Introduction Actuarial Science

Option pricing

Frank van Berkum
f.vanberkum@uva.nl

May 2026



University of Amsterdam

Motivation for option pricing

Insurance companies carry large liabilities that must be supported by investment portfolios. The value of these assets evolves over time and can be highly volatile.

This leads to several challenges:

- Market downturns can significantly reduce asset values
- Mismatches between assets and liabilities may arise
- Extreme scenarios may threaten solvency

Financial derivatives provide tools to manage these risks. To use them effectively, we need reliable methods to determine their value.

Aim for this lecture

In this part of the course you will learn about:

- Financial instruments such as bonds and stocks and how the value of such instruments depends on the state of the market;
- Financial derivatives such as options and futures that can be used to mitigate risk in an investment portfolio;
- Pricing of such financial derivatives through replicating portfolios using binomial trees, i.e. risk-neutral pricing.

Agenda

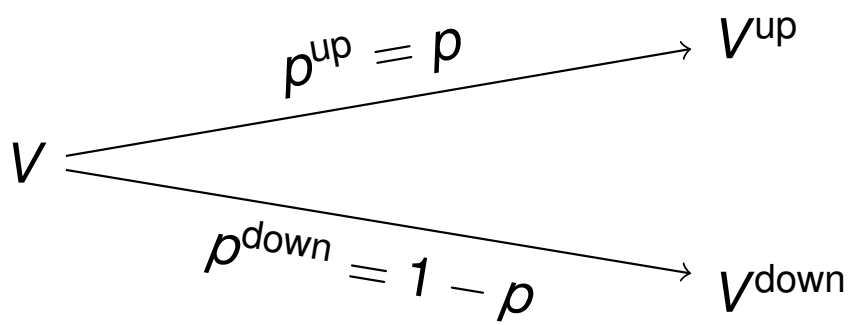
State-dependent market and financial instruments

Option pricing through replicating portfolios
- Introduction: numerical example
- Generalization in discrete time
- Generalization in continuous time

Other types of derivatives and their application in the insurance sector

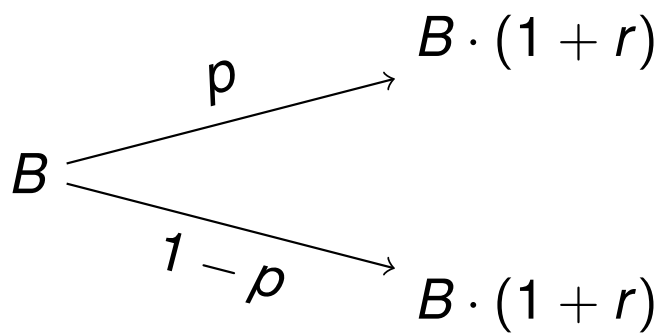
State-dependent market

- We consider assets that have a known value today, but an uncertain value in the future.
- To model this uncertainty, we use a simple framework:
  - In the next period, the economy can move **up** or **down**
  - The asset therefore takes one of two possible values
- This structure can be represented using a **binomial tree**, which will be the starting point for our analysis.



Bond

- A bond represents an agreement between an issuer and an investor. At inception, the investor provides an amount B.
- In return, the issuer promises a fixed payment at maturity:
  - The notional amount
  - Plus interest at rate r
- The payoff is therefore:



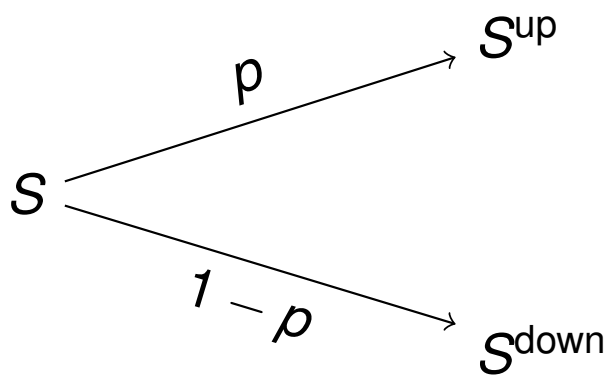
- This payment does not depend on how the market evolves. Bonds are therefore referred to as **fixed-income instruments**.

Stock

- A stock represents ownership in a company. Its value depends on how market conditions evolve.
- In our two-state setting, the stock price next period is either:
  - S^{up}, if the economy improves
  - S^{down}, if conditions deteriorate
- We assume both upside potential and downside risk:

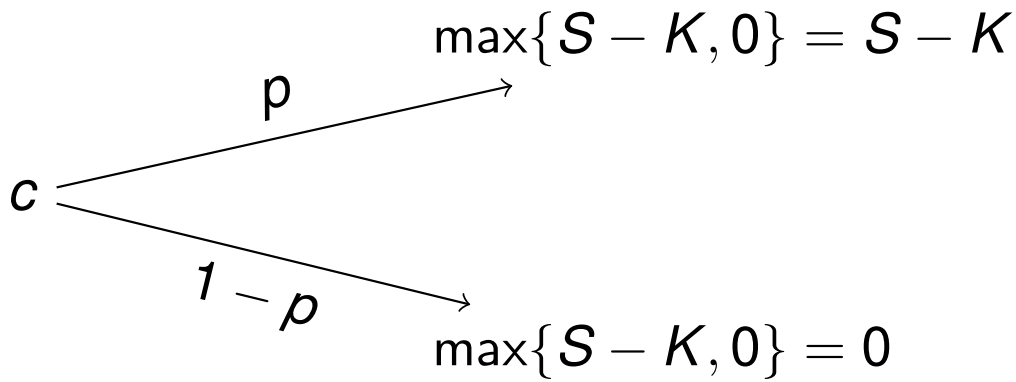
$$S^{down} < S < S^{up}$$

- This ensures a realistic setting in which investors face uncertainty.



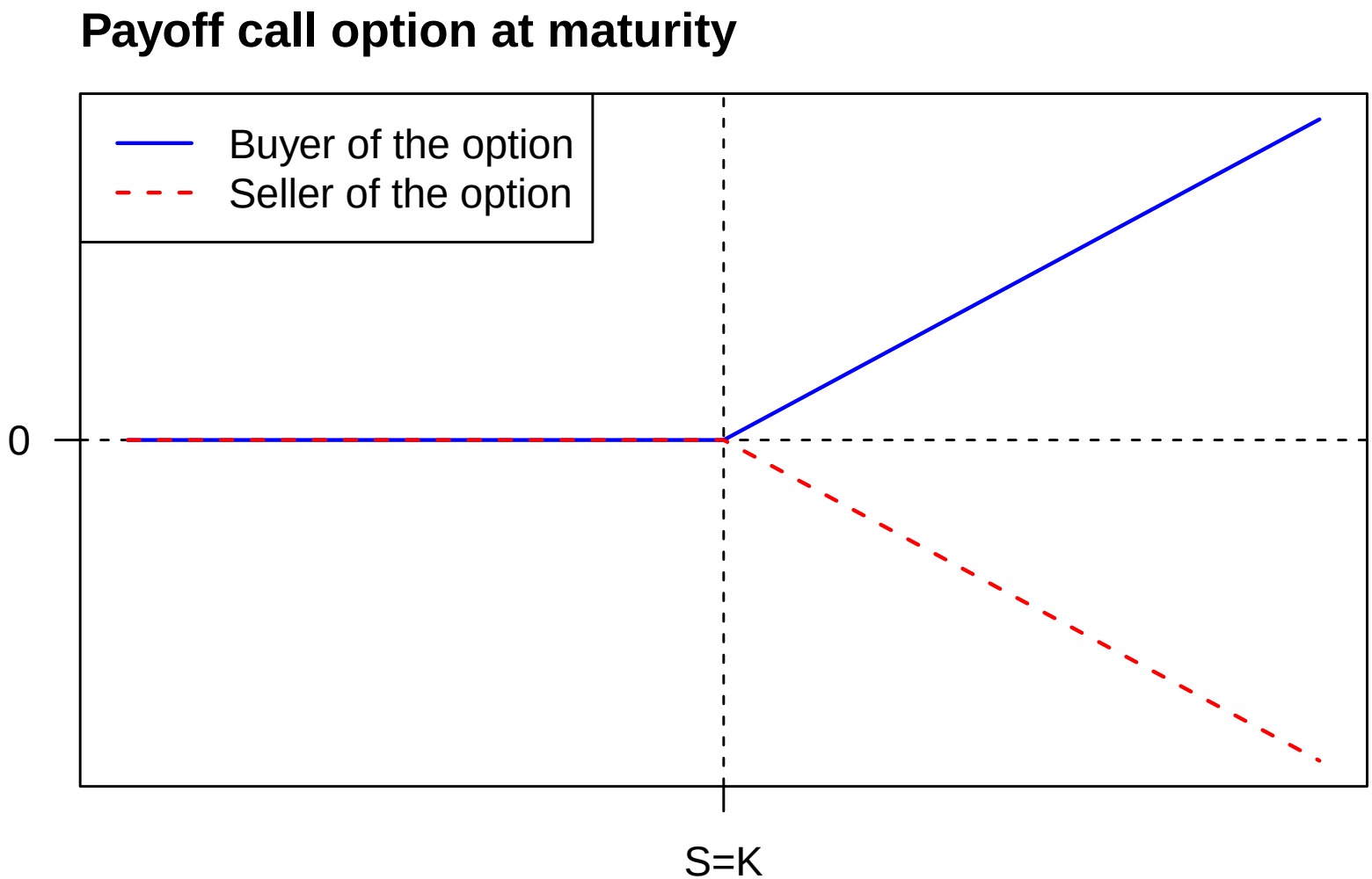
Call option

- A call option C is a financial derivative whose payoff depends on some underlying asset;
- A call option gives the owner the right (not the obligation!) **to buy** the underlying asset at the contractually agreed strike price K;
- If $S^{\text{down}} < K < S^{\text{up}}$, the payoff for the owner of the call option is given by the following scheme:



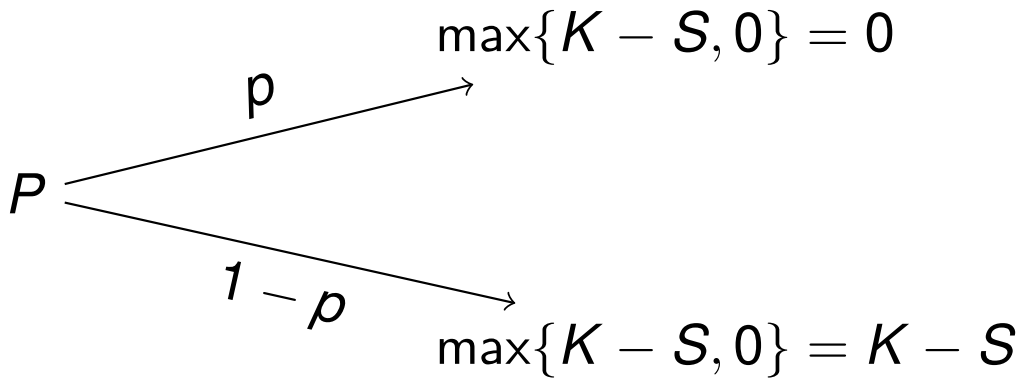
- The payoff for the issuer of the call option is mirrored to the scheme above.

Call option

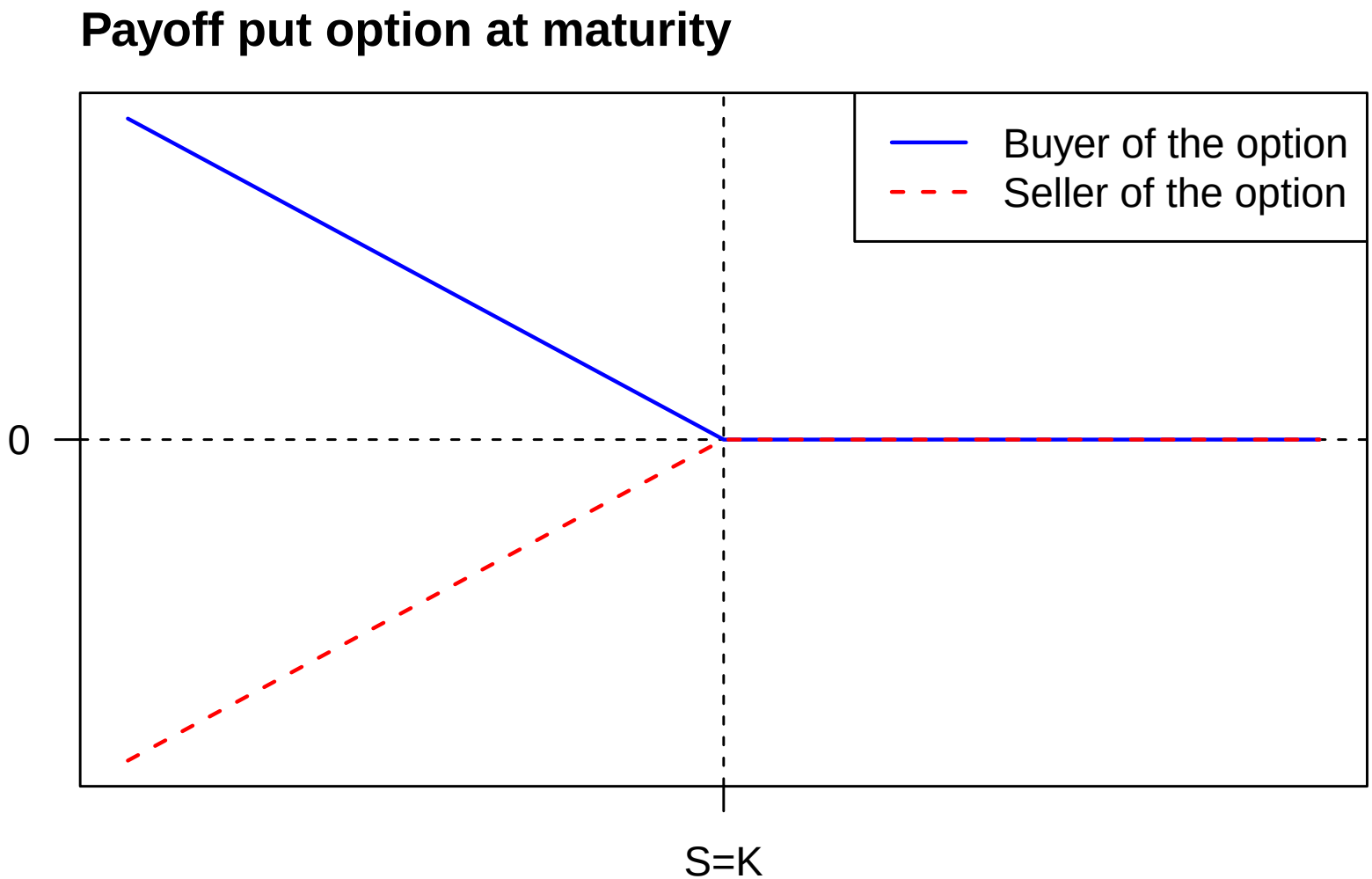

Payoff call option at maturity
Buyer of the option
Seller of the option
0
S=K

Put option

- A put option P is a financial derivative whose payoff depends on some underlying asset;
- A put option gives the owner the right (not the obligation!) **to sell** the underlying asset at the contractually agreed strike price K;
- If $S^{\text{down}} < K < S^{\text{up}}$, the payoff of the put option is given by the following scheme:



Put option


Payoff put option at maturity
Buyer of the option
Seller of the option
0
S=K

Summary of what we now know

- We have discussed bonds and stocks for which we assume the price today and the range of possible outcomes for next period to be known;
- For call and put options with given strike price and known values of the underlying, we can compute the payoff at maturity;
- We have not yet discussed how the price of these options should be determined!

Pricing of options

What could reasonable approaches be for the pricing of options?

Possibility 1 The fair price equals is computed as the discounted expected value of the stochastic payoff;

Possibility 2 The price is determined by equation supply and demand, taking into account everyone's risk preference.

As we will see, both approaches are wrong, but we will find that:

- There is a theoretical, unique price
- which is not simply the discounted expected value of the contract's payoffs at maturity
- and the price is universal: it has nothing to do with risk preferences.

Key ideas behind Option Pricing

- We will not try to compute option prices on a stand-alone basis, but instead in terms of the market price of the underlying assets (through replication);
- We will not calculate ordinary expectations (using p), but apply risk-neutral pricing;
- Risk-neutral expectations replace the ordinary expectations.

Consequences:

- Option price will turn out not to depend on mean rate of return of underlying asset during the lifetime of the option;
- Seller of the option does not need to take any risk at all!

Agenda

State-dependent market and financial instruments

Option pricing through replicating portfolios
- Introduction: numerical example
- Generalization in discrete time
- Generalization in continuous time

Other types of derivatives and their application in the insurance sector

A numerical example

Defining the financial instruments

We consider a simple market with three instruments:

- A bond with notional 100 and interest rate 4%
- A stock currently priced at 100
- A call option on the stock with strike $K = 100$

The stock value next year is uncertain:

- 120 if the economy performs well
- 90 if the economy performs poorly

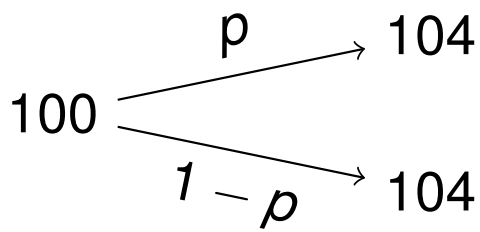
The payoff of the option is known, but its current price c is unknown.

A numerical example

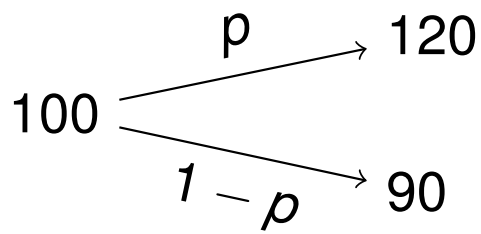
Constructing the binomial trees

The binomial trees for the bond, stock and the call option are:

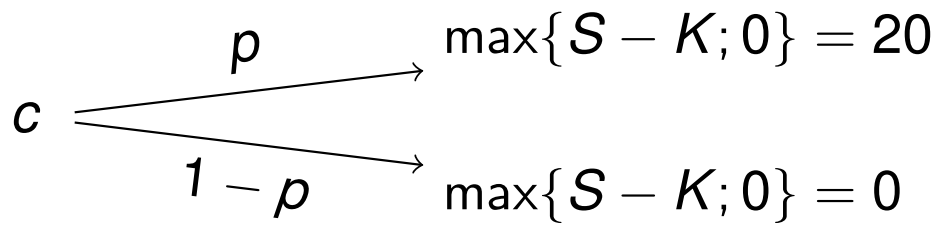
Bond



Stock



Call option



A numerical example

Finding a replicating portfolio

Now that we have constructed the binomial trees, the next step is to find a portfolio of ψ bonds and ϕ stocks that replicates the payoff of the call option in all states of economy:

	Bond	Stock	Call option
payoff in up-scenario:	$\psi \times 104$	$\phi \times 120$	20
payoff in down-scenario:	$\psi \times 104$	$\phi \times 90$	0

A numerical example

Finding a replicating portfolio

Now that we have constructed the binomial trees, the next step is to find a portfolio of ψ bonds and ϕ stocks that replicates the payoff of the call option in all states of economy:

	Bond	+ Stock	= Call option
payoff in up-scenario:	$\psi \times 104$	$+ \phi \times 120$	$= 20$
payoff in down-scenario:	$\psi \times 104$	$+ \phi \times 90$	$= 0$

A numerical example

Finding a replicating portfolio

Now that we have constructed the binomial trees, the next step is to find a portfolio of ψ bonds and ϕ stocks that replicates the payoff of the call option in all states of economy:

	Bond	+ Stock	= Call option
payoff in up-scenario:	$\psi \times 104$	$+ \phi \times 120$	$= 20$
payoff in down-scenario:	$\psi \times 104$	$+ \phi \times 90$	$= 0$

And this portfolio of bond and stock should then also have the same price as the call option right now:

price right now:	$\psi \times 100$	$+ \phi \times 100$	$= c$

A numerical example

Finding the price of the option

Solving the linear system of equations:

$$\begin{array}{lllll} \text{payoff in up-scenario:} & \psi \times 104 & + \phi \times 120 & = 20 \\ \text{payoff in down-scenario:} & \psi \times 104 & + \phi \times 90 & = 0 \\ \text{price right now:} & \psi \times 100 & + \phi \times 100 & = c, \end{array}$$

we find: $\phi = 2/3$, $\psi = -60/104$, and thus $c \approx 8.97$.

Key observations:

- We did not use the probability of up/down moves
- No assumptions were made about risk preferences

A numerical example

Generating a riskless portfolio

Consider the following investor that sells 100 options and buys 67 stocks:

Investor sells 100 options	+ 897
Investor buys $66\frac{2}{3}$ stocks	-6,667
Investor bank account	-5,770

In the up-scenario (if the price goes up), the people that bought the option from the investor will exercise their right and thus 100 people buy the stock from the investor for 100 when it costs 120:

Investor receives 100 people times 100	+ 10,000
Investor pays back loan (5,770 plus 4%)	-6,000
Investor buys $33\frac{1}{3}$ extra stocks (at a price of 120):	-4,000
	0

In the down-scenario (if the price goes down), the people that bought the option from the investor will not exercise the option:

Investor sells the stock at a price of 90	+ 6,000
Investor pays back loan (5,770 plus 4%)	-6,000
	0

Hence, if the investor sells N call options, buys $\phi \times N$ stocks and puts the remaning proceeds on the bank account, the payoff one year from now is completely predictable.

Agenda

State-dependent market and financial instruments

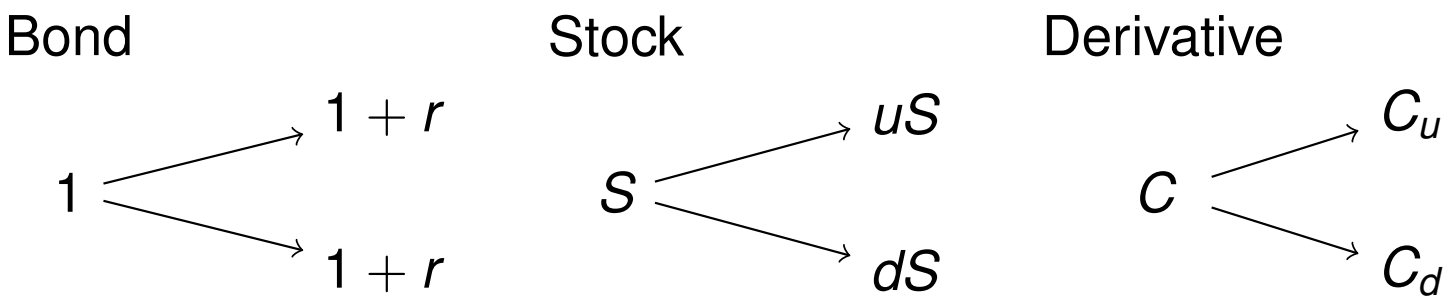
Option pricing through replicating portfolios
- Introduction: numerical example
- Generalization in discrete time
- Generalization in continuous time

Other types of derivatives and their application in the insurance sector

Generalizing the one-period two-state model

The numerical example is useful to understand the dynamics, but a generalization of the approach is useful to better understand wider dynamics

We generalize as follows (assuming $0 < d < 1 < u$):



For these generalized binomial trees, we can find the replicating portfolio in a similar way $\rightarrow$

Generic solution

The corresponding system of linear equations equals:

$$\begin{array}{lllll} \text{price right now:} & \psi \times 1 & + \phi \times S & = C \\ \text{payoff in up-scenario:} & \psi \times (1+r) & + \phi \times uS & = C_u \\ \text{payoff in down-scenario:} & \psi \times (1+r) & + \phi \times dS & = C_d, \end{array}$$

and solving this for ϕ and ψ yields

$$\phi = \frac{C_u - C_d}{S(u-d)} \quad \text{and} \quad \psi = \frac{uC_d - dC_u}{(1+r)(u-d)},$$

from which we find

$$C = \phi S + \psi 1 = \frac{C_u - C_d}{S(u-d)} S + \frac{uC_d - dC_u}{(1+r)(u-d)}.$$

Generic solution

You may already have seen that this expression can be simplified:

$$\begin{aligned} C = \phi S + \psi 1 &= \frac{C_u - C_d}{S(u-d)} S + \frac{uC_d - dC_u}{(1+r)(u-d)} \\ &= \frac{(1+r)(C_u - C_d)}{(1+r)(u-d)} + \frac{uC_d - dC_u}{(1+r)(u-d)} \\ &= \frac{1}{1+r} \left(q \cdot C_u + (1-q) \cdot C_d \right), \end{aligned}$$

with $q = \frac{(1+r)-d}{u-d}$. In here, we recognize the following structure:

$$C = \underbrace{\frac{1}{1+r}}_{\text{discounting}} \cdot \underbrace{\left(q \cdot C_u + (1-q) \cdot C_d \right)}_{\text{risk-neutral weighted average}},$$

but here we use the risk-neutral probability q rather than the ordinary p that was introduced before as the probability of moving up

Generic solution

Interpretation

The value of the option does not depend on the probability p but the risk-neutral probability q:

$\Rightarrow$ The current value of the stock S is a weighted average of the possible values next period

$\Rightarrow$ This weighting is using probabilities q rather than p!

The reason for this is that we are not valuing the option in absolute terms:

- Probabilities of going up or down are already reflected in the current stock price
- Probabilities of going up or down are not taken into account again when valuing the option
- We are calculating the option value in terms of the price of the underlying stock

Agenda

State-dependent market and financial instruments

Option pricing through replicating portfolios
- Introduction: numerical example
- Generalization in discrete time
- Generalization in continuous time

Other types of derivatives and their application in the insurance sector

Generalization in continuous time

Decreasing step sizes

So far, we have considered one-period (one year) binomial trees only which does not reflect reality:

- Financial assets can take on more than two values in future periods
- Financial assets can change almost any second

To better reflect reality, we now consider binomial trees of period $\Delta t < 1$:

- With $\Delta t = 1/2$, each binomial tree spans a period of half a year
- With $\Delta t = 1/365$, a binomial tree shows what can happen in one day

If we let $\Delta t \to 0^+$, we approach a continuous setting: financial assets can change at each instant!

Generalization in continuous time

Changing from discrete discounting to continuous discounting

When time steps become increasingly small, we should consider continuous discounting:

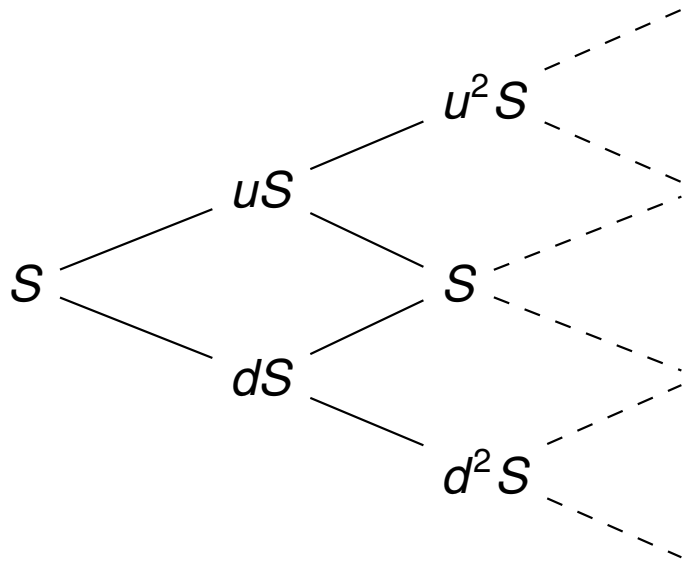
- Discounting for a one-year period using a discrete interest rate r^d is done through $(1 + r^d)^{-1}$ (two years would be $(1 + r^d)^{-2}$)
- Discounting for a period Δt (which may become arbitrarily small) using a continuous interest rate r^c is done through $\exp(-r^c \Delta t)$

Note that when $r^d = r^c$, the resulting discounting factor for a period of one year is practically the same, since $\exp(x) \approx 1 + x$ for small x.

Generalization in continuous time

Introducing symmetry in binomial trees of stock

It is convenient to impose $u = 1/d$ so that the binomial tree remains centered around S:



As a result, with T periods, there are $T + 1$ possible outcomes.
How many possible outcomes would there be if $u \neq 1/d$?

Generalization in continuous time

Choosing u and d to reflect stock dynamics

We still need to specify u and d, and preferably this is done in a way that reflects the actual dynamics of the stock

- Stock returns are often considered lognormally distributed, i.e. $\log(S_{t+\Delta t}/S_t) \sim \text{Normal}(\nu\Delta t, \sigma^2\Delta t)$ Why lognormal?
- Here, ν and σ^2 are the annualized logreturn and variance

Suppose again that p is the probability that the stock goes up and $1-p$ is the probability that it goes down

Can we find u, d and p as a function of ν, σ^2 and Δt?

Generalization in continuous time

Choosing u and d to reflect stock dynamics (cont.)

Mean return over a Δt-period:

$$\begin{aligned} \mathrm{E}\left[\ln(S_{t+\Delta t}/S_t)\right] &= \nu \Delta t \\ &= p \ln u + (1-p) \ln d \end{aligned}$$

Variance over a Δt-period: this result is based on $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$

$$\begin{aligned} \mathrm{Var}\left[\ln(S_{t+\Delta t}/S_t)\right] &= \sigma^2 \Delta t \\ &= p(\ln u)^2 + (1-p)(\ln d)^2 - (p \ln u + (1-p) \ln d)^2. \end{aligned}$$

Solving this system of two equations (subject to the constraint that $u = 1/d$) results in the following expressions for p, u and d:

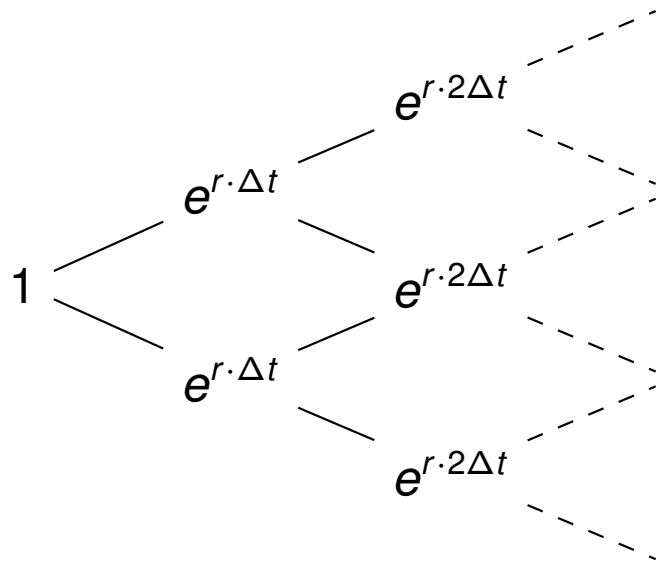
$$u \approx \exp(\sigma \sqrt{\Delta t}) \qquad p \approx \frac{1}{2} + \frac{1}{2} \left(\frac{\nu}{\sigma}\right) \sqrt{\Delta t}$$

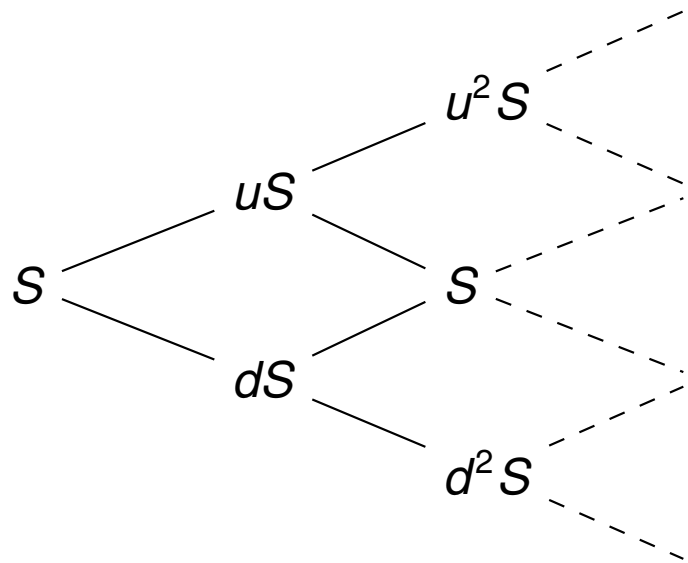
$$d \approx \exp(-\sigma \sqrt{\Delta t})$$

Generalization in continuous time

Finding option value in multi-period setting

We now have all the ingredients to construct the binomial trees for the bond and for the stock:





How should we compute the current price of a call option C on stock S with strike price K?

Generalization in continuous time

Finding option value in multi-period setting

Valuation of an option in a multi-period setting is done as follows:

1. Determine the payoff of the option at maturity in all states of the economy;
2. Compute the value of the option one period before maturity using the formula:

$$C = \exp(-r\Delta t)\left(q \cdot C_u + (1-q) \cdot C_d\right)$$

with

$$q = (\exp(r\Delta t) - d)/(u - d)$$

3. Continue until the starting node which provides the price of the option.

Generalization in continuous time

Example with two periods

Consider the following setting:

- A continuous, annualized risk free rate of 4%
- A stock S which is currently priced at 100 and has annualized volatility of 18%
- A call option with strike price $K = 100$ and maturity of 1 year

Determine the price of the call option using a binomial tree and consider two periods of half a year:

- Determine u and d and construct the binomial trees of stock and bond
- Determine the payoff of the call option at maturity in all states of the economy
- Work backwards to find the current price of the call option

Other financial derivatives

So far, we have only considered call and put options that can only be exercised at maturity, so-called European options:

- American options can also be exercised at intermediate points in time
- Given the same underlying stock S and strike price, what call option is worth more? European or American?

Future contracts give you the obligation (as opposed to an option) to purchase that asset at the strike price:

- Given the same underlying stock S and strike price K, what is worth more? A European call option or a future contract?

(Financial) derivatives in the insurance industry

The derivatives we considered in this lecture all relate to a stock S

Insurance companies may wish to purchase derivatives with a different underlying:

- In order to hedge interest rate sensitivities, insurance companies typically consider options, futures/swaps and swaptions on interest rates
- For example: if interest rates move down, liabilities go up. Interest rate derivatives may provide offsetting payments to the insurer
- Insurance companies may also want to hedge their longevity risk due to the mortality assumptions
- Longevity swaps provide payments to the insurance company if people live longer than expected