant. By a well-known result in pole placement, these poles are given by $\sigma(A+BF|\mathscr{V}^*/\mathscr{R}^*)$ $(=\sigma(A+GC|\mathscr{N}^*/\mathscr{S}^*))$. After deletion of the decoupling zeros, we obtain (14).

A matrix representation for Fig. 4 is obtained as follows. Let

$$\mathscr{X} = \bigoplus_{i=1}^{9} \mathscr{X}_i$$

such that

$$\langle A|\mathscr{B}\rangle = \mathscr{X}_3 \oplus \mathscr{X}_4 \oplus \mathscr{X}_6 \oplus \mathscr{X}_7 \oplus \mathscr{X}_8 \oplus \mathscr{X}_9$$

$$\langle \mathscr{K}|A\rangle = \mathscr{X}_1 \oplus \mathscr{X}_3 \oplus \mathscr{X}_7$$

$$\mathscr{S}^* = \mathscr{X}_7 \oplus \mathscr{X}_8 \oplus \mathscr{X}_9$$

$$\mathscr{V}^* = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{X}_3 \oplus \mathscr{X}_4 \oplus \mathscr{X}_7 \oplus \mathscr{X}_8$$

Let $\mathscr{U}=\mathscr{U}_1\oplus\mathscr{U}_2$ such that $\mathscr{U}_1=B^{-1}(\mathscr{V}^*)\cap\operatorname{Ker} D$ and let $\mathscr{Y}=\mathscr{Y}_1\oplus\mathscr{Y}_2$ such that $\mathscr{Y}_2=(C\mathscr{S}^*)+\operatorname{Im} D$. By the invariance of these spaces it is easy to check that in compatible bases we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{ccccccccc|cc} A_{11} & A_{12} & 0 & 0 & A_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 & A_{25} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & 0 & A_{38} & A_{39} & 0 & B_{32} \\ 0 & A_{42} & 0 & A_{44} & A_{45} & A_{46} & 0 & A_{48} & A_{49} & 0 & B_{42} \\ 0 & 0 & 0 & 0 & A_{55} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{62} & 0 & A_{64} & A_{65} & A_{66} & 0 & A_{68} & A_{69} & 0 & B_{62} \\ A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} & A_{78} & A_{79} & B_{71} & B_{72} \\ 0 & A_{82} & 0 & A_{84} & A_{85} & A_{86} & 0 & A_{88} & A_{89} & B_{81} & B_{82} \\ 0 & A_{92} & 0 & A_{94} & A_{95} & A_{96} & 0 & A_{98} & A_{99} & 0 & B_{92} \\ \hline 0 & 0 & 0 & 0 & C_{15} & C_{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{22} & 0 & C_{24} & C_{25} & C_{26} & 0 & C_{28} & C_{29} & 0 & D_{22} \end{array}\right]$$

which is a refinement of the matrix representation of van der Weiden and Bosgra (1979), except that now $D\neq 0$. The decoupling zeros can be identified as follows

$$\text{o.d.sys.}\equiv\sigma(A_{77})\quad \text{o.d.inv.}\equiv(A_{33})$$
$$\text{i.o.d.}\equiv\sigma(A_{11})\quad \text{i.d.inv.}\equiv\sigma(A_{22})$$
$$\text{i.d.sys.}\equiv\sigma(A_{55})$$

4. Pole placement

Here we investigate the freedom in pole placement for the class of state feedback and output injection mappings of the preceding sections, which leads to alternative characterizations of invariant and system zeros.

Proposition 2

Consider the set of $F\in\mathscr{F}(\mathscr{V}^*)$ such that $\langle\mathscr{K}\,|\,A\rangle\subset\operatorname{Ker} F$. Then $\sigma(A+BF+GC+GDF\,|\,\mathscr{R}^*/(\mathscr{S}^*\cap\langle\mathscr{K}\,|\,A\rangle))$ and $\sigma(A+BF+GC+GDF\,|\,\mathscr{S}^*/\mathscr{R}^*)$, where $G\in\mathscr{G}(\mathscr{S}^*)$, can be placed arbitrarily and independently by such F.

Proof

Decompose $\mathscr{X}=\bigoplus_{i=1}^{5}\mathscr{X}_i$ with $\mathscr{X}_1=\mathscr{R}^*\cap\langle\mathscr{K}\,|\,A\rangle(=\mathscr{S}^*\cap\langle\mathscr{K}\,|\,A\rangle)$, $\mathscr{X}_1\oplus\mathscr{X}_2=\mathscr{R}^*$, $\mathscr{X}_1\oplus\mathscr{X}_2\oplus\mathscr{X}_3=\mathscr{V}^*$, $\mathscr{X}_1\oplus\mathscr{X}_3\supset\langle\mathscr{K}\,|\,A\rangle$, and $\mathscr{X}_1\oplus\mathscr{X}_2\oplus\mathscr{X}_4=\mathscr{S}^*$. Also, decompose $\mathscr{U}=\mathscr{U}_1\oplus\mathscr{U}_2$ with $\mathscr{U}_1=B^{-1}(\mathscr{V}^*)\cap\operatorname{Ker} D$. Take $F\in\mathscr{F}(\mathscr{V}^*)$ such that $\langle \mathscr{K} | A \rangle \subset \operatorname{Ker} F$, and take $G \in \mathscr{G}(\mathscr{S}^*)$. Write $A_0 = A + BF + GC + GDF$, $B_0 = B + GD$, $C_0 = C + DF$. With respect to the chosen decompositions we have

$$A_0 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} \\ 0 & 0 & A_{33} & 0 & A_{35} \\ 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & A_{55} \end{bmatrix} \quad B_0 = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ 0 & 0 \\ 0 & B_{42} \\ 0 & 0 \end{bmatrix}$$

$$C_0 = [0 \quad 0 \quad 0 \quad C_4 \quad C_5] \quad D = [0 \quad D_2]$$

Because $\langle A_0 | B_0 \mathscr{U}_1 \rangle = \mathscr{R}^*$, it follows that the pair (A_{22}, B_{21}) is controllable. Also, because $\langle A_0 | \operatorname{Im} B_0 \rangle = \mathscr{S}^*$, the pair (A_{44}, B_{42}) is controllable. If we define $\tilde{F}$ by $\tilde{F} = F + F'$, where

$$F' = \begin{bmatrix} 0 & F_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{24} & 0 \end{bmatrix}$$

we obtain

$$\tilde{A}_0 = A + B\tilde{F} + GC + GD\tilde{F} = \begin{bmatrix} A_{11} & A_{12} + B_{11}F_{12} & A_{13} & A_{14} + B_{12}F_{24} & A_{15} \\ 0 & A_{22} + B_{21}F_{12} & A_{23} & A_{24} + B_{22}F_{24} & A_{25} \\ 0 & 0 & A_{33} & 0 & A_{35} \\ 0 & 0 & 0 & A_{44} + B_{42}F_{24} & A_{45} \\ 0 & 0 & 0 & 0 & A_{55} \end{bmatrix}$$

$$\tilde{C}_0 = C + D\tilde{F} = [0 \quad 0 \quad 0 \quad C_4 + D_2 F_{24} \quad C_5]$$

From this representation, it is clear that $\mathscr{V}^* = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{X}_3$ is invariant for $\tilde{A}_0$ and is contained in $\operatorname{Ker} \tilde{C}_0$, so that $\tilde{F} \in \mathscr{F}(\mathscr{V}^*)$. Moreover, we have $\langle \mathscr{K} | A \rangle \subset \mathscr{X}_1 \oplus \mathscr{X}_3 \subset \operatorname{Ker} \tilde{F}$. By the controllability results mentioned above, the eigenvalues of the matrices $A_{22} + B_{21}F_{12}$ and of $A_{44} + B_{42}F_{24}$ can be assigned arbitrarily and independently. Because

$$\sigma(A_{22} + B_{21}F_{12}) = \sigma(\tilde{A}_0 | \mathscr{R}^* / (\mathscr{R}^* \cap \langle \mathscr{K} | A \rangle))$$

and $\sigma(A_{44} + B_{42}F_{24}) = \sigma(\tilde{A}_0 | \mathscr{S}^* / \mathscr{R}^*)$, the proof is complete. □

As usual, the dual case holds also and will not be treated here. If we broaden the set of F by dropping the requirement that $\langle \mathscr{K} | A \rangle \subset \operatorname{Ker} F$, we can prove the following proposition.

Proposition 3

Let $G \in \mathscr{G}(\mathscr{S}^*)$. Then $\sigma(A+BF+GC+GDF|\mathscr{R}^*)$ and $\sigma(A+BF+GC+GDF|\mathscr{S}^*/\mathscr{R}^*)$ can be placed arbitrarily and independently by $F \in \mathscr{F}(\mathscr{V}^*)$.

Proof

The proof is completely analogous to that of Proposition 2, and can in fact be obtained from it by merging the subspaces $\mathscr{X}_1$ and $\mathscr{X}_2$. □

We can now characterize invariant and system zeros by the following theorem.

Theorem 3

The set of system zeros equals the set of zeros of G.C.D.$\{\chi(A+BF+GC+GDF)|F \in \mathscr{F}(\mathscr{V}^*),\ G \in \mathscr{G}(\mathscr{S}^*),\ \langle \mathscr{K}|A\rangle \subset \operatorname{Ker} F,\ \operatorname{Im} G \subset \langle A|\mathscr{B}\rangle\}$. The set of invariant zeros equals the set of zeros of G.C.D.$\{\chi(A+BF+GC+GDF)|F \in \mathscr{F}(\mathscr{V}^*),\ G \in \mathscr{G}(\mathscr{S}^*)\}$.

Proof

For the first case, note that the spectrum of $A+BF+GC+GDF$ restricted to $\langle \mathscr{K}|A\rangle$, $\mathscr{X}/\langle A|\mathscr{B}\rangle$ or $\mathscr{V}^*/\mathscr{R}^*$ is independent of F and G. Figure 4, together with Proposition 4 and its dual, then leads to the conclusion. For the second case, only the restriction to $\mathscr{V}^*/\mathscr{R}^*$ has a fixed spectrum, as is clear from Proposition 3 and its dual. ■

5. Structural invariants

In this section we shall demonstrate the relationships between certain subspaces and some structural invariants for the combined action of state feedback, output injection and input-, output- and state space transformations. First we shall consider the 'zeros at infinity', using the results of Commault and Dion (1982) for the strictly proper case.

Theorem 4

For the system (1) define $\nu_i = \dim(\mathscr{V}^* + \mathscr{S}^i) - \dim(\mathscr{V}^* + \mathscr{S}^{i-1})$ $(i=1, 2, \ldots)$. Let $\rho_i = \#\{j|\nu_j \geqslant i\}$ (see Fig. 6). Then the ρ_i are the orders of the zeros at infinity of the transfer function of the system, given by $C(s-A)^{-1}B+D$.

Proof

For arbitrary F and G we have

$$\begin{aligned} C(s-A)^{-1}B+D = \{I-(C+DF)(s-(A+BF))^{-1}G\}\{D+(C+DF) \\ \times (s-(A+BF+GC+GDF))^{-1}(B+GD)\} \\ \times \{I-F(s-A)^{-1}B\} \quad (16) \end{aligned}$$

This equation is of the form $G(s) = B_1(s)G_1(s)B_2(s)$, where $B_1(s)$ and $B_2(s)$ are bicausal isomorphisms (i.e. proper rational matrices having a proper inverse

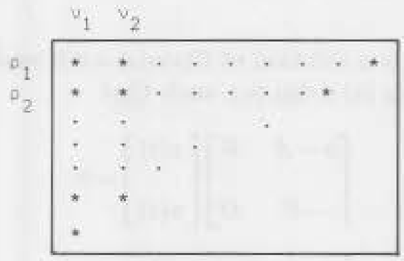

Figure 6. If ν_i equals the number of stars in column i, then ρ_j is the number of stars in row j.

(Hautus and Heymann 1978)), so $G(s)$ has the same zeros at infinity as $G_1(s)$. By choosing suitable bases and choosing F and G as in (11), we have

$$\begin{bmatrix} A+BF+GC+GDF & B+GD \\ C+DF & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & 0 & 0 \\ C_1 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

with B_1 monic and C_1 epic. The zeros at infinity of $\Sigma=(A, B, C, D)$ equal those of $\Sigma_1=(A_1, B_1, C_1, 0)$. Σ_1 has the same $\mathscr{V}^*$ and $\mathscr{S}^i$ as the original system, as was pointed out in § 2. For strictly proper systems the theorem was proved by Commault and Dion (1982), and application of this to Σ_1 yields the result. ∎

By this theorem the structure of $(\mathscr{V}^*+\mathscr{S}^*)/\mathscr{V}^*$ has been interpreted. So in Fig. 4 only $\mathscr{R}^*/(\langle\mathscr{K}|A\rangle\cap\mathscr{S}^*)$ and $(\mathscr{V}^*+\langle A|\mathscr{B}\rangle)/\mathscr{N}^*$ are left to be investigated. We shall study the former space, leaving the latter to duality. As is well known, a system is left invertible (or, equivalently, its transfer function is) if and only if $\mathscr{R}^*=\{0\}$ (Morse and Wonham 1971). It is thus expected that $\mathscr{R}^*$ is related to the kernel of the transfer function. Roughly speaking the unobservable part of $\mathscr{R}^*$ does not contribute to the transfer function, and we may therefore expect that $\mathscr{R}^*/(\langle\mathscr{K}|A\rangle\cap\mathscr{S}^*)$ is the only space involved. In the following we shall relate the subspaces $\mathscr{V}^*\cap\mathscr{S}^i$ to a minimal polynomial basis for the kernel of the transfer function (Forney 1975). This result will be generalized to the unobservable case later on.

Theorem 5

Let the system $\Sigma=(A, B, C, D)$ be an observable realization of the transfer matrix $G(s)$. Define $\gamma_i=\dim(\mathscr{V}^*\cap\mathscr{S}^i)-\dim(\mathscr{V}^*\cap\mathscr{S}^{i-1})$ $(i=1, 2, \ldots)$. Let $\{\delta_1, \delta_2, \ldots\}$ be the list of degrees, greater or equal to one, of the polynomials in a minimal polynomial basis for Ker G, arranged in non-increasing order. Then the lists $\{\gamma_1, \gamma_2, \ldots\}$ and $\{\delta_1, \delta_2, \ldots\}$ are related in the following way: $\delta_i=\#\{j|\gamma_j\geqslant i\}$ (cf. again Fig. 6).

To show this, we first need some lemmas where we will assume without further mentioning that the pair (C, A) is observable.

Lemma 7

The polynomial $u(s)$ is a solution of $G(s)u(s)=0$ if and only if there exists a polynomial $x(s)$, with $\deg(x) \leqslant \deg(u)$, such that

$$\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = 0$$

Proof

(' if ') we have $x(s) = -(s-A)^{-1}Bu(s)$ and $-Cx(s)+Du(s)=0$, i.e. $G(s)u(s)=0$.

(' only if ') suppose that $G(s)u(s)=0$. Define $x(s)=-(s-A)^{-1}Bu(s)$. Note that the degree of $x(s)$ (i.e. the highest power in its development around infinity having a non-zero coefficient) is at least 1 less than that of $u(s)$. Writing $u(s)=u_{-k}s^k+\ldots+u_0$ and $x(s)=x_{-k+1}s^{k-1}+\ldots+x_0+x_1s^{-1}+\ldots$, we obtain the following by expanding the equations $x(s)=-(s-A)^{-1}Bu(s)$ and $-Cx(s)+Du(s)=0$

$$\left.\begin{array}{ll} x_{-k+1} = \qquad -Bu_{-k} & y_{-k} = \qquad Du_{-k} = 0 \\ x_{-k+2} = Ax_{-k+1} - Bu_{-k+1} & y_{-k+1} = -Cx_{-k+1} + Du_{-k+1} = 0 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ x_0 = Ax_{-1} - Bu_{-1} & y_{-1} \quad -Cx_{-1} + Du_{-1} = 0 \\ x_1 = Ax_0 - Bu_0 & y_0 = -Cx_0 + Du_0 = 0 \\ x_2 = Ax_1 & y_1 = -Cx_1 \qquad = 0 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{array}\right\} \tag{18}$$

It follows that $C(s-A)^{-1}x_1=0$. Because the pair (C, A) is observable, it follows that $x_1=0$ and, in fact, $x_j=0$ for all $j \geqslant 1$, so $x(s)$ is a polynomial.

Moreover, we obviously have $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = 0.$ □

Lemma 8

(1) If $\left\{ \begin{bmatrix} x_1(s) \\ u_1(s) \end{bmatrix}, \ldots, \begin{bmatrix} x_k(s) \\ u_k(s) \end{bmatrix} \right\}$ is a polynomial basis for Ker $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}$, then $\{u_1(s), \ldots, u_k(s)\}$ is a polynomial basis for Ker G.

(2) Conversely, if $\{u_1(s), \ldots, u_k(s)\}$ is a polynomial basis for Ker G, then $\left\{ \begin{bmatrix} x_1(s) \\ u_1(s) \end{bmatrix}, \ldots, \begin{bmatrix} x_k(s) \\ u_k(s) \end{bmatrix} \right\}$ is a polynomial basis for Ker $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}$, where $x_i(s) = -(s-A)^{-1}Bu(s)$ $(i=1, \ldots, k)$.

(3) Moreover, the lists of degrees of the basis elements are the same.

Proof

(1) Suppose that $G(s)u(s)=0$. Then

$$\begin{bmatrix} -(s-A)^{-1}Bu(s) \\ u(s) \end{bmatrix} \in \operatorname{Ker} \begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}$$

so that $u(s)$ can be written as a linear combination of the $u_i(s)$ $(i=1, \ldots, k)$.

Suppose that $\alpha_1(s)u_1(s)+\ldots+\alpha_k(s)u_k(s)=0$. From $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}\begin{bmatrix} x_i(s) \\ u_i(s) \end{bmatrix}=0$ it follows that $x_i(s)=-(s-A)^{-1}Bu_i(s)$ $(i=1, \ldots, k)$. So we have

$$\alpha_1(s)\begin{bmatrix} x_1(s) \\ u_1(s) \end{bmatrix}+\ldots+\alpha_k(s)\begin{bmatrix} x_k(s) \\ u_k(s) \end{bmatrix}=0 \tag{19}$$

and consequently $\alpha_1(s)=\ldots=\alpha_k(s)=0$.

(2) Suppose that $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}=0$. Then $x(s)$ is a polynomial (as above) and $x(s)=-(s-A)^{-1}Bu(s)$, with $G(s)u(s)=0$. It follows that $\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}$ can be written as a linear combination of the $\begin{bmatrix} x_i(s) \\ u_i(s) \end{bmatrix}$ $(i=1, \ldots, k)$. If (19) holds, then $\alpha_1(s)u_1(s)+\ldots+\alpha_k(s)u_k(s)=0$, so $\alpha_1(s)=\ldots=\alpha_k(s)=0$.

(3) If $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}=0$, then $x(s)=-(s-A)^{-1}Bu(s)$; so $\deg(x)<\deg(u)$. Therefore $\deg\left(\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}\right)=\deg(u(s))$. □

Corollary 2

The list of degrees of a minimal polynomial basis for Ker $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}$ is the same as the list of degrees of a minimal polynomial basis for Ker $G(s)$.

Corollary 3

The degree-list of a minimal polynomial basis for Ker $G(s)$ is invariant under changes of basis in $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{Y}$, and under transformations $\Sigma\to\Sigma_F=(A+BF, B, C+DF, D)$ and $\Sigma\to\Sigma^G=(A+GC, B+GD, C, D)$.

Proof

The cited transformations correspond to multiplication of $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}$ from the left and/or right by constant invertible matrices. □

Lemma 9

If $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}=0$, with $\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}$ polynomial, then $x(s)$ is $\mathscr{R}^*$-valued (i.e. if $x(s)=x_{-k+1}s^{k-1}+\dots+x_0$, then $x_{-k+i}\in\mathscr{R}^*$ for $i=1,\dots,k$). In fact, we even have $x_{-k+i}\in\mathscr{V}^*\cap\mathscr{S}^i$ $(i=1,\dots,k)$.

Proof

Consider eqns. (18) with $x_1=0$. Obviously, we have $x_0\in\{x|\exists u$ such that $Ax+Bu\in\mathscr{V}^*$, $Cx+Du=0\}=\mathscr{V}^*$. From this, it follows that $x_{-1}\in\{x|\exists u$ s.t. $Ax+Bu\in\mathscr{V}^*$, $Cx+Du=0\}=\mathscr{V}^*$. Going on in this way, we find that $x_{-k+i}\in\mathscr{V}^*$ for $i=1,\dots,k$. On the other hand, we have $x_{-k+1}=-Bu_{-k}$ and $Du_{-k}=0$, so $x_{-k+1}\in\{Bu|Du=0\}=\mathscr{S}^1$. From this, it follows that $x_{-k+2}\in\{x|(\exists w\in\mathscr{S}^1)$ s.t. $Aw+Bu=x$, $Cw+Du=0\}=\mathscr{S}^2$. Going on in this way, we find that $x_{-k+i}\in\mathscr{S}^i\subset\mathscr{S}^*$ for all $i=1,\dots,k$. In all, we have

$$x_{-k+i}\in\mathscr{V}^*\cap\mathscr{S}^*=\mathscr{R}^* \quad \text{for } i=1,\dots,k$$ □

Lemma 10

Take $F\in\mathscr{F}(\mathscr{V}^*)$. Then $\mathscr{R}^*=\langle A+BF|\mathscr{S}^1\cap\mathscr{V}^*\rangle$

$$\left(\text{in fact, } \sum_{i=0}^{k}(A+BF)^i(\mathscr{S}^1\cap\mathscr{V}^*)=\mathscr{S}^{k+1}\cap\mathscr{V}^*\right).$$

The proof of this is by standard methods (Anderson 1975).

Proof of the Theorem

Decompose $\mathscr{U}=\mathscr{U}_1\oplus\mathscr{U}_2\oplus\mathscr{U}_3\oplus\mathscr{U}_4$ where $\mathscr{U}_1\oplus\mathscr{U}_2\oplus\mathscr{U}_4=\text{Ker } D$, $\mathscr{U}_4=\text{Ker } B\cap\text{Ker } D$, and $B\mathscr{U}_1=\mathscr{S}^1\cap\mathscr{V}^*\subset\mathscr{R}^*$. Note that $B\mathscr{U}_2\cap\mathscr{R}^*=\{0\}$. Decompose $\mathscr{X}=\mathscr{X}_1\oplus\mathscr{X}_2$ with $\mathscr{X}_1=\mathscr{R}^*$ and $\mathscr{Y}=\mathscr{Y}_1\oplus\mathscr{Y}_2$ with $\mathscr{Y}_2=\text{Im } D$. Take $F\in\mathscr{F}(\mathscr{V}^*)$; then we can write

$$A+BF=\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad B=\begin{bmatrix} B_{11} & 0 & B_{13} & 0 \\ 0 & B_{22} & B_{23} & 0 \end{bmatrix}$$

$$C+DF=\begin{bmatrix} 0 & C_{12} \\ 0 & C_{22} \end{bmatrix} \quad D=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & D_{23} & 0 \end{bmatrix}$$

where D_{23} is invertible, and B_{11} and B_{22} are monic. Suppose that $\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}$ is polynomial and

$$\begin{bmatrix} s-A_{11} & -A_{12} & B_{11} & 0 & B_{13} & 0 \\ 0 & s-A_{22} & 0 & B_{22} & B_{23} & 0 \\ 0 & -C_{12} & 0 & 0 & 0 & 0 \\ 0 & -C_{22} & 0 & 0 & D_{23} & 0 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \\ u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4(s) \end{bmatrix} = 0 \tag{20}$$

Then $x_2(s)=0$ according to Lemma 9. From $D_{23}u_3(s)=0$, it follows that $u_3(s)=0$. Also, from $B_{22}u_2(s)=0$ it follows that $u_2(s)=0$. We are left with the equation

$$[s-A_{11} \quad B_{11}] \begin{bmatrix} x_1(s) \\ u_1(s) \end{bmatrix} = 0 \tag{21}$$

where (A_{11}, B_{11}) is a controllable pair. We may assume that this pair is in the Brunovsky canonical form (see, for instance, Wonham (1979), p. 118). Then (21) breaks down into a number (equal to rank B_{11}) of equations of the form

$$\begin{bmatrix} s & -1 & & 0 \\ & \ddots & \ddots & \vdots \\ & & -1 & 0 \\ & & \ddots & \\ & & s & 1 \end{bmatrix} \begin{bmatrix} x_1(s) \\ \vdots \\ x_k(s) \\ u(s) \end{bmatrix} = 0 \tag{22}$$

which can be rewritten as

$$\left.\begin{aligned} x_2(s) &= sx_1(s) \\ &\cdots\cdots \\ x_k(s) &= sx_{k-1}(s) \\ u(s) &= sx_k(s) \end{aligned}\right\} \tag{23}$$

Obviously, a solution of minimal degree is obtained by setting $x_1(s)=1$, which gives $x_2(s)=s$, ..., $x_k(s)=s^{k-1}$, and $u(s)=-s^k$. The minimal degree is equal to k, the size of the corresponding Brunovsky block. The degree of these solutions are greater or equal to one, whereas the degree of the minimal solution of (22) is obviously zero. Therefore, the list of degrees greater or equal to one in a minimal polynomial basis for Ker $\begin{bmatrix} s-A & B \\ -C & D \end{bmatrix}$ (or Ker $G(s)$) is equal to the list of controllability indices of the pair (A_{11}, B_{11}). On the other hand, it is well known (see, for example, Wonham (1979), pp. 119–120) that the controllability indices relate to $\dim \sum_{i=0}^{l-1} A_{11}{}^i \operatorname{Im} B_{11} = \dim (\mathscr{V}^* \cap \mathscr{S}^l)$ $(l=1, 2, \ldots)$ in the way indicated in the statement of the theorem. ■

By reduction to the minimal case we are now able to prove the following theorem.

Theorem 6

Let Σ and $\bar{\bar{\Sigma}}$ be defined as in § 2. Define the indices $\gamma_j (j=1, 2, \ldots)$ by

$$\gamma_j = \alpha_j - \alpha_{j-1} \tag{24}$$

$$\alpha_j = \dim (\langle \mathscr{K} | A \rangle + (\mathscr{V}^*(\Sigma) \cap \mathscr{S}^j(\Sigma))) \tag{25}$$

Let $\{\delta_1, \delta_2, \ldots\}$ be the list of degrees of the non-constant polynomials in a minimal polynomial basis for Ker $G(s)$, arranged in non-increasing order. Then the lists $\{\gamma_1, \gamma_2, \ldots\}$ and $\{\delta_1, \delta_2, \ldots\}$ are related in the following way

$$\delta_l = \#\{j \mid \gamma_j \geq l\}, \quad l=1, 2, \ldots \tag{26}$$

Proof

By Theorem 5, all we have to show is that α_j and $\bar{\bar{\alpha}}_j$ differ by a constant integer $(\forall j)$, where

$$\bar{\bar{\alpha}}_j = \dim (\mathscr{V}^*(\bar{\bar{\Sigma}}) \cap \mathscr{S}^j(\bar{\bar{\Sigma}})) \tag{27}$$

The following simple facts from linear algebra will be needed. If $T : \mathscr{X}_1 \to \mathscr{X}_2$ is a linear mapping between (finite-dimensional) vector spaces, and if $\mathscr{V}$ is a subspace of $\mathscr{X}_1$ and $\mathscr{W}$ a subspace of $\mathscr{X}_2$, then

$$\dim (T\mathscr{V}) = \dim (\mathscr{V}) - \dim (\operatorname{Ker} T \cap \mathscr{V}) \tag{28}$$

$$\dim (T^{-1}\mathscr{W}) = \dim (\operatorname{Ker} T) + \dim (\operatorname{Im} T \cap \mathscr{W}) \tag{29}$$

Using this, together with the results of the Appendix, we can write

$$\begin{aligned}
\bar{\bar{\alpha}}_j &= \dim \{\pi i^{-1}(\mathscr{V}^*(\Sigma) \cap \mathscr{S}^j(\Sigma))\} \\
&= \dim \{i^{-1}(\mathscr{V}^*(\Sigma) \cap \mathscr{S}^j(\Sigma))\} - \dim \{i^{-1}(\mathscr{V}^*(\Sigma) \cap \mathscr{S}^j(\Sigma)) \cap i^{-1}(\langle \mathscr{K} | A \rangle)\} \\
&= \dim \{\mathscr{V}^*(\Sigma) \cap \mathscr{S}^j(\Sigma) \cap \langle A | \mathscr{B} \rangle\} - \dim \{i^{-1}(\mathscr{V}^*(\Sigma) \cap \mathscr{S}^j(\Sigma) \cap \langle \mathscr{K} | A \rangle)\} \\
&= \dim \{(\mathscr{V}^*(\Sigma) \cap \mathscr{S}^j(\Sigma)) / (\mathscr{V}^*(\Sigma) \cap \mathscr{S}^j(\Sigma) \langle \mathscr{K} | A \rangle)\} \\
&= \dim \{((\mathscr{V}^*(\Sigma) \cap \mathscr{S}^j(\Sigma)) + \langle \mathscr{K} | A \rangle) / \langle \mathscr{K} | A \rangle\} \\
&= a_j - \dim \{\langle \mathscr{K} | A \rangle\}
\end{aligned} \tag{30}$$

■

Remark

It is easily seen that $G(s)u_0=0$, for $u_0 \in \mathscr{U}$, if and only if $Bu_0 \in \langle \mathscr{K} | A \rangle$ and $Du_0=0$. Therefore, the number of independent constant solutions to the equation $G(s)u(s)=0$ (= the number of zero-degree polynomials in a minimal polynomial basis for Ker G) is equal to

$$\dim\left(\langle \mathscr{K} | A \rangle \cap \mathscr{S}^1(\Sigma)\right) + \dim\left(\operatorname{Ker}\begin{bmatrix} B \\ D \end{bmatrix}\right).$$

In particular, it follows that if $\tilde{\Sigma}=(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is an observable realization of $G(s)$, then $\dim\left(\operatorname{Ker}\begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix}\right) = \dim\left(\operatorname{Ker}\begin{bmatrix} B \\ D \end{bmatrix}\right) + \dim\left(\langle \mathscr{K} | A \rangle \cap \mathscr{S}^1(\Sigma)\right)$. This, together with the dual statements, gives a criterion for a state-space system to have a corresponding minimal system which is standard $\left(\text{i.e. } \begin{bmatrix} B \\ D \end{bmatrix} \text{ is monic and } [C \quad D] \text{ is epic}\right)$.

Finally, some remarks on invertibility of transfer functions can be made. One of the results of Dion and Commault (1982) is that $G(s)$ has a polynomial inverse if and only if $\mathscr{S}^* = \mathscr{X}$ and $\mathscr{V}^* = \{0\}$. This statement holds only for minimal systems. The general statement is contained in the following proposition.

Proposition 4

Let the system $\Sigma=(A, B, C, D)$ have transfer function $G(s)$. Then $G(s)$ is

left invertible iff $\mathscr{R}^* = \{0\}$ and $\begin{bmatrix} B \\ D \end{bmatrix}$ is monic

right invertible iff $\mathscr{N}^* = \mathscr{X}$ and $[C \quad D]$ is epic

invertible iff $\mathscr{S}^* \oplus \mathscr{V}^* = \mathscr{X}$, $[C \quad D]$ is epic and $\begin{bmatrix} B \\ D \end{bmatrix}$ is monic

Furthermore, these (left, right) inverses can be chosen polynomial iff additionally the conditions $\mathscr{V}^* \cap \langle A | \mathscr{B} \rangle \subset \langle \mathscr{K} | A \rangle$, $\langle A | \mathscr{B} \rangle \subset \mathscr{S}^* + \langle \mathscr{K} | A \rangle$ or both of these conditions, respectively, are satisfied. Also, $G(s)$ has a proper

left inverse iff $\mathscr{S}^* = \{0\}$ and $\begin{bmatrix} B \\ D \end{bmatrix}$ is monic

right inverse iff $\mathscr{V}^* = \mathscr{X}$ and $[C \quad D]$ is epic

inverse iff $\mathscr{S}^* = \{0\}$, $\mathscr{V}^* = \mathscr{X}$, $\begin{bmatrix} B \\ D \end{bmatrix}$ is monic and $[C \quad D]$ is epic

Proof

The condition for left invertibility has been mentioned already; the condition for right invertibility follows by duality, and invertibility holds if and only if both of these conditions are satisfied. Using the Smith–McMillan form it is easily seen that the inverses may be chosen polynomial if and only if there are no transmission zeros, which is equivalent to (Fig. 4): $\mathscr{V}^*\cap\langle A|\mathscr{B}\rangle\subset\mathscr{R}^*+\langle\mathscr{K}|A\rangle$, or $\mathscr{S}^*+\langle\mathscr{K}|A\rangle\supset\mathscr{N}^*\cap\langle A|\mathscr{B}\rangle$. For left invertibility $\mathscr{R}^*=\{0\}$, so in this case the first condition comes down to $\mathscr{V}^*\cap\langle A|\mathscr{B}\rangle\subset\langle\mathscr{K}|A\rangle$. Right invertibility requires $\mathscr{N}^*=\mathscr{X}$, and the second condition becomes: $\langle A|\mathscr{B}\rangle\subset\mathscr{S}^*+\langle\mathscr{K}|A\rangle$. For the last part, note that $G(s)$ can be written $G(s)=B_1(s)G_1(s)B_2(s)$ with $B_1(s)$ and $B_2(s)$ bicausal, and $G_1(s)=\begin{bmatrix}G_{11}(s) & 0\\ 0 & 0\end{bmatrix}$, where $G_{11}(s)=\operatorname{diag}\{s^{-\rho_1},\ldots,s^{-\rho_k},1,\ldots,1\}$. Then k is the number of zeros at infinity, and the ρ_i are their orders. As $\lim_{|s|\to\infty} B_i(s)$ is invertible $(i=1,2)$, we see that a necessary and sufficient condition for the existence of a (left/right) proper inverse is that a (left/right) inverse exists and there are no zeros at infinity, i.e. $\mathscr{S}^*=\mathscr{R}^*$ (or $\mathscr{N}^*=\mathscr{V}^*$). For left invertibility this becomes $\mathscr{S}^*=\{0\}$, $\begin{bmatrix}B\\ D\end{bmatrix}$ monic; and for right invertibility $\mathscr{V}^*=\mathscr{X}$, $[C \quad D]$ epic. □

6. Conclusions

The different kinds of zeros as defined by Rosenbrock have a very natural geometric interpretation. Together with a number of structural invariants they can be represented in one canonical lattice diagram. This diagram can be viewed as the product of the Morse decomposition and the Kalman decomposition. As for future research, the question as to whether a similar geometric interpretation can be obtained for non-proper rational matrices and generalized state-space systems would be of interest.

ACKNOWLEDGMENTS

The authors wish to thank Professor O. H. Bosgra and Dr. A. J. J. van der Weiden for their useful comments and suggestions during the preparation of this paper.

Appendix

Relations between the geometric structure of a given system and of the corresponding minimal system

In this appendix we shall use definitions (6) and (8) and show how they are related to their analogue for $\bar{\Sigma}$ in the situation of Fig. 2. Fortunately, this relation is as close as one might hope.

Proposition A 1

$\mathscr{V}^j(\bar{\Sigma})=\pi i^{-1}(\mathscr{V}^j(\Sigma))$, for all $j=0,1,2,\ldots$.

Proof

By induction. For $j=0$, we have $\mathscr{V}^0(\Sigma)=\mathscr{X}$ and $\pi i^{-1}(\mathscr{X})=\pi\bar{\mathscr{X}}=\bar{\bar{\mathscr{X}}}=\mathscr{V}^0(\bar{\bar{\Sigma}})$. because π is epic.

Let us now suppose that $\mathscr{V}^j(\bar{\bar{\Sigma}})=\pi i^{-1}(\mathscr{V}^j(\Sigma))$ for some defined j, and let us prove the same fact for $j+1$. We first show that $\pi i^{-1}(\mathscr{V}^{j+1}(\Sigma))\subset\mathscr{V}^{j+1}(\bar{\bar{\Sigma}})$. Take $\bar{x}\in\bar{\mathscr{X}}$ such that $i\bar{x}\in\mathscr{V}^{j+1}(\Sigma)$; we have to prove that $\pi\bar{x}\in\mathscr{V}^{j+1}(\bar{\bar{\Sigma}})$. Because $i\bar{x}\in\mathscr{V}^{j+1}(\Sigma)$, there exists $u\in\mathscr{U}$ such that

$$Ai\bar{x}+Bu\in\mathscr{V}^j(\Sigma),\quad Ci\bar{x}+Du=0 \tag{A 1}$$

Using the commutativity of Fig. 2, from (A 1) we get $i(\bar{A}\bar{x}+\bar{B}u)\in\mathscr{V}^j(\Sigma)$, from which it follows, that

$$\bar{\bar{A}}\pi\bar{x}+\bar{\bar{B}}u=\pi(\bar{A}\bar{x}+\bar{B}u)\in\pi i^{-1}(\mathscr{V}^j(\Sigma))=\mathscr{V}^j(\bar{\bar{\Sigma}}) \tag{A 2}$$

Also, we have from (A 1)

$$\bar{\bar{C}}\pi\bar{x}+\bar{\bar{D}}u=0 \tag{A 3}$$

From (A 2) and (A 3) it follows that, indeed, $\pi\bar{x}\in\mathscr{V}^{j+1}(\bar{\bar{\Sigma}})$.

Now, we show that $\mathscr{V}^{j+1}(\bar{\bar{\Sigma}})\subset\pi i^{-1}(\mathscr{V}^{j+1}(\Sigma))$. Take $\bar{\bar{x}}\in\mathscr{V}^{j+1}(\bar{\bar{\Sigma}})$. Because π is epic, there exists $\bar{x}\in\bar{\mathscr{X}}$ such that $\pi\bar{x}=\bar{\bar{x}}$. We would like to prove that $i\bar{x}\in\mathscr{V}^{j+1}(\Sigma)$. Since $\bar{\bar{x}}\in\mathscr{V}^{j+1}(\bar{\bar{\Sigma}})$, there exists u such that $\bar{\bar{A}}\bar{\bar{x}}+\bar{\bar{B}}u\in\mathscr{V}^j(\bar{\bar{\Sigma}})$ and $\bar{\bar{C}}\bar{\bar{x}}+\bar{\bar{D}}u=0$. Because $\mathscr{V}^j(\bar{\bar{\Sigma}})=\pi i^{-1}(\mathscr{V}^j(\Sigma))$, there exists $\bar{w}\in\bar{\mathscr{X}}$ with $i\bar{w}\in\mathscr{V}^j(\Sigma)$, such that $\bar{\bar{A}}\bar{\bar{x}}+\bar{\bar{B}}u=\pi\bar{w}$. Using commutativity, we have $\pi\bar{w}=\bar{\bar{A}}\pi\bar{x}+\bar{\bar{B}}u=\pi(\bar{A}\bar{x}+\bar{B}u)$, from which it follows that $\bar{A}\bar{x}+\bar{B}u=\bar{w}+\bar{w}_0$ with $\bar{w}_0\in\operatorname{Ker}\pi$. Consequently, we get $Ai\bar{x}+Bu=i(\bar{A}\bar{x}+\bar{B}u)=i\bar{w}+i\bar{w}_0$. We know that $i\bar{w}\in\mathscr{V}^j(\Sigma)$; moreover, $\bar{w}_0\in\operatorname{Ker}\pi=i^{-1}(\langle\mathscr{K}|A\rangle)$, so $i\bar{w}_0\in\langle\mathscr{K}|A\rangle\subset\mathscr{V}^j(\Sigma)$ for all j. So

$$Ai\bar{x}+Bu\in\mathscr{V}^j(\Sigma) \tag{A 4}$$

Furthermore

$$Ci\bar{x}+Du=\bar{\bar{C}}\pi\bar{x}+\bar{\bar{D}}u=\bar{\bar{C}}x+\bar{\bar{D}}u=0 \tag{A 5}$$

It follows from (A 4) and (A 5) that $i\bar{x}\in\mathscr{V}^{j+1}(\Sigma)$, as desired. □

Proposition A 2

$\mathscr{S}^j(\bar{\bar{\Sigma}})=\pi i^{-1}(\mathscr{S}^j(\Sigma))$ for all $j=0,1,2,\ldots$.

Proof

By induction. For $j=0$, we have $\mathscr{S}^0(\Sigma)=\{0\}$ and $\pi i^{-1}(\{0\})=\pi(\{0\})=\{0\}=\mathscr{S}^0(\bar{\bar{\Sigma}})$, because i is monic. Let us now suppose that $\mathscr{S}^j(\bar{\bar{\Sigma}})=\pi i^{-1}(\mathscr{S}^j(\Sigma))$ for some fixed j, and let us prove the same fact for $j+1$. We first show that $\mathscr{S}^{j+1}(\bar{\bar{\Sigma}})\subset\pi i^{-1}(\mathscr{S}^{j+1}(\Sigma))$. Take $\bar{\bar{x}}\in\mathscr{S}^{j+1}(\bar{\bar{\Sigma}})$. Then there exist vectors $\bar{\bar{w}}\in\mathscr{S}^j(\bar{\bar{\Sigma}})$ and $u\in\mathscr{U}$ such that $\bar{\bar{A}}\bar{\bar{w}}+\bar{\bar{B}}u=\bar{\bar{x}}$ and $\bar{\bar{C}}\bar{\bar{w}}+\bar{\bar{D}}u=0$. Because $\bar{\bar{w}}\in\mathscr{S}^j(\bar{\bar{\Sigma}})=\pi i^{-1}(\mathscr{S}^j(\Sigma))$, there exists $\bar{w}\in\bar{\mathscr{X}}$ such that $\pi\bar{w}=\bar{\bar{w}}$ and $i\bar{w}\in\mathscr{S}^j(\Sigma)$. Define $\bar{x}$ by $\bar{x}=\bar{A}\bar{w}+\bar{B}u$. Then we have $\pi\bar{x}=\pi\bar{A}\bar{w}+\pi\bar{B}u=\bar{\bar{A}}\pi\bar{w}+\bar{\bar{B}}u=\bar{\bar{A}}\bar{\bar{w}}+\bar{\bar{B}}u=\bar{\bar{x}}$. So $\pi\bar{x}=\bar{\bar{x}}$, and it remains to prove that $i\bar{x}\in\mathscr{S}^j(\Sigma)$. This follows from $i\bar{x}=i\bar{A}\bar{w}+i\bar{B}u=Ai\bar{w}+Bu$ (note that $i\bar{w}\in\mathscr{S}^j(\Sigma)$), and $Ci\bar{w}+Du=\bar{\bar{C}}\pi\bar{w}+\bar{\bar{D}}u=\bar{\bar{C}}\bar{\bar{w}}+\bar{\bar{D}}u=0$). Next, we prove that $\pi i^{-1}(\mathscr{S}^{j+1}(\Sigma))\subset\mathscr{S}^{j+1}(\bar{\bar{\Sigma}})$. Let $\bar{x}\in\bar{\mathscr{X}}$

be such that $i\bar{x}\in\mathscr{S}^{j+1}(\Sigma)$. We want to show that $\pi\bar{x}\in\mathscr{S}^{j+1}(\bar{\bar{\Sigma}})$. Because $i\bar{x}\in\mathscr{S}^{j+1}(\Sigma)$, there exist vectors $w\in\mathscr{S}^{j}(\Sigma)$ and $u\in\mathscr{U}$ such that

$$i\bar{x}=Aw+Bu,\quad Cw+Du=0 \tag{A 6}$$

Because $\mathscr{S}^j(\Sigma)\subset\langle A\,|\,\mathscr{B}\rangle=\operatorname{Im} i$ for all j, there exists $\bar{w}\in\bar{\mathscr{X}}$ such that $i\bar{w}=w$. Then we have, from (A 6), $i\bar{x}=Ai\bar{w}+Bu=i\bar{A}\bar{w}+i\bar{B}u$. Because i is monic, it follows that $\bar{x}=\bar{A}\bar{w}+\bar{B}u$, so that $\pi\bar{x}=\pi\bar{A}\bar{w}+\pi\bar{B}u=\bar{\bar{A}}\pi\bar{w}+\bar{\bar{B}}u$, where

$$\pi\bar{w}\in\pi i^{-1}(\mathscr{S}^j(\Sigma))=\mathscr{S}^j(\bar{\bar{\Sigma}}).$$

Moreover

$$\bar{\bar{C}}\pi\bar{w}+\bar{\bar{D}}u=Ci\bar{w}+Du=Cw+Du=0.$$

This shows that $\pi x\in\mathscr{S}^{j+1}(\bar{\bar{\Sigma}})$, and so the proof is complete. □

To proceed, we need the following lemma, which is easily proved by the standard methods of linear algebra.

Lemma A 1

Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be vector spaces, and let $T:\mathscr{X}_1\to\mathscr{X}_2$ be a linear mapping. Suppose that $\mathscr{V}$ and $\mathscr{W}$ are subspaces of $\mathscr{X}_1$.

Then we have

$$T(\mathscr{V}+\mathscr{W})=T\mathscr{V}+T\mathscr{W} \tag{A 7}$$

We also have

$$T(\mathscr{V}\cap\mathscr{W})=T\mathscr{V}\cap T\mathscr{W} \tag{A 8}$$

if and only if the distributive rule $(\mathscr{V}\cap\mathscr{W})+\operatorname{Ker}T=(\mathscr{V}+\operatorname{Ker}T)\cap(\mathscr{W}+\operatorname{Ker}T)$ holds. Now suppose that $\mathscr{V}$ and $\mathscr{W}$ are subspaces of $\mathscr{X}_2$. Then we have

$$T^{-1}(\mathscr{V}\cap\mathscr{W})=T^{-1}\mathscr{V}\cap T^{-1}\mathscr{W} \tag{A 9}$$

We also have

$$T^{-1}(\mathscr{V}+\mathscr{W})=T^{-1}\mathscr{V}+T^{-1}\mathscr{W} \tag{A 10}$$

if and only if the distributive rule $(\mathscr{V}+\mathscr{W})\cap\operatorname{Im}T=(\mathscr{V}\cap\operatorname{Im}T)+(\mathscr{W}\cap\operatorname{Im}T)$ holds.

Proposition A 3

For all $k=0,1,2,\ldots$ and $l=0,1,2,\ldots$, the following is true

$$\mathscr{V}^k(\bar{\bar{\Sigma}})\cap\mathscr{S}^l(\bar{\bar{\Sigma}})=\pi i^{-1}(\mathscr{V}^k(\Sigma)\cap\mathscr{S}^l(\Sigma)) \tag{A 11}$$

Proof

Because $\operatorname{Ker}\pi=i^{-1}(\langle\mathscr{K}\,|\,A\rangle)$ is contained in $i^{-1}(\mathscr{V}^k(\Sigma))$ for all k, we can write down

$$\begin{aligned}\mathscr{V}^k(\bar{\bar{\Sigma}})\cap\mathscr{S}^l(\bar{\bar{\Sigma}})&=\pi i^{-1}(\mathscr{V}^k(\Sigma))\cap\pi i^{-1}(\mathscr{S}^l(\Sigma))\\&=\pi(i^{-1}(\mathscr{V}^k(\Sigma))\cap i^{-1}(\mathscr{S}^l(\Sigma)))=\pi i^{-1}(\mathscr{V}^k(\Sigma)\cap\mathscr{S}^l(\Sigma))\end{aligned}$$ □

In a completely analogous way one proves the following proposition.

Proposition A 4

For all $k = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$, the following is true

$$\mathscr{V}^k(\bar{\Sigma}) + \mathscr{S}^l(\bar{\bar{\Sigma}}) = \pi i^{-1}(\mathscr{V}^k(\Sigma) + \mathscr{S}^l(\Sigma)) \qquad \text{(A 12)}$$

By letting the indices in Propositions (A 1)–(A 4) be large enough (for instance, equal to dim $(\mathscr{X})$), we get the following special cases

$$\left.\begin{aligned} \mathscr{V}^*(\bar{\bar{\Sigma}}) &= \pi i^{-1}(\mathscr{V}^*(\Sigma)) \\ \mathscr{S}^*(\bar{\bar{\Sigma}}) &= \pi i^{-1}(\mathscr{S}^*(\Sigma)) \\ \mathscr{R}^*(\bar{\bar{\Sigma}}) &= \pi i^{-1}(\mathscr{R}^*(\Sigma)) \\ \mathscr{N}^*(\bar{\bar{\Sigma}}) &= \pi i^{-1}(\mathscr{N}^*(\Sigma)) \end{aligned}\right\} \qquad \text{(A 13)}$$

References

Anderson, B D. O., 1975, *Proc. IFAC 6th World Congress*, Boston/Cambridge, Massachusetts, U.S.A., paper 43.6.

Bosgra, O. H., and van der Weiden, A. J. J., 1981 a, *Int. J. Control*, **33**, 393 ; 1981 b, *Proc. Joint Automatic Control Conf.* TA-4E.1, American Automatic Control Council, Charlottesville, VA.

Commault, C., and Dion, J. M., 1982, *I.E.E.E. Trans. autom. Control*, **27**, 693.

Desoer, C. A., and Schulman, J. D., 1974, *I.E.E.E. Trans. Circuits Syst.*, **21**, 3.

Dion, J. M., and Commault, C., 1982, *Syst. Control Lett.*, **1**, 312.

Forney, G. D., 1975, *SIAM J. Control*, **13**, 493.

Francis, B. A., and Wonham, W. M., 1975, *Int. J. Control*, **22**, 657.

Gantmacher, F. R., 1959, *The Theory of Matrices—vols. I and II* (New York : Chelsea Publishing).

Hautus, M. L. J., and Heymann, M., 1978, *SIAM J. Control Optim.*, **16**, 83.

Hosoe, S., 1975, *I.E.E.E. Trans. autom. Control*, **20**, 799.

Kailath, T., 1980, *Linear Systems* (Englewood Cliffs, N.J. : Prentice-Hall).

MacFarlane, A. G. J., and Karcanias, N., 1976, *Int. J. Control*, **24**, 33.

Molinari, B. P., 1976 a, *I.E.E.E. Trans. autom. Control*, **21**, 136 ; 1976 b, *Ibid.*, **21**, 761 ; 1978, *Int. J. Control*, **28**, 493.

Moore, B. C., and Silverman, L. M., 1974, *Proc. Joint Automatic Control Conf.*, Texas, paper B2-5, p. 186.

Morse, A. S., 1973, *SIAM J. Control*, **11**, 446 ; 1976, *Lecture Notes in Economics and Mathematical Systems Vol. 131* (Berlin : Springer-Verlag), pp. 61–74.

Morse, A. S., and Wonham, W. M., 1971, *I.E.E.E. Trans. autom. Control*, **16**, 568.

Rosenbrock, H. H., 1970, *State-Space and Multivariable Theory* (London : Nelson) ; 1973, *Int. J. Control*, **18**, 297 ; 1974, *Ibid.*, **20**, 525.

Schumacher, J. M., 1981, *Dynamic Feedback in Finite and Infinite Dimensional Linear Systems*, Doctoral diss. Tract Series no. 143, Mathematical Centre, Amsterdam.

Silverman, L. M., 1969, *I.E.E.E. Trans. autom. Control*, **14**, 270.

Thorp, J. S., 1973, *Int. J. Control*, **18**, 577.

van der Weiden, A. J. J., 1983, *The Use of Structural Properties in Linear Multivariable Control System Design*, Doctoral diss., Delft University of Technology.

van der Weiden, A. J. J., and Bosgra, O. H., 1979, *Int. J. Control*, **29**, 835 ; 1980, *Ibid.*, **32**, 489.

Willems, J. C., 1981, *I.E.E.E. Trans. autom. Control*, **26**, 235.

Wonham, W. M., 1979, *Linear Multivariable Control : a Geometric Approach* (New York : Springer-Verlag).