Chapter 3
Introduction

3.1 Why Hedge?

In many markets, companies face risks that are imposed from outside. For instance, a company producing toys and selling them abroad is faced with a currency risk. To protect the company from bankruptcy caused by this kind of risk, the company might look for trading strategies that reduce this risk. A trading strategy that is designed to reduce risk is called a hedging strategy. To reduce risk, hedgers can trade futures, forward, and option contracts. Both *futures* and *forward contracts* are agreements to buy or sell an asset at a future time T for a certain price (the so-called *strike price*). Thus both parties commit themselves to some action at time T. The difference between both contracts is that forward contracts are agreements between private institutions/persons, whereas futures contracts are contracts that are traded on an exchange. An *option contract* gives the holder the right to buy/sell an asset by a certain date T for a certain price. An option that gives the holder the right to buy an asset is called a *call option*, and one that gives the holder the right to sell an asset is called a *put option*. Unlike with futures and forward contracts, holders of an option are not obligated to exercise their right. For instance, with a call option, say the right to buy some raw material at time T for a price of 2, if it turns out that at time T the actual price of the material is 1, then a company holding this option will not exercise its right to buy the material for a price of 2.

Forward contracts are designed to neutralize risk by fixing the price that the hedger will pay or receive for the underlying asset. Option contracts provide insurance. With an option a company can protect itself against, for example, unfavorable price swings while benefiting from favorable ones. As in the preceding example, the company holding the call option insures itself that it will not have to pay more than 2 for its raw material at time T, and it can buy the raw material for the actual price at time T if it is smaller than 2.

Another distinction between futures/forward contracts and option contracts is that it costs nothing to enter into a futures contract, whereas the holder of an option contract has to pay a price for it up front.

P. Bernhard et al., *The Interval Market Model in Mathematical Finance*, Static & Dynamic Game Theory: Foundations & Applications, DOI 10.1007/978-0-8176-8388-7_3,

Notice that a contract always involves two parties – the one writing the contract and the one buying the contract. An important point to make about the smooth functioning of the futures, forwards, and options markets is that there is a mechanism to guarantee that both parties of a contract will honor the contract. That is, there are mechanisms (like daily settlements) in place so that if one of the parties does not live up to the agreement, the other party will not have to resort to costly lawsuits. Furthermore, the markets should be such that for each side of a contract there is someone that is prepared to take the opposite position in the contract. Usually this means that in futures markets two other types of traders take positions too, i.e., speculators and arbitrageurs. *Speculators* are willing to take on the risk of a contract. *Arbitrageurs* take offsetting positions in different markets to lock in a profit without taking any risk.

Hedging is used to avoid unpleasant surprises in price movements. This can be appropriate if one owns an asset and expects to sell it at some future time T (like a farmer who grows grain) or if one has to buy a certain asset at time T and wants to lock in a price now (like the company who needs raw material at time T). Another reason for hedging can be that one is planning to hold a portfolio for a long period of time and would like to protect oneself against short-term market uncertainties. High transaction costs of selling and buying the portfolio back later might be a reason to use this strategy. In that case one can use stock index futures to hedge market risk.

However, in practice many risks are not hedged. One reason is that risk hedging usually costs money. Another reason is that one should look at all the implications of price changes for a company's profitability. It may happen that different effects of a price change on the profitability of a firm will offset each other. That is, the company is already hedged internally for this price change.

Problems that may arise in hedging include the hedger's not knowing the exact date the asset will be bought or sold, a mismatch between the expiration date of the contract and the date required by the hedger, a hedger's ability to hedge only a proxy of the asset on the market.

Also, situations exist where one would like to mitigate a risk that will arise far into the future at time T but there exist no futures contracts to hedge this risk (like a pension fund that makes commitments to pay pensions in the distant future). A usual approach to tackling such a situation is to roll the hedge forward by closing out one futures contract and taking the same position in a futures contract with a later delivery date and repeating this procedure until one arrives at time T.

As indicated previously, the main reason that hedging was introduced was to reduce trading risk, that is, to shift (a part of) the risk to another trader who either has greater expertise in dealing with that risk or who has the capability to shoulder the risk. An important issue in the context of the latter case is that for large traders it is in practice not always clear what the exact risk position is they have taken. Clearly one should try to improve on this situation. One should avoid situations where large traders cannot meet their commitments. How to improve on this is an ongoing discussion. One line of thinking is to formulate more explicit rules traders must follow. Within this context one should keep in mind that optimal trading strategies often occur at the boundaries of what is allowed. So these rules should anticipate such behavior.

3.2 A Simplistic Hedging Scheme: The Stop-Loss Strategy

A well-known simple hedging strategy is the so-called *stop-loss strategy*. To illustrate the basic idea, consider a hedger who has written a call option with a strike price of X to buy one unit of a stock. To hedge his position, the simplest procedure the hedger could follow is to buy one unit of the stock when its price rises above X and to sell this unit again when its price drops below X. In this way the hedger makes sure that at the *expiration time* T of the option he will be in a position where he owns the stock if the stock price is greater than X. Figure 3.1 illustrates the selling and buying procedure.

Note that basically four different situations can occur. Denoting the stock price at time t by $S(t)$, (1) $S(0)$ and $S(T)$ are less than X; (2) $S(0)$ and $S(T)$ are greater than X; (3) $S(0) > X$ and $S(T) < X$; or (4) $S(0) < X$ and $S(T) > X$. Denoting $[K]^+ = \max\{K, 0\}$, it follows directly that the total revenues from hedging and closure under these four different scenarios are as follows:

$$\begin{aligned}
&(1)\ -[S(T)-X]^+ = 0,\\
&(2)\ -S(0)+S(T)-[S(T)-X]^+ = X - S(0),\\
&(3)\ -S(0)+X-[S(T)-X]^+ = X - S(0),\\
&(4)\ -X+S(T)-[S(T)-X]^+ = 0,
\end{aligned}$$

respectively. Notice that in cases (1) and (4), $S(0) < X$. Therefore, we can rewrite the total revenues from hedging and closure in compact form as $-[S(0)-X]^+$. We state this result formally in a theorem.

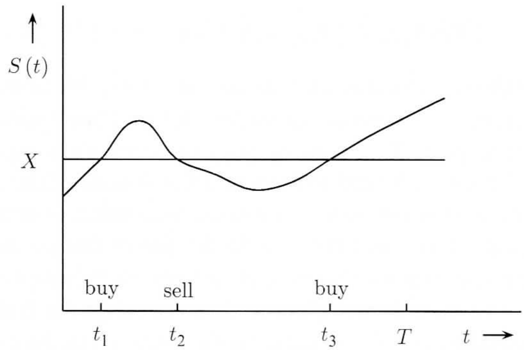


Fig. 3.1 Stop-loss strategy

Theorem 3.1. *The total costs of hedging and closure for a call option using a stop-loss strategy is* $Q^{\text{stop-loss}}(S_0) = [S(0) - X]^+$. □

Or, stated differently, the total cost of hedging and closure equals the intrinsic value of the option.

However, notice that we ignored transaction costs associated with buying and selling the stock under this strategy. Furthermore, if we assume that trading takes place continuously in time, then an important issue is that the hedger cannot know whether, when the stock price equals X, it will then rise above or fall below X. These issues imply that in practice this hedging scheme usually does not work as well as one might have hoped. For a further discussion on this issue we refer the reader to, for example, [88].

3.3 Risk-Free Hedging in the Binomial Tree Model

In this section we recall the well-known binomial tree model that was analyzed by Cox et al. [57] to price options under the assumption that there exist *no arbitrage opportunities*.[1] For a more extensive treatment of this subject, we refer the reader to, for example, Hull [88, Chap. 12].

Consider a market with a single underlying asset. Assume a discrete-time setting where time points are indicated by $t_j,\ j = 0, 1, 2, \ldots$. The price of the asset at time t_j will be denoted by S_j. An *asset price path* is a sequence

$$S = \{S_0, \ldots, S_N\},$$

where the initial price S_0 is fixed throughout and t_N represents the time horizon, which is also assumed to be fixed. The *binomial tree* model $\mathbb{B}^{u,d}$ consists of all price paths that just allow one specific upward and downward price movement at any point in time:

$$\mathbb{B}^{u,d} := \{\mathscr{S} \mid S_{j+1} \in \{d_j S_j, u_j S_j\} \text{ for } j = 0, 1, \ldots, N-1\}.$$

Here u_j and d_j are the *proportional jump factors* at time t_j. We depict this in Fig. 3.2.

Now consider an initial portfolio of a trader who sold one option contract at time t_0 to buy the asset at price X at time t_N (i.e., he went *short* one *European*[2] *call option* with a strike price of X) and who owns a fraction Δ_0 of the asset. Within this binomial model framework one can easily price this option over time and design a trading strategy on the asset such that the final value of this portfolio, where Δ_0 is chosen in a specific way that will become clear later on, is independent of the price path of the asset. That is, if at any point in time we adapt the fraction of the asset in our portfolio according to this, so-called *delta hedging*, trading strategy, then the (net present) value of the portfolio will remain the same. Thus this trading strategy

[1] That is, it is not possible to earn a profit on securities that are mispriced relative to each other.

[2] A European style option contract can be exercised only at the option's expiration date.

tells us at any point in time how many units of the stock we should hold for each option contract in order to create a portfolio whose value does not change over time. Such a *risk-free portfolio* can be set up because the price of the asset and option contract have the same underlying source of uncertainty: the change in asset prices.

To determine this option contract's price and a strategy to trade it over time, we proceed as follows. Let $f_j(S_j)$ denote the value (price) of the option contract at time t_j if the price of the asset at time t_j is S_j. Assume that our portfolio consists at time t_j of Δ_j shares of the asset and the option contract. Then, since the trader has the obligation to pay the buyer of the option contract the value of the contract at t_N, the value of his portfolio at time t_{j+1} is

$$\Delta_j u_j S_j - f_{j+1}(u_j S_j) \text{ if } S_{j+1} = u_j S_j \text{ and } \Delta_j d_j S_j - f_{j+1}(d_j S_j) \text{ if } S_{j+1} = d_j S_j,$$

if the stock price moves up/down, respectively.

Thus the portfolio has the same value in both scenarios if $\Delta_j u_j S_j - f_{j+1}(u_j S_j) = \Delta_j d_j S_j - f_{j+1}(d_j S_j)$, that is, if we choose Δ_j as follows:

$$\Delta_j = \frac{1}{S_j} \frac{f_{j+1}(u_j S_j) - f_{j+1}(d_j S_j)}{u_j - d_j}. \tag{3.1}$$

Stated differently, if we choose Δ_j as the ratio of the change in the price of the stock option contract to the change in the price of the underlying stock [cf. (3.1)], then the portfolio is risk free and must therefore earn the *risk-free*[3] *interest rate* r_j. Thus, denoting the time elapsed between t_{j+1} and t_j by Δt_j, we obtain the present value of the portfolio at time t_j as

$$(\Delta_j u_j S_j - f_{j+1}(u_j S_j))\mathrm{e}^{-r_j \Delta t_j}.$$

On the other hand, we know that this value equals $\Delta_j S_j - f_j(S_j)$. So we get

$$(\Delta_j u S_j - f_{j+1}(u_j S_j))\mathrm{e}^{-r_j \Delta t_j} = \Delta_j S_j - f_j(S_j).$$

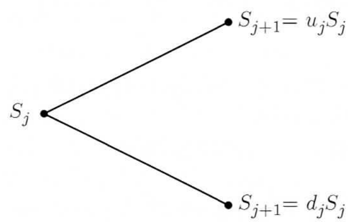


Fig. 3.2 Asset price movements in binomial tree

[3] Usually this is the interest rate at which banks will lend to each other.

Substitution of Δ_j from (3.1) then yields the following backward recursion formula for the option price f_j:

$$f_j(S_j) = \frac{f_{j+1}(u_jS_j)(1-d_j\mathrm{e}^{-r_j\Delta t_j}) + f_{j+1}(d_jS_j)(u_j\mathrm{e}^{-r_j\Delta t_j}-1)}{u_j-d_j}$$

$$= q_jf_{j+1}(u_jS_j) + (\mathrm{e}^{-r_j\Delta t_j} - q_j)f_{j+1}(d_jS_j), \text{ with} \tag{3.2}$$

$$f_N(S_N) = [S_N - X]^+. \tag{3.3}$$

Here $q_j := \frac{1-d_j\mathrm{e}^{-r_j\Delta t_j}}{u_j-d_j}$.

From this recursion formula for the price (3.2) we can now also derive directly the following recursion formula for the corresponding *delta-hedging* trading strategy:

$$\Delta_j(S_j) = \lambda_j\Delta_{j+1}(u_jS_j) + (1-\lambda_j)\Delta_{j+1}(d_jS_j), \text{ with} \tag{3.4}$$

$$\Delta_{N-1}(S_{N-1}) = \frac{[u_{N-1}S_{N-1} - X]^+ - [d_{N-1}S_{N-1} - X]^+}{(u_{N-1} - d_{N-1})S_{N-1}}. \tag{3.5}$$

Here $\lambda_j = u_jq_j$.

We will just show the correctness of (3.4). That (3.5) is correct is easily verified. Substitution of (3.2) into (3.1) gives

$$\begin{aligned}
\Delta_j(S_j) &= \frac{1}{S_j}\frac{f_{j+1}(u_jS_j) - f_{j+1}(d_jS_j)}{u_j - d_j} \\
&= \frac{1}{S_j(u_j-d_j)}\Big\{q_jf_{j+2}(u_j^2S_j) + (\mathrm{e}^{-r_j\Delta t_j} - q_j)f_{j+2}(u_jd_jS_j) \\
&\quad -q_jf_{j+2}(u_jd_jS_j) - (\mathrm{e}^{-r_j\Delta t_j} - q_j)f_{j+2}(d_j^2S_j)\Big\} \\
&= \frac{q_j}{S_j(u_j-d_j)}\{f_{j+2}(u_j^2S_j) - f_{j+2}(u_jd_jS_j)\} \\
&\quad + \frac{\mathrm{e}^{-r_j\Delta t_j} - q_j}{S_j(u_j-d_j)}\{f_{j+2}(u_jd_jS_j) - f_{j+2}(d_j^2S_j)\} \\
&= \frac{u_jq_j}{(u_jS_j)(u_j-d_j)}\{f_{j+2}(u_j(u_jS_j)) - f_{j+2}(d_j(u_jS_j))\} + \frac{d_j(\mathrm{e}^{-r_j\Delta t_j} - q_j)}{(d_jS_j)(u_j-d_j)} \\
&\quad \{f_{j+2}(u_j(d_jS_j)) - f_{j+2}(d_j(d_jS_j))\} \\
&= \lambda_j\Delta_{j+1}(u_jS_j) + (1-\lambda_j)\Delta_{j+1}(d_jS_j).
\end{aligned}$$

Remark 3.2. (1) Notice that for all j, $0 \leq \Delta_j \leq 1$.

(2) The same procedure can also be used to value an option to sell an asset at a certain price at time t_N (European put option) and to determine a trading strategy

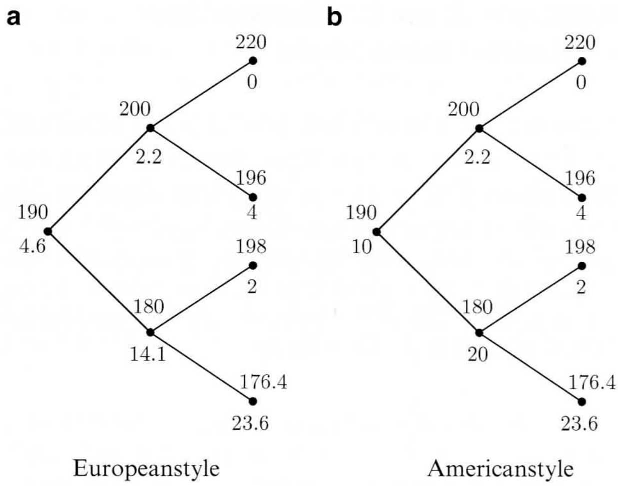


Fig. 3.3 Put option valuation in binomial tree

such that the net present value of a portfolio consisting of the option and a number of shares does not change over time.

(3) The presented formulas can be used also to value so-called *American style option contracts*, i.e., option contracts that can be exercised at any point t_j in time. Working backward in time, the value of such an option at time t_j is the maximum of the value given by (3.2) at t_j and the payoff from exercise at t_j. We illustrate this in Example 3.3 below. □

Example 3.3. Since European and American call options (with no dividend payments for the stock) yield the same price, we will consider in this example the valuation of a put option contract in a two-step binomial model. The initial price of the corresponding stock is 190 euros and the strike price is 200 euros. We assume that each time step is 3 months long and the risk-free annual interest rate is 12%. In the first time step the price may go up by a factor $u_0 = \frac{20}{19}$ and down by a factor $d_0 = \frac{18}{19}$. In the second time step the potential growth factor is $u_1 = 1.1$ and the potential decline factor is $d_1 = 0.98$. This leads to the stock prices illustrated in Fig. 3.3. The upper number at each node indicates the stock price.

The payoff from the European put option is at time t_N given by $[X - S_N]^+$. At time t_j, $j < N$ its value is determined by the backward recursion (3.2), where $q_1 = 0.408$, $q_0 = 0.766$, and $r_j \Delta_j = 0.03$. At each node of the tree the lower number indicates the option price. In Fig. 3.3a the price of the European style option is indicated. Figure 3.3b shows how prices are affected if early exercise of the option is allowed. □

3.4 Relationship with the Continuous-Time Black–Scholes–Merton Model

The binomial model is often used numerically to value options and other derivatives. This is motivated from the well-known Black–Scholes (or Black–Scholes–Merton or Samuelson) model (see [46]). In their seminal paper, Black and Scholes assumed that the relative return on a stock (with no dividend payments) in a short period of time was normally distributed. Assuming that μ is the expected return on the stock and σ is the standard deviation (*volatility*) of the stock price S, the expected return over the time interval $[t_0, t_0 + \Delta t]$ is then $\mu \Delta t$, whereas the standard deviation of the return over this time interval is $\sigma\sqrt{\Delta t}$. That is,

$$\frac{\Delta S}{S} \sim N(\mu \Delta t, \sigma^2 \Delta t), \tag{3.6}$$

where ΔS is the change in the stock price S from $t = t_0$ to $t = t_0 + \Delta t$, μ is the expected return on the stock, and σ is the standard deviation of the stock price.

Following Merton's approach (e.g., [118]) this can be motivated as follows. Assuming that the expectation mentioned below exists, consider the random variable

$$\Delta W_j = (S_j - S_{j-1}) - E_{j-1}[S_j - S_{j-1}].$$

Here, $E_{j-1}[S]$ is the expectation of S conditional on the information that is available at time t_{j-1}.

Thus ΔW_j is the part in $S_j - S_{j-1}$ that cannot be predicted given the available information at time t_{j-1}. Moreover, we assume that ΔW_j can be observed at time t_j, that is, $E_j[\Delta W_j] = \Delta W_j$, and that the ΔW_j are uncorrelated across time. ΔW_j is called the *innovation term* of the stock price because

$$S_j = S_{j-1} + E_{j-1}[S_j - S_{j-1}] + \Delta W_j.$$

Now let $V_j = E_0[(\Delta W_j)^2]$ denote the variance of ΔW_j and $V = E_0[(\sum_{j=1}^N \Delta W_j)^2]$ the variance of the cumulative errors. Since the ΔW_j are uncorrelated across time, it follows that

$$V = \sum_{j=1}^{N} V_j.$$

In finance the next three assumptions on V_k and V are widely accepted.

Assumption 3.4. Consider a fixed time interval $[t_0, t_0 + T]$, where stock prices are observed at N equidistant points in time t_j, $j = 0, \ldots, N$. Then there exist three positive constants $c_i > 0$, $i = 1, 2, 3$, that are independent of the number of points N such that:

1. $V \geq c_1 > 0$, that is, increasing the number of observations of stock prices will not completely eliminate risk. There always remains uncertainty about stock prices.

2. $V \leq c_2 < \infty$, that is, if more observations of stock prices, and therefore more trading, occurs, then the pricing system will not become unstable.
3. $\frac{V_j}{\max\{V_j,\ j=1,\ldots,N\}} \geq c_3,\ j=1,\ldots,N$, that is, market uncertainty is not concentrated in some special periods. Whenever markets are open, there is at least some volatility. □

Merton [118] used these three assumptions to prove that the innovation term ΔW_j has a variance that is proportional to the length of the time interval Δt that has elapsed between t_j and t_{j-1} (see also [121] for a proof of the next result).

Theorem 3.5. *Under Assumption 3.4 there exist finite constants σ_j that are independent of Δt such that $V_j = \sigma_j^2 \Delta t$. The σ_j depend on the available information at time t_{j-1}.* □

The next step to motivate (3.6) is to give an approximation for the conditional expectation of the change in stock prices $E_{j-1}[S_j - S_{j-1}]$. Notice that this expectation depends both on the available information at time t_{j-1}, which we will denote by I_j, and on the length of the time interval Δt. Assuming that this is a smooth function $f(I_{j-1},\Delta t)$ we can use Taylor's theorem to approximate this expectation as follows:

$$E_{j-1}[S_j - S_{j-1}] = f(I_{j-1},0) + \frac{\partial f(I_{j-1},\Delta t)}{\partial \Delta t}\Delta t + h((\Delta t)^2),$$

where $h(.)$ contains the higher-order terms in Δt. Now, if $\Delta t = 0$, then time will not pass and the predicted change in stock prices will be zero, i.e., $f(I_{j-1},0) = 0$. Therefore, neglecting the higher-order terms in Δt we have that

$$E_{j-1}[S_j - S_{j-1}] \approx \frac{\partial f(I_{j-1},\Delta t)}{\partial \Delta t}\Delta t. \tag{3.7}$$

Therefore, assuming additionally that the increments have a normal distribution,[4] we arrive at (3.6).

Assumption (3.6) implies that the stock price $S(t)$ has a *lognormal distribution*. That is, given the price of the stock at time $t = 0$ is S_0, the distribution of the natural logarithm of the stock at time t is

$$\ln(S(t)) \sim N(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t).$$

Thus the expectation and variance of $S(t)$ are

$$E[S(t)] = S_0 e^{\mu t} \text{ and } \sigma^2[S(t)] = S_0^2 e^{2\mu t (e^{\sigma^2 t} - 1)}, \text{ respectively.}$$

[4]Together with the previous assumptions made on W_j this implies that W_j is a *Brownian motion*.

Furthermore, we conclude with 95% confidence that

$$\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t - z_{\frac{\alpha}{2}}\sigma\sqrt{t} < \ln(S(t)) < \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + z_{\frac{\alpha}{2}}\sigma\sqrt{t},$$

where $z_{\frac{\alpha}{2}} \approx 1.96$ is the number that satisfies $\Phi(z_{\frac{\alpha}{2}}) = \frac{\alpha}{2} = 0.975$. Here $\Phi(d)$ is the *cumulative standard normal distribution* evaluated at d.[5] This implies that

$$S_0 * d := \mathrm{e}^{\ln(S_0)+(\mu-\frac{\sigma^2}{2})t - z_{\frac{\alpha}{2}}\sigma\sqrt{t}} < S(t) < \mathrm{e}^{\ln(S_0)+(\mu-\frac{\sigma^2}{2})t + z_{\frac{\alpha}{2}}\sigma\sqrt{t}} =: S_0 * u.$$

Thus, there is a 95% probability that the stock price will lie between $S_0 d$ and $S_0 u$. These numbers u and d give, then, some educated guesses for the corresponding numbers in the interval model we will discuss in Sect. 3.5.2.

In practice when the binomial model is used to value derivatives, and consequently Δt is small, one often uses $u = \frac{1}{d} = \mathrm{e}^{\sigma\sqrt{\Delta t}}$. This choice has the advantage that the tree recombines at the nodes, that is, an up movement followed by a down movement leads to the same stock prices as a down movement followed by an up movement. Furthermore, since $ud = 1$, one can easily calculate the price at any node. Notice that within the foregoing context with $z_{\frac{\alpha}{2}} = 1$, this choice implies that there is a 16% probability that the stock price will be lower, a 16% probability that it will be higher, and a 68% probability that it will be between these upper and lower bounds.

Black, Scholes, and Merton also derived pricing formulas for European calls and puts under the assumption that stock prices change continuously under the assumption of (3.6). They showed that the corresponding unique arbitrage-free prices for call and put options are

$$\begin{aligned} f_0(S_0) &= \Phi(d_1)S_0 - \mathrm{e}^{-r(t_N - t_0)}\Phi(d_2)X \text{ and} \\ f_0(S_0) &= \mathrm{e}^{-r(t_N - t_0)}\Phi(-d_2)X - S_0\Phi(-d_1), \end{aligned} \tag{3.8}$$

respectively, where $d_1 = \frac{\ln(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})(t_N - t_0)}{\sigma\sqrt{t_N - t_0}}$, $d_2 = d_1 - \sigma\sqrt{t_N - t_0}$.

Example 3.6. Consider the pricing of a European call option when both the stock and strike prices are 50 euros, the risk-free interest rate is 10% per year, the volatility is 40% per year, and the contract ends in 3 months. Then, with $r = 0.1$, $\sigma = 0.4$, $t_N - t_0 = 3/12$, and $S_0 = X = 50$, the price of this call option is, according to (3.8), $f_0 = 4.58$. In Fig. 3.4 we illustrate the pricing of this option using (3.2), (3.3) in a corresponding binomial tree with $N = 3$, which implies $\Delta t = 1/12$ and $u = \frac{1}{d} = \mathrm{e}^{\sigma\sqrt{\Delta t}} = 1.1224$. The price that results in this case is $f_0 = 4.77$. If we take a smaller grid $N = 6$, implying $\Delta = 1/24$ and $u = \frac{1}{d} = \mathrm{e}^{\sigma\sqrt{\Delta t}} = 1.1224$, a price of 4.42 results.

[5] Or, the probability that a variable with a standard normal distribution will be less than d.

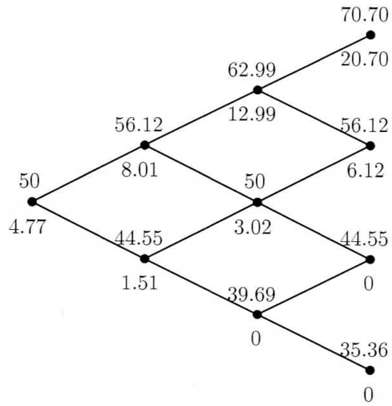


Fig. 3.4 Call option valuation in binomial tree

It can be shown in general that by increasing the number of grid points N, the price in the binomial model will converge to the continuous-time-model price (3.8). □

3.5 Risk Assessment Models

3.5.1 Current Models

Since the publication of the Black–Scholes formula [46], the theory of option pricing has gone through extensive developments in both theory and applications. Today it is the basis of a multibillion-dollar industry that covers not only stock options but also contracts written on interest rates, exchange rates, and so on. The theory has implications not only for the pricing of derivatives, but also for the way in which the risks associated with these contracts can be hedged by taking market positions in related assets. In fact the two sides of the theory are linked together inextricably since the theoretical price of an option is usually based on model assumptions that imply that all risk can be eliminated by suitable hedging. In daily financial practice, hedging is a theme that is at least as important as pricing; indeed, probably greater losses have been caused by misconstrued hedging schemes than by incorrect pricing.

Given the size of the derivatives markets, it is imperative that the risks associated with derivative contracts be properly quantified. The idealized model assumptions that usually form the basis of hedging constructions are clearly not enough to create a reliable assessment of risk. *Value-at-Risk* (VaR) was introduced by Morgan [119] as a way of measuring the sensitivity of the value of a portfolio to typical changes in asset prices. Although the VaR concept has been criticized on theoretical grounds (see, for instance, Artzner et al. [2]), it has become a standard that is used by regulatory authorities worldwide. For portfolios with a strong emphasis on derivative contracts, the normality assumptions underlying the VaR methodology may not be suitable, and additional ways of measuring risk are called for to generate a more complete picture.

Often, *stress testing* is recommended, in particular by practitioners, as a method that should supplement other measures to create a full picture of portfolio risk (see, for instance, Basel Committee [24], Laubsch [107], and Greenspan [78]). The method evaluates the performance of given strategies under fairly extreme scenarios. In particular, in situations where worst-case scenarios are not easily identified, stress testing on the basis of a limited number of selected scenarios may be somewhat arbitrary, however. It would be more systematic, although also more computationally demanding, to carry out a comprehensive worst-case search among all scenarios that satisfy certain limits.

Major concerns associated with worst-case analysis are firstly, as already mentioned, the computational cost and, secondly, the dependence of the results on the restrictions placed on scenarios. The latter problem cannot be avoided in any worst-case setting; in the absence of restrictions on scenarios, the analysis would not lead to meaningful results. To some extent, the second problem may be obviated (at the cost of increased computational complexity) by looking at the results as a function of the imposed constraints. Among an array of risk management tools that are likely to be used jointly in practice, worst-case analysis may be valued as a method that is easily understood also by nonexperts.

In the standard Black–Scholes model, there is one parameter that is not directly observable, *volatility*. When the value of this parameter is inferred from actual option prices, quite a bit of variation is seen both over time and across various option types. It is therefore natural that uncertainty modeling in the context of option pricing and hedging has concentrated on the volatility parameter. In particular, the so-called *uncertain volatility model* has been considered by a number of authors [18, 108, 146]. In this model, volatility is assumed to range between certain given bounds, and prices and hedges are computed corresponding to a worst-case scenario.

The uncertain volatility model as proposed in the cited references assumes continuous trading, which is of course an idealization. In the following sections, we consider a discrete-time version that we call the *interval model*. In this model, the relative price changes of basic assets from one point in time to the next are bounded below and above, but no further assumptions concerning price movements are made.

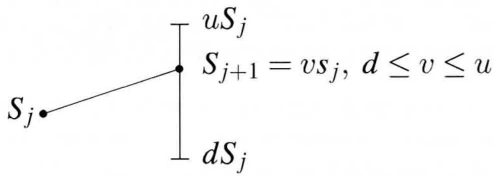
3.5.2 *Interval Model*

Interval models naturally arise in the context of markets where uncertainty instead of risk plays a dominant role, that is, if the uncertainty cannot be quantified in, e.g., a probability distribution. For instance if one would like to launch a completely new product for which there as yet no market, it is almost impossible to assess the involved risk of price changes. Also, in actuarial science well-known variables that are uncertain are, for example, life expectancy, evolution of wages, and interest rates. Further, in general, model risk cannot always be quantified in a stochastic framework. Therefore, we will approach price uncertainty here differently. We will assume that tomorrow's prices can fluctuate between some upper and lower bounds, which are given. For the rest we do not have a clue as to which price in this interval will be realized.

Formally, an *interval model* is a model of the form

$$\mathbb{I}^{u,d} := \{\mathscr{S} \mid S_{j+1} \in [dS_j, uS_j] \text{ for } j = 0, 1, 2, \ldots\}, \tag{3.9}$$

where u and d are given parameters satisfying $d < 1 < u$. The following figure illustrates a typical step in the price path of an interval model.



The model parameters u and d denote respectively the maximal and minimal growth factor over each time step.

An important issue is how these models relate to the binomial tree model and the continuous-time Black–Scholes–Merton model considered in Sect. 3.3.

The interval model may be compared to the standard binomial tree model with parameters u and d [57]:

$$\mathbb{B}^{u,d} := \{\mathscr{S} \mid S_{j+1} \in \{dS_j, uS_j\} \text{ for } j = 0, 1, 2, \ldots\}.$$

The binomial tree model just allows one specific upward and downward price movement. It provides boundary paths for the interval model $\mathbb{I}^{u,d}$. As already mentioned in Sect. 3.3, binomial models are motivated mainly by the fact that they can be used to approximate continuous-time models by letting the time step tend to zero. In contrast, the interval model may be taken seriously on its own, even for time steps that are not small.

Compared to the continuous-time modeling framework of Black, Scholes, and Merton, we recall from Sect. 3.4 that the continuous time models postulate a lognormal distribution for future prices. That is, with $t_0 = 0$,

$$\ln\left(\frac{S(t)}{S_0}\right) \sim N\left(t\left(\mu - \frac{\sigma^2}{2}\right), \sigma\sqrt{t}\right).$$

The stepwise comparison with interval models is straightforward. For a given price S_0 at time $t_0 = 0$, the statement on the next price S_1 at time t_1 is

$$\ln\left(\frac{S_1}{S_0}\right) \in [\ln(d), \ln(u)] \tag{3.10}$$

according to the interval model, while the continuous-time model postulates

$$\ln\left(\frac{S_1}{S_0}\right) \sim N\left(\mu - \frac{\sigma^2}{2}, \sigma\right). \tag{3.11}$$

The first statement is nondeterministic, though it may be interpreted in a stochastic sense, with σ-field $\{\emptyset, [\ln(d), \ln(u)]\}$ and their complements in $\mathbb{R}$, and probability one assigned to the interval. Under Assumption (3.11), the statement (3.10) is true with probability $\Phi(\frac{\ln(u)-\mu}{\sigma} - \frac{\sigma^2}{2}) - \Phi(\frac{\ln(d)-\mu}{\sigma} - \frac{\sigma^2}{2})$. In particular, under the extra symmetry condition $ud = 1$, u and d are fixed by specifying a confidence level for that probability. It is harder to compare the models globally over several time steps.