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The role of the dissipation matrix in singular optimal control *

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The optimal cost in the regular stationary linear-quadratic optimal control problem is given by the maximal solution of an algebraic Riccati equation. This solution minimizes the rank of a certain matrix, which we call the dissipation matrix. The rank minimization problem is also meaningful in the singular case. Does it provide the optimal cost for the singular control problem? This question was posed by J.C. Willems in 1971. The answer is: yes.

Keywords: Singular optimal control, Dissipation inequality, Algebraic Riccati equation, Right invertibility, Spectral factorization.

1. Introduction

As is very well known, the solution of the linear-quadratic time-invariant optimal control problem on an infinite time interval can be obtained through the algebraic Riccati equation, in case the problem is regular, i.e. every nonzero control action gives rise to a nonzero cost. This paper addresses the question of what takes the place of the ARE in the singular case. In this, we follow a suggestion of J.C. Willems in [1].

To describe the issue more precisely, let us first introduce some notation. We consider the finite-dimensional time-invariant linear system over $\mathbb{R}$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

with *control function* u and *instantaneous cost function* y. The associated *cost functional* is

$$J(x_0, u) = \int_0^\infty \|y(t)\|^2 \mathrm{d}t. \tag{1.2}$$

Throughout the paper, we shall consider this system under the following *standing assumptions*: the mapping $\binom{B}{D}$ is injective, the mapping $(C\ D)$ is surjective, the pair (A, B) is stabilizable, and the pair (C, A) is detectable.

Following [1], we shall say that a real symmetric matrix K satisfies the *dissipation inequality* if the inequality

$$x_1'Kx_1 - x_0'Kx_0 + \int_{t_0}^{t_1} \|y(t)\|^2 \mathrm{d}t \geqslant 0 \tag{1.3}$$

holds along a trajectories of (1.1), i.e. (1.3) must hold whenever there exists a pair of functions $(x(\cdot), y(\cdot))$ on $[t_0, t_1]$ such that $x(t_0) = x_0$, $x(t_1) = x_1$, and such that (1.1) holds for some control function $u(\cdot)$. This inequality is an obvious necessary condition for the form $x_0'Kx_0$ to represent the optimal cost

$$\inf_u J(x_0, u)$$

under any conditions related to the long-term behavior of the system (1.1). The equivalent differential form of (1.3) is obtained as

$$\begin{aligned} &(Ax + Bu)'Kx + x'K(Ax + Bu) \\ &\quad + (Cx + Du)'(Cx + Du) \geqslant 0 \end{aligned} \tag{1.4}$$

and this should hold for all x, u. A more concise form is

$$(x'\ u') \begin{pmatrix} A'K + KA + C'C & KB + C'D \\ BK + D'C & D'D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \geqslant 0 \quad \forall x, u. \tag{1.5}$$

So if we define

$$F(K) = \begin{pmatrix} A'K + KA + C'C & KB + C'D \\ B'K + D'C & D'D \end{pmatrix}, \tag{1.6}$$

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then the condition for K to satisfy the dissipation inequality is simply that $F(K)$ should be nonnegative definite. We shall call $F(K)$ the *dissipation matrix.*

It was noted in [1] (Remark 10) that the solutions of the algebraic Riccati equation

$$A'K+KA+C'C - (KB+C'D)(D'D)^{-1}(B'K+D'C)=0 \quad (1.7)$$

are 'boundary' solutions of the linear matrix inequality $F(K)\geqslant 0$ in the sense that they make $F(K)$ of minimal rank. Willems asked whether the solutions of the linear matrix inequality which minimize the rank of $F(K)$ are related to the solutions of the singular optimal control problem. We shall present here a partial and affirmative answer to this question.

In the next section, we shall show that a lower bound for the rank of $F(K)$ is given by the rank (over the field of rational functions) of the transfer matrix of (1.1):

$$W(s)=C(sI-A)^{-1}B+D. \quad (1.8)$$

This lower bound is attained if and only if a transformed system, which is determined by the matrix K and which is defined only if $F(K)\geqslant 0$, is right invertible. To show that there exists a matrix K such that rank $F(K)=$ rank W, we concentrate, in Section 3, on the matrix K^+ associated with the optimal cost under the endpoint condition $x(\infty)=0$. The transformed system defined by K^+ has zero optimal cost, and it follows from results obtained via regularization that this system must then be right invertible. So we can conclude that the minimal rank of $F(K)$ is equal to rank W, and that one solution to the rank minimization problem is given by the matrix K^+. As in the regular case, this matrix is maximal among the symmetric matrices that satisfy the dissipation inequality. So it turns out that the rank minimization problem for the dissipation matrix is the proper general formulation which reduces to the algebraic Riccati equation in the regular case.

2. A lower bound for the rank of the dissipation matrix

Obviously, the mapping $\alpha\to\bar\alpha$ defined by

$$\bar\alpha(s)=\alpha(-s) \quad (2.1)$$

is an automorphism of order 2 (i.e. a mapping whose square is the identity) on the field of real rational functions $\mathbb{R}(s)$. For $x, y\in\mathbb{R}^k(s)$, we define the form

$$\langle x,y\rangle=\sum_{j=1}^{k}x_j\bar y_j. \quad (2.2)$$

It is easily verified that this is a sesquilinear form on the rational vector space $\mathbb{R}^k(s)$ with respect to the automorphism given by (2.1) (see [2], [3] Ch. XIV). (This means that the form is linear in the first variable but antilinear in the second, i.e. $\langle x,\alpha y\rangle=\bar\alpha\langle x,y\rangle$.) Moreover, the form turns out to be definite.

Lemma 1. *If* $\langle x,x\rangle=0$, *for* $x\in\mathbb{R}^k(s)$, *then* $x=0$.

Proof. By inserting $s=\mathrm{i}\omega$ ($\omega\in\mathbb{R}$), we find

$$\sum_{j=1}^{k}|x_j(\mathrm{i}\omega)|^2=\sum_{j=1}^{k}x_j(\mathrm{i}\omega)x_j(-\mathrm{i}\omega)=0. \quad (2.3)$$

So the rational functions $x_j(s)$ must all be zero on the imaginary axis; but then they must be zero everywhere.

Now consider two rational vector spaces $\mathbb{R}^m(s)$ and $\mathbb{R}^p(s)$ with associated forms (2.2). Let W be a linear mapping from $\mathbb{R}^m(s)$ to $\mathbb{R}^p(s)$. By the definiteness of the forms, we can uniquely define a mapping

$$W^*:\mathbb{R}^p(s)\to\mathbb{R}^m(s)$$

by requiring

$$\langle x, Wy\rangle=\langle W^*x,y\rangle \quad \forall x\in\mathbb{R}^p(s), y\in\mathbb{R}^m(s). \quad (2.4)$$

In fact, it is not hard to see that W^* is given by

$$W^*(s)=W'(-s). \quad (2.5)$$

The following lemma will be needed.

Lemma 2. *For any linear mapping* $W:\mathbb{R}^p(s)\to\mathbb{R}^m(s)$, *we have the following equality between subspaces of* $\mathbb{R}^p(s)$:

$$\ker W=\ker W^*W. \quad (2.6)$$

Proof. Suppose that $W^*Wx=0$. Then, in particular, $\langle W^*Wx, x\rangle=0$ which implies $\langle Wx, Wx\rangle=0$, so that $Wx = 0$, according to Lemma 1. The converse is trivial, of course.

We now turn to the dissipation matrix. For any K such that $F(K) \geqslant 0$, we can find matrices C_K and D_K of full row rank such that

$$(C_K \; D_K)'(C_K \; D_K) = F(K). \tag{2.7}$$

Note that the number of rows of $(C_K \; D_K)$ is equal to the rank of $F(K)$. We write

$$W_K(s) = C_K(sI - A)^{-1}B + D_K. \tag{2.8}$$

The following lemma contains the key observation of this paper.

Lemma 3. *Let the system* (1.1) *be given, and suppose that K is a symmetric matrix such that $F(K) \geqslant 0$. Then*

$$\operatorname{rank} F(K) \geqslant \operatorname{rank} W \tag{2.9}$$

with equality if and only if W_K is right invertible.

Proof. Write rank $F(K) = r$. Noting that W_K is an $r \times m$-matrix, we see that

$$\operatorname{rank} F(K) \geqslant \operatorname{rank} W_K \tag{2.10}$$

with equality if and only if W_K is right invertible. So it is sufficient to prove that rank $W_K =$ rank W. By direct computation, one verifies that $W_K^* W_K = W^* W$. It then follows from Lemma 2 that ker W = ker W_K. But

$$\begin{aligned}\operatorname{rank} W &= \operatorname{codim} \ker W \\ &= \operatorname{codim} \ker W_K = \operatorname{rank} W_K,\end{aligned}$$

so that the proof is done.

We have shown that the rank of the transfer matrix is a lower bound for the rank of the dissipation matrix, but it remains to be proven that this lower bound can actually be achieved. This will be taken up in the next section.

3. Right invertibility and zero optimal cost

Consider the system (1.1) with cost functional (1.2). The optimal cost under the endpoint condition

$$\lim_{t \to \infty} x(t) = 0$$

depends quadratically on the initial value x_0, so (following [1]) we can define a symmetric matrix K^+ by

$$x_0' K^+ x_0 = \inf\{J(x_0, u) \mid u \text{ loc. int.}, x(\infty) = 0\}. \tag{3.1}$$

It is obvious that K^+ satisfies the dissipation inequality and so we also have $F(K^+) \geqslant 0$. In fact, K^+ can be characterized as the maximal element in the set of solutions of the linear matrix inequality $F(K) \geqslant 0$.

Lemma 4 [1]. *If $F(K) \geqslant 0$, then $K \leqslant K^+$.*

Proof. Let $F(K) \geqslant 0$. Define $(C_K \; D_K)$ as in (2.7), and let J_K be the associated cost functional:

$$J_K(x_0, u) = \int_0^\infty \|C_K x(t) + D_K u(t)\|^2 \mathrm{d}t. \tag{3.2}$$

Computation shows that, for every control function u such that

$$\lim_{t \to \infty} x(t) = 0,$$

we have

$$J_K(x_0, u) = J(x_0, u) - x_0' K x_0. \tag{3.3}$$

Taking infima on both sides, we obtain

$$\begin{aligned}0 &\leqslant \inf\{J_K(x_0, u) \mid u \text{ s.t. } x(\infty) = 0\} \\ &= x_0' K^+ x_0 - x_0' K x_0.\end{aligned} \tag{3.4}$$

The proof also shows that the optimal cost, under the condition $x(\infty) = 0$, for the system (1.1) with cost functional J_{K^+} is equal to zero for every initial value x_0. We can connect this to right invertibility of the transfer matrix W_{K^+} by making use of results on 'cheap control'. First, we need the following lemma.

Lemma 5. *Define $F(K; \varepsilon)$ by*

$$F(K; \varepsilon) = \begin{pmatrix} A'K + KA + C'C & KB + C'D \\ B'K + D'C & D'D + \varepsilon^2 I \end{pmatrix}. \tag{3.5}$$

For $\varepsilon > 0$, the matrix

$$K^+(\varepsilon) = \max\{K \mid K \text{ symmetric}, F(K; \varepsilon) \geqslant 0\} \tag{3.6}$$

is well defined, and $K^+(\varepsilon)$ is non-decreasing as a function of ε. Moreover

$$\lim_{\varepsilon \to 0} K^+(\varepsilon) = K^+. \tag{3.7}$$

Proof. Consider the system (1.1) with 'regularized' cost functional J_ε defined through

$$C_\varepsilon = \begin{pmatrix} C \\ 0 \end{pmatrix}, \qquad D_\varepsilon = \begin{pmatrix} D \\ \varepsilon I \end{pmatrix}. \tag{3.8}$$

It is verified immediately that the associated dissipation matrix $F_\varepsilon(K)$ is equal to $F(K;\, \varepsilon)$ as defined above. So $K^+(\varepsilon)$ has the interpretation of representing the optimal cost, under the condition $x(\infty) = 0$, for the regularized system. This makes it obvious that $K^+(\varepsilon)$ is a non-decreasing function of ε. It follows that

$$\lim_{\varepsilon \to 0} K^+(\varepsilon)$$

exists; let us temporarily write the limit as K_0^+. Since the matrix-valued function F defined in (3.5) is jointly continuous in K and in ε, we have, as $K^+(\varepsilon) \to K_0^+$ and $\varepsilon \to 0$,

$$0 \leqslant F(K^+(\varepsilon);\, \varepsilon) \to F(K_0^+;\, 0). \tag{3.9}$$

This shows that $F(K_0^+) \geqslant 0$, which implies, by Lemma 4, that $K_0^+ \leqslant K^+$. On the other hand, it is clear that $K^+ \leqslant K^+(\varepsilon)$ for all $\epsilon > 0$, so that

$$K^+ \leqslant \lim_{\varepsilon \to 0} K^+(\varepsilon) = K_0^+.$$

It follows that $K_0^+ = K^+$.

The following result, which we shall use as a lemma, provides the link between 'right invertibility' and 'zero optimal cost'. The statement is actually more precise than that.

Lemma 6. *Consider the system* (1.1) *with cost functional* (1.2). *If* $K^+ = 0$, *then the transfer matrix* $W(s)$ *has a right inverse with poles in the closed left half plane.*

The most complete proof for this fact has been given, as far as the author knows, by Francis [4]. The result there is stated in terms of

$$\lim_{\varepsilon \to 0} K_\varepsilon^+,$$

but according to Lemma 5, this is K^+. Actually, Francis considers only the 'totally singular' case ($D = 0$, in our notation), but this is inessential. (In particular, the result is also true in the regular case, as the reader will easily be able to verify.) An earlier version was given in [5]. It has been argued in the literature that the result is trivial [6], but if this is true, it may well be that we have here one of those trivialities that do not allow easy proofs.

We shall use the lemma as follows. As noted before, the optimal cost for the system with modified cost functional defined by $(C_{K^+}\ D_{K^+})$ is equal to zero. This implies that the associated transfer matrix W_{K^+} has a (stable) right inverse. By the results of Section 2, this means that the rank of $F(K^+)$ must be equal to the lower bound rank W. This closes the circle, except for one point, which is taken care of by the following lemma.

Lemma 7. *Suppose that* $K \geqslant 0$ *and* $F(K) \geqslant 0$. *Define* $(C_K\ D_K)$ *as in* (2.7). *Then the pair* (C_K, A) *is detectable.*

Proof. The matrices C_K and C are related via

$$C_K' C_K = A'K + KA + C'C. \tag{3.10}$$

Suppose now that there would be a real unstable (C_K, A)-unobservable eigenvalue λ. Then there would exist an $x \neq 0$ such that $Ax = \lambda x$ and $C_K x = 0$. By (3.10), this would imply

$$\begin{aligned} 0 &= x'A'Kx + x'KAx + x'C'Cx \\ &= 2\lambda x'Kx + x'C'Cx. \end{aligned} \tag{3.11}$$

Since $K \geqslant 0$ and $\lambda \geqslant 0$, it would follow that $Cx = 0$. But this would contradict the standing assumption that the pair (C, A) is detectable. A similar proof can be given for the case in which there is a pair of conjugate complex unobservable unstable eigenvalues.

So our reasoning can be completed by the simple observation that the matrix K^+ is, by its definition, nonnegative definite.

4. Main results

We collect the results of our considerations in three theorems.

Theorem 1. *The minimal rank of the dissipation matrix* $F(K)$, *where* K *varies over the symmetric matrices satisfying* $F(K) \geqslant 0$, *is equal to the rank of the transfer matrix*

$$W(s) = C(sI - A)^{-1}B + D.$$

Theorem 2. *The matrix K^+, which defines the optimal cost under the endpoint condition*

$$\lim_{t\to\infty} x(t) = 0$$

(see (3.1)), can be found as the maximal element among the set of all symmetric matrices K which satisfy the conditions $F(K) \geqslant 0$ and rank $F(K) =$ rank W.

Theorem 3. *Let $W(s)$ be a given transfer matrix. Then there exists a transfer matrix $\hat{W}(s)$ which satisfies*

$$\hat{W}'(-s)\hat{W}(s) = W'(-s)W(s),$$

and which has a right inverse having poles in the closed left half plane.

In the regular case, the condition that the rank of the dissipation matrix be equal to the rank of the transfer matrix can immediately be reformulated as the algebraic Riccati equation. So it is reasonable to say that the rank minimization problem for the dissipation matrix provides the proper generalization of the algebraic Riccati equation to the not necessarily regular case.

Theorem 3 follows from the theory developed here by taking any stabilizable and detectable realization of $W(s)$, and by setting $\hat{W} = W_{K^+}$, where W_{K^+} is defined through (3.1) and (2.8). The result is a version of a theorem of Youla on spectral factorization of rational matrices [7].

5. Conclusions

Needless to say, many questions remain in connection with the minimization of the rank of the dissipation matrix. In particular, it would be interesting to describe the set of all matrices which minimize this rank and to develop a geometry for the right inverses, as was done for the regular case in [1]. In the singular situation, the optimal control function does not exist unless impulses are allowed, and, in either case, it cannot be obtained from a feedback control law (cf. the discussion in [8]). It is possible, though, to construct approximating sequences of 'high-gain' feedback control laws. This can be done via regularization (see Lemma 5), but it would be more interesting to develop procedures that are based on a direct computation of the optimal cost, using the characterization of Theorem 2. This would require a method to make this characterization numerically effective, which is an interesting problem of its own.

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